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Differential geometry

A Hopf algebra associated with a Lie pair ☆

Une algèbre de Hopf associée à une paire de Lie

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A R T I C L E   I N F O

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In tribute to Alan Weinstein on the occasion of his seventieth birthday

A B S T R A C T

The quotient $L/A[-1]$ of a pair $A \hookrightarrow L$ of Lie algebroids is a Lie algebra object in the derived category $D^b(\mathcal{A})$ of the category $\mathcal{A}$ of left $\mathcal{U}(A)$-modules, the Atiyah class $\alpha_{L/A}$ being its Lie bracket. In this note, we describe the universal enveloping algebra of the Lie algebra object $L/A[-1]$ and we prove that it is a Hopf algebra object in $D^b(\mathcal{A})$.

R É S U M É

Le quotient $L/A[-1]$ d'une paire $A \hookrightarrow L$ d'algébroïdes de Lie est un objet algèbre de Lie dans la catégorie dérivée $D^b(\mathcal{A})$ de la catégorie $\mathcal{A}$ des modules à gauche sur $\mathcal{U}(A)$. Dans cette note, nous décrivons l'algèbre enveloppante universelle de l'objet algèbre de Lie $L/A[-1]$ et nous prouvons que celle-ci constitue un objet algèbre de Hopf dans $D^b(\mathcal{A})$.

Version française abrégée

Soit L une algébroïde de Lie sur une variété différentiable M et A une sous-algébroïde de L – nous dirons que (L, A) est une paire de Lie. La classe d'Atiyah α_E d'un A-module E relative à la paire (L, A), qui fut introduite par [1], est l'obstruction à l'existence d'une L-connexion A-*compatible* sur E. Elle unifie les classes définies par Atiyah pour les fibrés vectoriels holomorphes et par Molino pour les variétés feuilletées.

Soit $\mathcal{A}$ la catégorie (abélienne) des A-modules, c'est-à-dire les modules sur l'algèbre enveloppante universelle $\mathcal{U}(A)$ de l'algébroïde de Lie A. Le quotient L/A d'une paire de Lie (L, A) – plus précisément, son espace de sections C^∞ – constitue un A-module [1]. Sa classe d'Atiyah $\alpha_{L/A}$ relative à la paire (L, A) détermine un morphisme $L/A[-1] \otimes L/A[-1] \to L/A[-1]$ de la catégorie dérivée $D^b(\mathcal{A})$, faisant de $L/A[-1]$ un objet algèbre de Lie dans $D^b(\mathcal{A})$.

Dans cette note, nous construisons l'algèbre universelle enveloppante de l'objet algèbre de Lie $L/A[-1]$ dans $D^b(\mathcal{A})$ et nous prouvons que celle-ci est un objet algèbre de Hopf dans $D^b(\mathcal{A})$. Pour ce faire, nous établissons une application de type Hochschild–Kostant–Rosenberg et, reprenant les idées de Ramadoss [6], remplaçons $L/A[-1]$ par le complexe $L(\mathcal{D}^1_{\mathrm{poly}})$, qui est, quant à lui, quasi-isomorphe.

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Le cas particulier de $T_X[-1] \in D^b(X)$ correspondant à la paire de Lie $(T_X \otimes \mathbb{C}, T_X^{0,1})$ associée à une variété complexe X joue un rôle important dans la théorie des invariants de Rozansky–Witten des variétés de dimension 3 et en théorie de l'indice. Des applications du résultat ci-dessus dans ces domaines seront développées ailleurs.

L'algèbre enveloppante universelle $\mathcal{U}(L)$ d'une algébroïde de Lie réelle L est un bimodule sur $R := C^\infty(M)$, qui s'identifie à l'algèbre des opérateurs différentiels sur le groupoïde de Lie local $\mathscr{L}$ intégrant L tangents aux s-fibres et invariants à gauche. De plus, $\mathcal{U}(L)$ admet une comultiplication cocommutative et coassociative $\Delta : \mathcal{U}(L) \to \mathcal{U}(L) \tilde{\otimes} \mathcal{U}(L)$. Ici, $\tilde{\otimes}$ désigne le produit tensoriel de modules à gauche sur R. Enfin, $\mathcal{U}(L)$ est un L-module, puisque tout élément l de $\Gamma(L)$ agit sur $\mathcal{U}(L)$ par multiplication à gauche : $\nabla_l u = l \cdot u$, pour tout $u \in \mathcal{U}(L)$.

