C. R. Acad. Sci. Paris, Ser. I 352 (2014) 907–912

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Optimal control

Source identification for the wave equation on graphs

Identification de sources pour l'équation des ondes sur des graphes

Sergei Avdonin [a], Serge Nicaise [b]

[a] *Dept. of Mathematics and Statistics, University of Alaska, Fairbanks, AK 99775, USA*
[b] *Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, 59313 Valenciennes cedex 9, France*

ARTICLE INFO

Article history:
Received 24 July 2014
Accepted 12 September 2014
Available online 2 October 2014

Presented by Gilles Lebeau

ABSTRACT

We consider source identification problems for the wave equation on an interval and on trees. The main advantage of our approach is its *locality*. Our algorithm reduces essentially to the resolution of a linear integral Volterra equation of the second kind and is new even for an interval.

RÉSUMÉ

Nous considérons un problème d'identification de sources pour l'équation des ondes sur un intervalle ou sur des arbres. L'avantage principal de notre approche est sa *localité*. Notre algorithme se réduit essentiellement à la résolution d'une équation intégrale de Volterra du second ordre et est nouveau, même pour un intervalle.

Version française abrégée

Introduction

Dans cette Note, nous considérons un problème d'identification de sources pour l'équation des ondes sur un intervalle ou sur des arbres. L'avantage principal de notre approche est sa *localité* : pour reconstruire les fonctions inconnues sur une partie du graphe, nous utilisons seulement des informations sur ce sous-graphe. Ce point de vue nous permet de proposer un algorithme d'identification très efficace et qui est nouveau, même pour un intervalle, et qui est beaucoup plus simple que les algorithmes proposés dans [4,5,2,3]. Notre algorithme se réduit essentiellement à la résolution d'une équation intégrale de Volterra du second ordre. À partir du problème d'identification de sources sur un intervalle, nous montrons comment l'étendre au cas des graphes.

E-mail addresses: s.avdonin@alaska.edu (S. Avdonin), Serge.Nicaise@univ-valenciennes.fr (S. Nicaise).

http://dx.doi.org/10.1016/j.crma.2014.09.008
1631-073X/

Le cas d'un intervalle

Nous considérons le problème aux limites (1), où f est une fonction donnée et g est l'inconnue qui est du type (a) ou (b). Notre problème est de reconstruire g à partir de l'observation $u_x(0,t), t \in [0,T]$. Cela est possible dans le temps optimal $T = l$.

En utilisant une représentation spectrale de la solution u de (1), on peut montrer que l'observation admet la représentation (7) dans le cas (a) et (8) dans le cas (b), valide pour $0 \le t \le l$ avec la régularité spécifiée dans le Théorème 2.1, la fonction w étant de la forme (9). Cette représentation permet l'identification of g. Dans le cas (a), pour obtenir un algorithme d'identification stable, nous devons supposer que $f \in H^1(0,T)$ et que $f(0) \neq 0$. De (7) et (9), nous déduisons que $u_x(0,\cdot) \in \mathcal{H}^1 := \{\mu \in H^1(0,l), \mu(0) = 0\}$. En posant $H(x) = \int_0^x g(\xi)\,d\xi$ et en intégrant par parties dans (10), nous obtenons l'équation intégrale de Volterra du second ordre (11). L'application $H \mapsto u_x(0,\cdot)$ étant bornée et inversible dans $\mathcal{H}^1$, l'application $g \mapsto u_x(0,\cdot)$ est aussi bornée et inversible de $L^2(0,l)$ dans $\mathcal{H}^1$.

Pour l'identification de g dans le cas (b), nous supposons $f \in C[0,T]$ et $f(0) \neq 0$. En substituant (9) dans (8), nous obtenons (12) – où nous étendons f par zéro sur $t < 0$. Cette formule permet de trouver ξ_i et α_i de manière itérative – voir (13) et (14).

Le problème d'identification sur des graphes étoilés

Un graphe étoilé Γ est constitué de N arêtes e_j identifiées aux intervalles $[0, l_j]$, $j = 1,\dots,N$, connectés en un nœud identifé avec l'extrémité gauche des intervalles. Sans restriction, nous pouvons supposer que $l_1 = \max_{i=1,\dots,N-1}\{l_i\}$. Sur chaque intervalle, on considère le problème (15) et au nœud intérieur, on impose les conditions standard de Kirchhoff–Neumann (16).

