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Partial differential equations

Some remarks on the paper “On the blow up criterion of 3D Navier–Stokes equations” by J. Benameur

Quelques remarques sur l’article « On the blow up criterion of 3D Navier–Stokes equations » par J. Benameur

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ABSTRACT

We indicate some important simplifications and extensions of the analysis recently given by J. Benameur to derive blow-up estimates for strong solutions to 3D incompressible Navier–Stokes equations in homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^3)$, $s > 1/2$, in case of finite-time existence.

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RÉSUMÉ

Nous indiquons quelques simplifications et extensions importantes des résultats obtenus par J. Benameur concernant des estimations inférieures pour l’explosion des solutions fortes des équations de Navier–Stokes incompressibles dans les espaces de Sobolev homogènes $\dot{H}^s(\mathbb{R}^3)$, $s > 1/2$, en cas d’existence non globale.

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In [2], J. Benameur introduced an interesting approach to derive lower bound estimates for the potential blow-up in Sobolev spaces $H^s \equiv H^s(\mathbb{R}^3)$ of strong solutions to the incompressible Navier–Stokes equations

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \nu\Delta u, & x\in\mathbb{R}^3,\ t>0,\\ \operatorname{div} u(\cdot,t) = 0, & x\in\mathbb{R}^3,\ t>0,\\ u(0) = u^0 \in H^{s_0}, & \operatorname{div} u^0 = 0, \end{cases} \tag{NS$_\nu$}$$

where $\nu > 0$, $s_0 > 5/2$. Local existence results in Kato [5] give a (unique) solution $u(t) \in \mathcal{C}([0,T_\nu^*), H^{s_0}) \cap \mathcal{C}^1([0,T_\nu^*), H^{s_0-1})$ on some maximal interval $[0,T_\nu^*)$, with $\lim_{t\nearrow T_\nu^*}\|u(t)\|_{\dot{H}^{s_0}} = \infty$ if $T_\nu^* < \infty$. Many similar blow-up properties are also known, see, e.g., [1,3,4,7,9,11,12]. The main result in [2, Theorem 1.3] is the following:

Theorem A. *Let $u(t) \in \mathcal{C}([0,T_\nu^*), H^s)$, $s > 5/2$, be the maximal solution above. If $T_\nu^* < \infty$, then, for every $0 \le t < T_\nu^*$:*

(*i*) $\|u(t)\|_{L^2}^{s-1}\|u(t)\|_{\dot{H}^s} \ge \frac{c(s)\nu^{3s/4}}{(T_\nu^*-t)^{s/4}}$,

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(ii) $\|u(t)\|_{L^2}^{\frac{2s}{3}-1}\|u(t)\|_{\dot H^s} \geq \frac{c(s)\nu^{s/3}}{(T_\nu^*-t)^{s/3}}$,

(iii) $\|\nabla u(t)\|_{L^2} \geq \frac{c\nu^{3/4}}{(T_\nu^*-t)^{1/4}}$ *[Leray's property],*

(iv) $\|\hat u(t)\|_{L^1} \geq \frac{(\nu/2)^{1/2}}{(T_\nu^*-t)^{1/2}}$.

(Here, $c(s) > 0$ is some constant depending only on s, with distinct values in (i) and (ii), and $\hat u(t)$ denotes the Fourier transform of $u(t)$. For more on this notation, see [2].)

We make the following remarks about Theorem A and its proof as provided in [2]. These remarks are important because the analysis in [2] can be applied to other similar problems, like incompressible magnetohydrodynamics [6,10] or the micropolar fluid equations [8] in $\mathbb{R}^3$:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = (\nu+\chi)\Delta u + \chi\nabla\times w, \\ w_t + u\cdot\nabla w = \gamma\Delta w + \kappa\nabla \operatorname{div} w + \chi\nabla\times u - 2\chi w, \\ \operatorname{div} u(\cdot,t) = 0, \quad x\in\mathbb{R}^3,\ t>0, \\ u(0) = u^0 \in H^{s_0}, \quad \operatorname{div} u^0 = 0, \quad w(0) = w^0 \in H^{s_0}, \end{cases} \tag{μ}$$

for which (i)–(iv) can be similarly derived (with $\nu > 0$ replaced by $\min\{\nu,\gamma\} > 0$).

For better clarity, let us reset $u(t)$, $0 \leq t < T_\nu^*$, as the maximal Kato's solution to (NS_ν) corresponding to some (arbitrary) initial state $u^0 \in H^{s_0}$, $s_0 > 5/2$. (Actually, by Theorem 1.1 in [13], it is sufficient to have $s_0 > 3/2$.)