Étant donné une paire de Lie (L, A), considérons le quotient $\mathcal{D}^1_{\mathrm{poly}} = \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ de $\mathcal{U}(L)$ par son idéal à gauche engendré par $\Gamma(A)$. On vérifie aisément que la comultiplication de $\mathcal{U}(L)$ induit une comultiplication $\Delta : \mathcal{D}^1_{\mathrm{poly}} \to \mathcal{D}^1_{\mathrm{poly}} \tilde{\otimes} \mathcal{D}^1_{\mathrm{poly}}$ sur $\mathcal{D}^1_{\mathrm{poly}}$ et que l'action de L sur $\mathcal{U}(L)$ détermine une action de A sur $\mathcal{D}^1_{\mathrm{poly}}$. En fait, $\mathcal{D}^1_{\mathrm{poly}}$ est une coalgèbre cocommutative coassociative sur R, dont la comultiplication est compatible avec la A-action.

Soit $\mathrm{Ch}^b(\mathcal{A})$ la catégorie des complexes de $\mathcal{A}$ bornés, et $D^b(\mathcal{A})$ la catégorie dérivée de $\mathcal{A}$. Notons $\mathcal{D}^n_{\mathrm{poly}}$ la n-ième puissance tensorielle $\mathcal{D}^1_{\mathrm{poly}} \tilde{\otimes} \cdots \tilde{\otimes} \mathcal{D}^1_{\mathrm{poly}}$ de $\mathcal{D}^1_{\mathrm{poly}}$ et, pour $n = 0$, posons $\mathcal{D}^0_{\mathrm{poly}} = R$. L'opérateur $d : \mathcal{D}^\bullet_{\mathrm{poly}} \to \mathcal{D}^{\bullet+1}_{\mathrm{poly}}$ défini par l'équation (1) fait de $\mathcal{D}^\bullet_{\mathrm{poly}} = \bigoplus_{n=0}^\infty \mathcal{D}^n_{\mathrm{poly}}$ un objet de $\mathrm{Ch}^b(\mathcal{A})$. L'application de Hochschild–Kostant–Rosenberg

$$\mathrm{HKR} : \Gamma\big(S^\bullet\big(L/A[-1]\big)\big) \to \mathcal{D}^\bullet_{\mathrm{poly}}$$

est obtenue à partir de l'inclusion naturelle $\Gamma(L/A) \subset \mathcal{D}^1_{\mathrm{poly}}$ par symétrisation (voir l'équation (2)).

Proposition 0.1. *L'application* $\mathrm{HKR} : (S^\bullet(L/A[-1]), 0) \to (\mathcal{D}^\bullet_{\mathrm{poly}}, d)$ *est, dans* $\mathrm{Ch}^b(\mathcal{A})$*, un quasi-isomorphisme.*

Théorème 0.2.

(i) *L'objet* $\mathcal{D}^\bullet_{\mathrm{poly}}$ *est une algèbre de Hopf dans* $D^b(\mathcal{A})$*.*
(ii) *L'algèbre associative* $(\mathcal{D}^\bullet_{\mathrm{poly}}, \tilde{\otimes})$ *est l'algèbre enveloppante universelle de* $L/A[-1]$ *dans* $D^b(\mathcal{A})$*.*

1. Introduction

Let A be a Lie algebroid over a manifold M. Its space of smooth sections $\Gamma(A)$ is a Lie–Rinehart algebra over the commutative ring $R = C^\infty(M)$. By an A-module, we mean a module over the Lie–Rinehart algebra corresponding to the Lie algebroid A, i.e. a module over the associative algebra $\mathcal{U}(A)$.

Recall that the universal enveloping algebra $\mathcal{U}(A)$ of a Lie algebroid A over M is simultaneously an associative algebra and an R-bimodule. In case the Lie algebroid A is real, $\mathcal{U}(A)$ is canonically identified to the algebra of left-invariant s-fiberwise differential operators on the local Lie groupoid $\mathscr{A}$ integrating A. Let us recall its construction.