Le problème est de reconstruire les fonctions inconnues g_j, $j = 1,\dots,N$, à partir des observations $\mu_j(t) := u_x^j(l_j,t)$, $j = 1,\dots,N-1$, $t \in [0,T]$. Nous prouvons un résultat d'unicité et fournissons un algorithme de reconstruction dans l'intervalle de temps optimal $T = l_1 + l_N$.

Dans une première étape, nous reconstruisons les fonctions g_j, $j = 1,\dots,N-1$ en utilisant les observations correspondantes $\mu_j(t)$, $j = 1,\dots,N-1$ à l'aide d'un algorithme similaire à celui décrit dans la Section « Le cas d'un intervalle ». Ces identifications s'opèrent en un intervalle de temps de longueur l_1. Comme les fonctions g_j, $j = 1,\dots,N-1$ sont maintenant connues, on peut considérer la solution d'un problème similaire à (15), où maintenant $g_j = 0$, pour tout $j = 1,\dots,N-1$. En utilisant l'astuce décrite à la Remarque 2.2, on se ramène au problème d'identification pour le problème (18). Comme à la section précédente, en utilisant une représentation spectrale de la solution de (18), on obtient (19), où $w_N^j(x,t)$ désigne la restriction à l'arête e_N de la solution w^j du problème (16)–(20). Une formule explicite de $w_N^1(x,t)$ peut être obtenue, ce qui permet de transformer (19) en (21). Cette identité permet de trouver $g_N(x)$ pour $0 < x < \lambda := \min_{i=1,\dots,N}\{l_i\}$ en utilisant l'algorithme décrit dans la Section « Le cas d'un intervalle ». Nous soustrayons ensuite au problème la partie de $g_N(x)$ déjà trouvée et répétons la procédure pour trouver $g_N(x)$ sur l'intervalle $\lambda < x < 2\lambda$ en utilisant (22), où $\tilde{\mu}_1(t)$ est l'observation obtenue après la soustraction correspondante. On trouve g_N sur tout l'intervalle $[0, l_N]$ en utilisant une nombre fini de telles étapes.

Le problème d'identification sur des arbres peut se faire de la même manière, avec des observations en tous les nœuds extérieurs, sauf un. Pour des graphes avec circuits, une observation supplémentaire par boucle doit être ajoutée. Nous revoyons à [1] pour les détails.

1. Introduction

In this Note we consider source identification problems for the wave equation on an interval and on trees. The main advantage of our approach is its *locality*: to recover unknown functions on a part of the graph, we use a piece of information relevant only to this subgraph. This feature of our method allows us to propose a very efficient identification algorithm, which is new, even for an interval, and much simpler than algorithms proposed in the papers [4,5,2,3]. Our algorithm reduces essentially to the resolution of a linear integral Volterra equation of the second kind. Starting with source identification problems on an interval, we extend our approach to equations on graphs.

2. The case of one interval

We consider the initial boundary value problem:

$$\begin{cases} u_{tt} - u_{xx} = f(t)g(x), & 0 < x < l,\ 0 < t < T, \\ u(0,t) = u(l,t) = 0, & 0 < t < T, \\ u(x,0) = u_t(x,0) = 0, & 0 < x < l. \end{cases} \tag{1}$$

Here f is a known function, while g is an unknown function. We will consider the cases:

$$\text{(a) } g \in L^2(0,l), \qquad \text{(b) } g(x) = \sum_{j=1}^{N} \alpha_j \delta(x - \xi_j), \quad 0 < \xi_1 < \xi_2 < \ldots < \xi_N < l.$$

The problem is to recover g from the observation $u_x(0,t), t \in [0,T]$. We will prove that it is possible for $T = l$, and this is generally the minimal observation time.

Note that similar results can be obtained if in (1) a term qu with a known function $q \in L^\infty(0,l)$ is added to the differential equation, the details are available in [1].

2.1. Two representations and regularity of the observation $u_x(0,t)$

Let $\phi_n = \sqrt{2}\sin(n\pi \cdot /l)$ and $\lambda_n = n^2\pi^2/l^2$, $n \in \mathbb{N}^*$, be the eigenfunctions and eigenvalues of the Laplace operator in $(0,l)$ with Dirichlet boundary conditions. We represent the solution u to (1) in the form of series

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x), \quad \text{with } c_n(t) = \gamma_n \int_0^t f(\tau) \frac{\sin \omega_n (t-\tau)}{\omega_n} \, d\tau, \tag{2}$$

where $\omega_n = n\pi/l$, $\gamma_n = \int_0^l g(x)\phi_n(x)\,dx$ in the case (a) and $\gamma_n = \sum_{j=1}^N \alpha_j \phi_n(\xi_j)$ in the case (b).

From (2), we obtain the spectral representation of the observation:

$$u_x(0,t) = \sum_{n=1}^{\infty} \gamma_n \phi_n'(0) \int_0^t f(\tau) \frac{\sin \omega_n (t-\tau)}{\omega_n} \, d\tau. \tag{3}$$

We notice now that the solution to the initial boundary value problem:

$$\begin{cases} w_{tt} - w_{xx} = 0, & 0 < x < l, \ 0 < t < T, \\ w(0,t) = f(t), \quad w(l,t) = 0, & 0 < t < T, \\ w(x,0) = w_t(x,0) = 0, & 0 < x < l, \end{cases} \tag{4}$$

can be presented under the form:

$$w(x,t) = \sum_{n=1}^{\infty} \lambda \left[\phi_n'(0) \int_0^t f(\tau) \frac{\sin \omega_n (t-\tau)}{\omega_n} \, d\tau \rho \right] \phi_n(x). \tag{5}$$

Changing formally the order of summation and integration with respect to x in (3), we get:

$$u_x(0,t) = \int_0^l g(x) w(x,t)\,dx, \quad 0 \le t \le T. \tag{6}$$

Since the speed of the wave propagation in the system (1) is equal to one, only $g(x)$ with $x \le t$ gives an input to $u_x(0,t)$. (Equivalently, we can use the fact that $w(x,t) = 0$ for $x > t$.) Therefore, we obtain the second, dynamical, representation of the observation which is valid for $0 \le t \le l$:

$$u_x(0,t) = \int_0^t g(x) w(x,t)\,dx. \tag{7}$$

In the case (b), formula (7) takes the form:

$$u_x(0,t) = \sum_{j:\, \xi_j \le t} \alpha_j w(\xi_j, t), \quad 0 \le t \le l. \tag{8}$$

Our arguments are valid for functions g that are finite linear combinations of ϕ_n and are extended to all g using a density argument. Hence our results can be summarized as follows.

Theorem 2.1. *(a) If in the IBVP (1) $f \in L^2(0,T)$ and $g \in L^2(0,l)$, then $u_x(0,\cdot) \in C[0,T]$ and can be represented by formulas (6) and (7).*

(b) If $f \in C[0,T]$ and g is a linear combination of δ-functions, then $u_x(0,\cdot) \in C[0,T]$ and can be represented by formula (8).

2.2. Identification of g

For $t \le l$ the solution to the IBVP (4) has the form:

$$w(x,t) = f(t-x), \quad \text{for } 0 < x \le t; \qquad w(x,t) = 0, \quad \text{for } x > t. \tag{9}$$

First, we discuss the identification procedure in the case (a). Substituting (9) into (7), we get:

$$u_x(0,t) = \int_0^t f(t-x) g(x)\,\mathrm{d}x. \tag{10}$$

At this point, to obtain a stable algorithm for the identification of g, we will require additionally that $f \in H^1(0,T)$ and $f(0) \neq 0$. Then from (7) and (9), it follows that $u_x(0,\cdot) \in \mathcal{H}^1 := \{\mu \in H^1(0,l), \mu(0) = 0\}$.

We set $H(x) = \int_0^x g(\xi)\,\mathrm{d}\xi$. Integrating by parts in (10), we obtain

$$u_x(0,t) = f(0)H(t) + \int_0^t f'(t-x)H(x)\,\mathrm{d}x. \tag{11}$$

This is a Volterra equation of a second kind on the interval $0 < t < l$ for $H(t)$. The map $H \mapsto u_x(0,\cdot)$ is bounded and boundedly invertible in $\mathcal{H}^1$, and so the map $g \mapsto u_x(0,\cdot)$ is bounded and boundedly invertible from $L^2(0,l)$ to $\mathcal{H}^1$.