The vector space $\mathfrak{g} = R \oplus \Gamma(A)$ admits a natural Lie algebra structure given by the Lie bracket:

$$[f + X, g + Y] = \rho(X)g - \rho(Y)f + [X, Y],$$

where $f, g \in R$ and $X, Y \in \Gamma(A)$. Here ρ denotes the anchor map. Let i denote the natural inclusion of $\mathfrak{g}$ into its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and let $\mathcal{V}(\mathfrak{g})$ denote the subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by $i(\mathfrak{g})$. The universal enveloping algebra $\mathcal{U}(A)$ of the Lie algebroid A is the quotient of $\mathcal{V}(\mathfrak{g})$ by the two-sided ideal generated by the elements of the form $i(f) \otimes i(g + Y) - i(fg + fY)$ with $f, g \in R$ and $Y \in \Gamma(A)$.

When A is a Lie algebra, $\mathcal{U}(A)$ is indeed the usual universal enveloping algebra. On the other hand, when A is the tangent bundle TM, $\mathcal{U}(A)$ is the algebra of differential operators on M.

We use the symbol $\mathcal{A}$ to denote the Abelian category of A-modules. Abusing terminology, we say that a vector bundle E over M is an A-module if $\Gamma(E) \in \mathcal{A}$.

Given a Lie pair (L, A) of algebroids, *i.e.* a Lie algebroid L with a Lie subalgebroid A, the Atiyah class α_E of an A-module E relative to the pair (L, A) is defined as the obstruction to the existence of an *A-compatible* L-connection on the vector bundle E. An L-connection ∇ on an A-module E is said to be A-compatible if it extends the given flat A-connection on E and satisfies $\nabla_a \nabla_l - \nabla_l \nabla_a = \nabla_{[a,l]}$ for all $a \in \Gamma(A)$ and $l \in \Gamma(L)$. This fairly recently defined class (see [1]) has as double origin, the Atiyah class of holomorphic vector bundles and the Molino class of foliations.

The quotient L/A of any Lie pair (L, A) is an A-module [1]. Its Atiyah class $\alpha_{L/A}$ can be described as follows. Choose an L-connection ∇ on L/A extending the A-action. Its curvature is the vector bundle map $R^\nabla : \wedge^2 L \to \mathrm{End}(E)$ defined by $R^\nabla(l_1, l_2) = \nabla_{l_1}\nabla_{l_2} - \nabla_{l_2}\nabla_{l_1} - \nabla_{[l_1,l_2]}$, for all $l_1, l_2 \in \Gamma(L)$. Since L/A is an A-module, R^∇ vanishes on $\wedge^2 A$ and, therefore, determines a section $R^\nabla_{L/A}$ of $A^* \otimes (L/A)^* \otimes \mathrm{End}(L/A)$. It was proved in [1] that $R^\nabla_{L/A}$ is a 1-cocycle for the Lie algebroid A with values in the A-module $(L/A)^* \otimes \mathrm{End}(L/A)$ and that its cohomology class $\alpha_{L/A} \in H^1(A; (L/A)^* \otimes \mathrm{End}(L/A))$ is independent of the choice of the connection.

Let $\mathrm{Ch}^b(\mathcal{A})$ denote the category of bounded complexes in $\mathcal{A}$ and let $D^b(\mathcal{A})$ denote the corresponding derived category. We write $L/A[-1]$ to denote the quotient L/A regarded as a complex in $\mathcal{A}$ concentrated in degree 1.

The following was proved in [1].

Proposition 1.1. *(See [1].) Let (L, A) be a Lie algebroid pair. The Atiyah class $\alpha_{L/A}$ of the quotient L/A relative to the pair (L, A) determines a morphism $L/A[-1] \otimes L/A[-1] \to L/A[-1]$ in the derived category $D^b(\mathcal{A})$ making $L/A[-1]$ a Lie algebra object in $D^b(\mathcal{A})$.*

It is well known that every ordinary Lie algebra $\mathfrak{g}$ admits a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, which is a Hopf algebra. We are thus led to the following natural questions: does there exist a universal enveloping algebra for $L/A[-1]$ in $D^b(\mathcal{A})$ and, if so, is it a Hopf algebra object?