For the identification of g in the case (b), we assume that $f \in C[0,T]$ and $f(0) \neq 0$. Substituting (9) into (8) we get:

$$u_x(0,t) = \sum_{j:\,\xi_j \le t} \alpha_j f(t-\xi_j), \quad 0 \le t \le l, \tag{12}$$

where we extended $f(t)$ by zero to negative t. This formula allows us to find ξ_i and α_i according to the following algorithm. First we find ξ_1 and α_1:

$$\xi_1 = \inf\{t \ge 0 : u_x(0,t) \neq 0\}, \qquad \alpha_1 = u_x(0,\xi_1)/f(0). \tag{13}$$

Then we set $\mu_2(t) = u_x(0,t) - \alpha_1 f(t-\xi_1)$, and find ξ_2 and α_2:

$$\xi_2 = \inf\{t \ge 0 : \mu_2(t) \neq 0\}, \qquad \alpha_2 = \mu_2(\xi_2)/f(0), \tag{14}$$

and so on.

Remark 2.2. If the function g is known on any subinterval of $(0,l)$ we can easily reduce the identification problem to the case when $g = 0$ on this subinterval. Indeed, if g is known on $(0,a)$, we consider the problem:

$$v_{tt} - v_{xx} = f(t)g_a(x), \qquad g_a(x) := g(x), \quad \text{for } 0 < x < a, \qquad g_a(x) := 0, \quad \text{for } a \le x < l,$$

with the same boundary and initial conditions as in (1). Then the difference $\hat{u} := u - v$ satisfies:

$$\hat{u}_{tt} - \hat{u}_{xx} = f(t)\hat{g}(x), \qquad \hat{g}(x) := 0, \quad \text{for } 0 < x < a, \qquad \hat{g}(x) := g(x), \quad \text{for } a \le x < l.$$

3. Identification problem on star graphs

A star graph Γ consists of N edges e_j identified with intervals $[0, l_j]$, $j = 1, \ldots, N$, connected with an internal vertex that we identify with the set of the left end points of the intervals. The boundary vertices are identified with the right end points of the corresponding intervals. Without loss of generality, we can assume that $l_1 = \max_{i=1,\ldots,N-1}\{l_i\}$. The following initial boundary value problem is considered on each interval:

$$\begin{cases} u^j_{tt} - u^j_{xx} = f_j(t)g_j(x), & 0 < x < l_j,\ 0 < t < T, \\ u^j(l_j,t) = 0, & 0 < t < T, \\ u^j(x,0) = u^j_t(x,0) = 0, & 0 < x < l_j. \end{cases} \tag{15}$$

At the internal vertex, we impose the standard (Kirchhoff–Neumann) matching conditions:

$$u^1(0,t) = \ldots = u^N(0,t), \qquad \sum_{j=1}^{N} u_x^j(0,t) = 0, \quad 0 < t < T. \tag{16}$$

The problem is to recover unknown functions g_j, $j = 1, \ldots, N$, from the observations $\mu_j(t) := u_x^j(l_j, t)$, $j = 1, \ldots, N-1$, $t \in [0, T]$. We will prove the uniqueness result and provide the reconstruction algorithm in the sharp time interval: $T = l_1 + l_N$. In the process of solving the problem, we obtain also a regularity result similar to Theorem 2.1.

In the first step, we recover the functions g_j, $j = 1, \ldots, N-1$, using correspondingly observations $\mu_j(t)$, $j = 1, \ldots, N-1$, with the help of a similar algorithm as the one described in Section 2. It can be done in the time interval of the length l_1. Indeed to recover g_j, we use the solution w to the wave equation on the graph with zero right-hand side and boundary control $w(l_j, t) = f_j(t)$, $w(l_i, t) = 0$ for $i \neq j$ (see the definition of a similar function below in (20)). This function is certainly different from the one used for one interval in (4), but our identification procedure requires only the observation at the point l_j in the time interval $[0, l_j]$. Boundary observation in this time interval "does not feel" the other edges of the graph; therefore, the identification algorithm is the same as that for one interval.

Then since the g_j, $j = 1, \ldots, N-1$ are known, we can consider the solution to the initial boundary value problem similar to (15), where instead of the first line of (15) we have:

$$v_{tt}^j - v_{xx}^j = (1 - \delta_{jN}) f_j(t) g_j(x), \quad \text{for } 0 < x < l_j,\ 0 < t < T, \tag{17}$$

with $\delta_{jN} = 1$ if $j = N$ and 0 else. Now we use the trick described in Remark 2.2. Subtracting the solution to (17) from that to (15), we reduce our identification problem to the case where all g_j except g_N are equal to zero:

$$\begin{cases} u_{tt}^j - u_{xx}^j = \delta_{jN} f_j(t) g_j(x), & 0 < x < l_j,\ 0 < t < T, \\ u^j(l_j, t) = 0, & 0 < t < T, \\ u^j(x, 0) = u_t^j(x, 0) = 0, & 0 < x < l_j. \end{cases} \tag{18}$$

As in the previous section, we can obtain a spectral representation of the solution to (18) and get

$$\mu_1(t) = u_x^1(l_1, t) = \int_0^{t-l_1} g_N(x) w_N^1(x, t)\, dx, \tag{19}$$

where $w_N^j(x,t)$ denotes the restriction to the edge e_N of the solution, $w^j(x,t)$ of the wave equation on the graph Γ with the Dirichlet boundary control f_N applied to the boundary vertex γ_j:

$$\begin{cases} w_{tt}^i - w_{xx}^i = 0, & i = 1, \ldots, N, \\ w^j(l_j, t) = f_N(t), \quad w^i(l_i, t) = 0, \quad i \neq j, & 0 < t < T, \\ w^i(x, 0) = w_t^i(x, 0) = 0, & 0 < x < l_i,\ i = 1, \ldots, N. \end{cases} \tag{20}$$

This solution satisfies also the Kirchhoff–Neumann matching conditions (16). As in the previous section, one can derive an explicit formula for $w_N^1(x,t)$. Substituting this expression into (19), we have:

$$\frac{N}{2} \mu_1(\tau + l_1) = \int_0^{\tau} g_N(x) f_N(\tau - x)\, dx, \quad 0 < \tau < \lambda := \min_{i=1,\ldots,N} \{l_i\}. \tag{21}$$

From this identity, one can now find $g_N(x)$ for $0 < x < \lambda$ using the algorithm described in Section 2.

Then we subtract the solution of the IBVP corresponding to the found part of $g_N(x)$ as described after Eq. (17), repeat the procedure and find $g_N(x)$ on the interval $\lambda < x < 2\lambda$ using the equation similar to (21):

$$\frac{N}{2} c \tilde{\mu}_1(\tau + l_j) = \int_{\lambda}^{\tau} g_N(x) f_N(\tau - x)\, dx, \quad \lambda < \tau < 2\lambda, \tag{22}$$

where $\tilde{\mu}_1(t)$ is the observation obtained after the corresponding subtraction.

We will find $g_N(x)$ on the whole interval $[0, l_N]$ using a finite number of such steps.

Remark 3.1. The identification problem on trees can be done in a similar fashion with observations at all external nodes except one. Our approach works also for equations with variable (x-dependent) coefficients and for graphs with cycles. For graphs with cycles, one additional observation by loop has to be added. We refer to [1] for the details.

References

[1] S. Avdonin, S. Nicaise, Source identification for the wave equation on graphs, Technical report, Lamav, Université de Valenciennes, France, 2014, in preparation.
[2] G. Bruckner, M. Yamamoto, On the determination of point sources by boundary observations: uniqueness, stability and reconstruction, Technical report 252, WIAS, Berlin, 1996.
[3] S. Nicaise, O. Zair, Identifiability, stability and reconstruction results of point sources by boundary measurements in heterogeneous trees, Rev. Mat. Complut. 16 (2003) 151–178.
[4] M. Yamamoto, Well-posedness of an inverse hyperbolic problem by the Hilbert uniqueness method, J. Inverse Ill-Posed Probl. 2 (4) (1994) 349–368.
[5] M. Yamamoto, Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method, Inverse Probl. 11 (2) (1995) 481–496.