In this Note, we give a positive answer to the questions above. For a complex manifold X, the Atiyah class of the Lie pair $(T_X \otimes \mathbb{C}, T_X^{0,1})$ is simply the usual Atiayh class of the holomorphic tangent bundle T_X recently exploited by Kapranov [2]. It was proved that the universal enveloping algebra of the Lie algebra object $T_X[-1]$ in $D^b(X)$ is the Hochschild cochain complex $(\mathcal{D}^\bullet_{\mathrm{poly}}(X), d)$ [5–7]. This result played an important role in the study of several aspects of complex geometry including the Riemann–Roch theorem [5], the Chern character [6] and the Rozansky–Witten invariants [7,8]. Applications of our result will be developed elsewhere.

2. Hochschild–Kostant–Rosenberg map

It is known [9] that the universal enveloping algebra $\mathcal{U}(L)$ of a Lie algebroid L admits a cocommutative coassociative coproduct $\Delta : \mathcal{U}(L) \to \mathcal{U}(L)\tilde{\otimes}\mathcal{U}(L)$, which is defined on generators as follows: $\Delta(f) = f\tilde{\otimes}1 = 1\tilde{\otimes}f$, $\forall f \in R$ and $\Delta(l) = l\tilde{\otimes}1 + 1\tilde{\otimes}l$, $\forall l \in \Gamma(L)$. Here, and in the sequel, $\tilde{\otimes}$ stands for the tensor product of left R-modules. Moreover, $\mathcal{U}(L)$ is an L-module since each section l of L acts on $\mathcal{U}(L)$ by left multiplication: $\nabla_l u = l \cdot u$, $\forall u \in \mathcal{U}(L)$.

Now, given a Lie pair (L, A), consider the quotient $\mathcal{D}^1_{\mathrm{poly}}$ of $\mathcal{U}(L)$ by the left ideal generated by $\Gamma(A)$. It is straightforward to see that the comultiplication on $\mathcal{U}(L)$ induces a comultiplication $\Delta : \mathcal{D}^1_{\mathrm{poly}} \to \mathcal{D}^1_{\mathrm{poly}}\tilde{\otimes}\mathcal{D}^1_{\mathrm{poly}}$ on $\mathcal{D}^1_{\mathrm{poly}}$ and the action of L on $\mathcal{U}(L)$ determines an action of A on $\mathcal{D}^1_{\mathrm{poly}}$.

Lemma 2.1. *The quotient $\mathcal{D}^1_{\mathrm{poly}} = \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is simultaneously a cocommutative coassociative R-coalgebra and an A-module. Moreover, its comultiplication is compatible with its A-action:*

$$\nabla_X(\Delta p) = \Delta(\nabla_X p), \quad \forall X \in \Gamma(A), p \in \mathcal{D}^1_{\mathrm{poly}}.$$

Let $\mathcal{D}^n_{\mathrm{poly}}$ denote the n-th tensorial power $\mathcal{D}^1_{\mathrm{poly}}\tilde{\otimes}\cdots\tilde{\otimes}\mathcal{D}^1_{\mathrm{poly}}$ of $\mathcal{D}^1_{\mathrm{poly}}$ and, for $n = 0$, set $\mathcal{D}^0_{\mathrm{poly}} = R$. We define a coboundary operator $d : \mathcal{D}^\bullet_{\mathrm{poly}} \to \mathcal{D}^{\bullet+1}_{\mathrm{poly}}$ on $\mathcal{D}^\bullet_{\mathrm{poly}} = \bigoplus_{n=0}^\infty \mathcal{D}^n_{\mathrm{poly}}$ by

$$\begin{aligned} d(p_1\tilde{\otimes}\cdots\tilde{\otimes}p_n) = {} & 1\tilde{\otimes}p_1\tilde{\otimes}\cdots\tilde{\otimes}p_n - (\Delta p_1)\tilde{\otimes}\cdots\tilde{\otimes}p_n + p_1\tilde{\otimes}(\Delta p_2)\tilde{\otimes}\cdots\tilde{\otimes}p_n - \cdots \\ & + (-1)^n p_1\tilde{\otimes}\cdots\tilde{\otimes}p_{n-1}\tilde{\otimes}(\Delta p_n) + (-1)^{n+1} p_1\tilde{\otimes}\cdots\tilde{\otimes}p_n\tilde{\otimes}1, \end{aligned} \tag{1}$$

for any $p_1, p_2, \ldots, p_n \in \mathcal{D}^1_{\mathrm{poly}}$. Since the comultiplication Δ is compatible with the action of A, the operator d is a morphism of A-modules. Moreover, Δ being coassociative, d satisfies $d^2 = 0$. Thus $(\mathcal{D}^\bullet_{\mathrm{poly}}, d)$ is an object in $\mathrm{Ch}^b(\mathcal{A})$.

When endowed with the trivial coboundary operator, the space of sections of

$$S^\bullet\big(L/A[-1]\big) = \bigoplus_{k=0}^\infty S^k\big(L/A[-1]\big) = \bigoplus_{k=0}^\infty \big(\wedge^k L/A\big)[-k]$$

is a complex of A-modules:

$$0 \to R \xrightarrow{0} \Gamma(L/A) \xrightarrow{0} \Gamma\big(\wedge^2(L/A)\big) \xrightarrow{0} \Gamma\big(\wedge^3(L/A)\big) \xrightarrow{0} \cdots$$

The natural inclusion $\Gamma(L/A) \hookrightarrow \mathcal{D}^1_{\mathrm{poly}}$ extends naturally to the Hochschild–Kostant–Rosenberg map

$$\mathrm{HKR} : \Gamma\big(S^\bullet\big(L/A[-1]\big)\big) \to \mathcal{D}^\bullet_{\mathrm{poly}}$$

by skew-symmetrization:

$$\mathrm{HKR}(b_1 \wedge \cdots \wedge b_n) = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) b_{\sigma(1)}\tilde{\otimes}b_{\sigma(2)}\tilde{\otimes}\cdots\tilde{\otimes}b_{\sigma(n)}, \quad \forall b_1, \ldots, b_n \in \Gamma(L/A). \tag{2}$$

Proposition 2.2. *In* $\mathrm{Ch}^b(\mathcal{A})$*, HKR is a quasi-isomorphism from* $(\Gamma(S^\bullet(L/A[-1])), 0)$ *to* $(\mathcal{D}^\bullet_{\mathrm{poly}}, d)$*.*

Sketch of proof Assuming L and A are real Lie algebroids, let $\mathscr{L}$ and $\mathscr{A}$ be local Lie groupoids integrating L and A respectively. The source map $s : \mathscr{L} \to M$ induces a surjective submersion $J : \mathscr{L}/\mathscr{A} \to M$. The right quotient $\mathscr{L}/\mathscr{A}$ is a left $\mathscr{L}$-homogeneous space with momentum map J [4]. Therefore, it admits an infinitesimal L-action, and hence an infinitesimal A-action. The coalgebra $\mathcal{D}^1_{\mathrm{poly}}$ may be regarded as the space of distributions on the J-fibers of $\mathscr{L}/\mathscr{A}$ supported on M. Its A-module structure then stems from the infinitesimal A-action on $\mathscr{L}/\mathscr{A}$. The n-th tensorial power $\mathcal{D}^n_{\mathrm{poly}}$ may be viewed as the space of n-differential operators on the J-fibers of $\mathscr{L}/\mathscr{A}$ evaluated along M and the differential d as the Hochschild coboundary. The conclusion follows from the classical Hochschild–Kostant–Rosenberg theorem. To prove the proposition for complex Lie algebroids, it suffices to consider formal groupoids instead of local Lie groupoids [3].

3. Universal enveloping algebra of $L/A[-1]$ in $D^b(\mathcal{A})$

Following Markarian [5], Ramadoss [6], and Roberts–Willerton [7], we introduce the following:

Definition 3.1. *If it exists, the universal enveloping algebra of a Lie algebra object* $\mathcal{G}$ *in* $D^b(\mathcal{A})$ *is an associative algebra object* $\mathcal{H}$ *in* $D^b(\mathcal{A})$ *together with a morphism of Lie algebras* $i : \mathcal{G} \to \mathcal{H}$ *in* $D^b(\mathcal{A})$ *satisfying the following universal property: given any associative algebra object* $\mathcal{K}$ *and any morphism of Lie algebras* $f : \mathcal{G} \to \mathcal{K}$ *in* $D^b(\mathcal{A})$*, there exists a unique morphism of associative algebras* $f' : \mathcal{H} \to \mathcal{K}$ *in* $D^b(\mathcal{A})$ *such that* $f = f' \circ i$*.*

In view of the similarity between $(\mathcal{D}^\bullet_{\mathrm{poly}}, d)$ and the Hochschild cochain complex, we define a cup product $\cup$ on $\mathcal{D}^\bullet_{\mathrm{poly}}$ by setting $P \cup Q = P \tilde{\otimes} Q$, for all $P, Q \in \mathcal{D}^\bullet_{\mathrm{poly}}$. It is simple to check that

$$d(P \cup Q) = dP \cup Q + (-1)^{|P|} P \cup dQ,$$

for all homogeneous $P, Q \in \mathcal{D}^\bullet_{\mathrm{poly}}$.

Proposition 3.2. *For any Lie pair* (L, A) *of algebroids,* $(\mathcal{D}^\bullet_{\mathrm{poly}}, d, \cup)$ *is an associative algebra object in* $D^b(\mathcal{A})$*, which is in fact the universal enveloping algebra of the Lie algebra* $L/A[-1]$ *in* $D^b(\mathcal{A})$*.*

Consider the inclusion $\eta : R \hookrightarrow \mathcal{D}^n_{\mathrm{poly}}$, the projection $\varepsilon : \mathcal{D}^n_{\mathrm{poly}} \twoheadrightarrow R$, and the maps $t : \mathcal{D}^\bullet_{\mathrm{poly}} \to \mathcal{D}^\bullet_{\mathrm{poly}}$ and $\tilde{\Delta} : \mathcal{D}^\bullet_{\mathrm{poly}} \to \mathcal{D}^\bullet_{\mathrm{poly}} \underset{R}{\otimes} \mathcal{D}^\bullet_{\mathrm{poly}}$ defined, respectively, by

$$t(p_1 \tilde{\otimes} p_2 \tilde{\otimes} \cdots \tilde{\otimes} p_n) = (-1)^{\frac{n(n-1)}{2}} p_n \tilde{\otimes} p_{n-1} \tilde{\otimes} \cdots \tilde{\otimes} p_1$$

and

$$\tilde{\Delta}(p_1 \tilde{\otimes} p_2 \tilde{\otimes} \cdots \tilde{\otimes} p_n) = \sum_{i+j=n} \sum_{\sigma \in \mathfrak{S}^j_i} \mathrm{sgn}(\sigma)(p_{\sigma(1)} \tilde{\otimes} \cdots \tilde{\otimes} p_{\sigma(i)}) \otimes (p_{\sigma(i+1)} \tilde{\otimes} \cdots \tilde{\otimes} p_{\sigma(n)}),$$

where $\mathfrak{S}^j_i$ denotes the set of (i, j)-shuffles.[1]

Theorem 3.3. *For any Lie pair* (L, A) *of algebroids,* $(\mathcal{D}^\bullet_{\mathrm{poly}}, d)$ *with the multiplication* $\cup$*, the comultiplication* $\tilde{\Delta}$*, the unit* η*, the counit* ε*, and the antipode* t*, is a Hopf algebra object in* $D^b(\mathcal{A})$*.*

4. Ramadoss's approach: $L(\mathcal{D}^1_{\mathrm{poly}})$

To prove Proposition 3.2 and Theorem 3.3, we essentially follow Ramadoss's approach [6]. Let $L(\mathcal{D}^1_{\mathrm{poly}})$ be the (graded) free Lie algebra generated over R by $\mathcal{D}^1_{\mathrm{poly}}$ concentrated in degree 1. In other words, $L(\mathcal{D}^1_{\mathrm{poly}})$ is the smallest Lie subalgebra of $\mathcal{D}^\bullet_{\mathrm{poly}}$ containing $\mathcal{D}^1_{\mathrm{poly}}$. The Lie bracket of two vectors $u \in \mathcal{D}^i_{\mathrm{poly}}$ and $v \in \mathcal{D}^j_{\mathrm{poly}}$ is the vector $[u, v] = u \tilde{\otimes} v - (-1)^{ij} v \tilde{\otimes} u \in \mathcal{D}^{i+j}_{\mathrm{poly}}$. Actually, $L(\mathcal{D}^1_{\mathrm{poly}})$ is made of all linear combinations of elements of the form $[p_1, [p_2, [\cdots, [p_{n-1}, p_n] \cdots]]]$ with $p_1, \ldots, p_n \in \mathcal{D}^1_{\mathrm{poly}}$. One checks that $L(\mathcal{D}^1_{\mathrm{poly}})$ is a d-stable A-submodule of $\mathcal{D}^\bullet_{\mathrm{poly}}$ and that its Lie bracket is a chain map with respect to the coboundary operator d. Therefore $(L(\mathcal{D}^1_{\mathrm{poly}}), d)$ is a Lie algebra object in $\mathrm{Ch}^b(\mathcal{A})$.

[1] An (i, j)-shuffle is a permutation σ of the set $\{1, 2, \ldots, i+j\}$ such that $\sigma(1) \le \sigma(2) \le \cdots \le \sigma(i)$ and $\sigma(i+1) \le \sigma(i+2) \le \cdots \le \sigma(i+j)$.

Let $S^\bullet(L(\mathcal{D}^1_{\text{poly}}))$ be the symmetric algebra of $L(\mathcal{D}^1_{\text{poly}})$ and let $I : S^\bullet(L(\mathcal{D}^1_{\text{poly}})) \to \mathcal{D}^\bullet_{\text{poly}}$ be the symmetrization map:

$$I(z_1 \odot \cdots \odot z_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma; z_1, \ldots, z_n) z_{\sigma(1)} \tilde{\otimes} z_{\sigma(2)} \tilde{\otimes} \cdots \tilde{\otimes} z_{\sigma(n)}.$$

The Koszul sign $\operatorname{sgn}(\sigma; z_1, \ldots, z_n)$ of a permutation σ of the (homogeneous) vectors $z_1, z_2, \ldots, z_n \in S^\bullet(L(\mathcal{D}^1_{\text{poly}}))$ is determined by the relation $z_{\sigma(1)} \odot z_{\sigma(2)} \odot \cdots \odot z_{\sigma(n)} = \operatorname{sgn}(\sigma; z_1, \ldots, z_n) z_1 \odot z_2 \odot \cdots \odot z_n$.

Lemma 4.1. *The symmetrization $I : S^\bullet(L(\mathcal{D}^1_{\text{poly}})) \to \mathcal{D}^\bullet_{\text{poly}}$ is an isomorphism in $\mathrm{Ch}^b(\mathcal{A})$.*

Using Lemma 4.1 and the HKR quasi-isomorphism, one can prove that the composition $\beta : \Gamma(L/A[-1]) \to L(\mathcal{D}^1_{\text{poly}})$ of the inclusions $\Gamma(L/A[-1]) \subset \mathcal{D}^1_{\text{poly}} \subset L(\mathcal{D}^1_{\text{poly}})$ is a quasi-isomorphism in $\mathrm{Ch}^b(\mathcal{A})$, which intertwines the Lie brackets on $\Gamma(L/A[-1])$ and $L(\mathcal{D}^1_{\text{poly}})$.

Proposition 4.2.

(i) *The inclusion $\beta : \Gamma(L/A[-1]) \to L(\mathcal{D}^1_{\text{poly}})$ is a quasi-isomorphism in $\mathrm{Ch}^b(\mathcal{A})$.*
(ii) *The inclusion $\beta : \Gamma(L/A[-1]) \to L(\mathcal{D}^1_{\text{poly}})$ is an isomorphism of Lie algebra objects in $D^b(\mathcal{A})$ as the diagram*

$$\begin{array}{ccc} \Gamma(L/A[-1]) \tilde{\otimes} \Gamma(L/A[-1]) & \xrightarrow{\beta \otimes \beta} & L(\mathcal{D}^1_{\text{poly}}) \tilde{\otimes} L(\mathcal{D}^1_{\text{poly}}) \\ \downarrow{\scriptstyle \alpha_{L/A}} & & \downarrow{\scriptstyle [,]} \\ \Gamma(L/A[-1]) & \xrightarrow[\beta]{} & L(\mathcal{D}^1_{\text{poly}}) \end{array}$$

commutes in $D^b(\mathcal{A})$.

Proposition 3.2 and Theorem 3.3 now follow immediately.

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