(1) Clearly, $u(t) \in \mathcal{C}([0,T_\nu^*), H^s)$ for each $0 < s \leq s_0$, because of the interpolation result $\|\theta\|_{\dot H^s} \leq C(s)\|\theta\|_{L^2}^{1-s/s_0}\|\theta\|_{\dot H^{s_0}}^{s/s_0}$. It is also true that $u(t) \in \mathcal{C}((0,T_\nu^*), H^s)$ for $s > s_0$ — this follows, e.g., from the expression (3.3) derived in the proof of (i) in [2].

In the latter, the only restriction is that $s \geq 1$, since one needs to use (cf. [2, p. 722]) the general estimate

$$\|\theta\|_{\dot H^1} \leq k(s)\|\theta\|_{L^2}^{1-1/s}\|\theta\|_{\dot H^s}^{1/s}, \quad \theta\in H^s, \tag{A.1}$$

which holds for $s \geq 1$ only. Therefore, the argument in [2] proves (i) for *any* $s \geq 1$, irrespectively of the value of s_0. In particular, there is no need for an extra proof of Leray's property (iii), as this is simply the case $s = 1$ already examined.

(2) The following minor change in the proof of (i) gives off plenty: for $0 < \delta < 1$, using Chemin's Lemma (i.e., Lemma 3.1 in [2]) with $\eta = 1/2+\delta$, $\eta' = s+1-\delta$ (instead of $\eta = 1$, $\eta' = s + 1/2$ as in [2, p. 721]), where $s \geq 1/2+\delta$ is arbitrary, together with the interpolation result:

$$\|\theta\|_{\dot H^{s+1-\delta}} \leq \|\theta\|_{\dot H^s}^{\delta}\|\nabla\theta\|_{\dot H^s}^{1-\delta}, \quad \theta\in H^s \tag{A.2}$$

(instead of (A.1) above), one obtains, by the very same argument used for (i) in [2], the more general estimate:

$$\|u(t)\|_{L^2}^{p(s,\delta)}\|u(t)\|_{\dot H^s} \geq \frac{c(s,\delta)\nu^{r(s,\delta)}}{(T_\nu^*-t)^{q(s,\delta)}} \quad \forall 0\leq t<T_\nu^* \tag{A.3a}$$

for all $s \geq 1/2+\delta$, where

$$p(s,\delta) = \frac{2s}{1+2\delta} - 1, \qquad q(s,\delta) = \frac{\delta s}{1+2\delta}, \qquad r(s,\delta) = \frac{(2-\delta)s}{1+2\delta}. \tag{A.3b}$$

Choosing $\delta = 1/2$ gives back (i) above; and, when $1/2 < s < 3/2$, taking $\delta := s - 1/2$ in (A.3a)–(A.3b) produces the fundamental estimate:

$$\|u(t)\|_{\dot H^s} \geq \frac{c(s)\nu^{(5-2s)/4}}{(T_\nu^*-t)^{s/2-1/4}} \quad \forall 0\leq t<T_\nu^* \quad (1/2<s<3/2). \tag{A.4}$$

Also, for $s \geq 3/2$ and $0 < \epsilon < s/3$ arbitrary, taking $\delta = (s-3\epsilon)/(s+6\epsilon)$ in (A.3a)–(A.3b) gives:

$$\|u(t)\|_{L^2}^{\frac{2s}{3}-1+4\epsilon}\|u(t)\|_{\dot H^s} \geq c(s,\epsilon)\nu^{s/3+5\epsilon}\big[(T_\nu^*-t)\big]^{-\frac{s}{3}+\epsilon} \quad \forall 0\leq t<T_\nu^*, \tag{A.5}$$

where $c(s,\epsilon) \to 0$ as $\epsilon \to 0$. For $s = 3/2$, one falls therefore short of showing that

$$\|u(t)\|_{\dot H^{3/2}} \geq \frac{c\nu^{1/2}}{(T_\nu^*-t)^{1/2}} \quad \forall 0\leq t<T_\nu^*, \tag{A.6}$$

a result that is believed to be true, but has never been obtained to date. On the other hand, for $s > 3/2$, taking (formally) $\epsilon = 0$ in (A.5) suggests the estimate (ii) above, which can be derived by various methods (see, e.g., [2,9]). We note in passing that (A.4) has been recently extended to the range $3/2 < s < 5/2$ in [9], with the case $s = 5/2$ still open. For $s > 5/2$, the appropriate optimal results (if any) are not yet known, but estimates stronger than (ii) have already been established [9, §VI].

(3) Taking $\delta = 0$ and proceeding as in the derivation of (A.3a)–(A.3b) above, one gets:

$$\partial_t \|u(t)\|^2_{\dot{H}^s} + 2\nu \|\nabla u(t)\|^2_{\dot{H}^s} \le c(s) \|u(t)\|_{\dot{H}^{1/2}} \|\nabla u(t)\|^2_{\dot{H}^s}, \quad 0 < t < T^*_\nu \tag{A.7}$$

for $s > 0$ arbitrary. We thus get the familiar result that, for some $\epsilon > 0$ appropriately small, $u(t)$ is globally defined ($T^*_\nu = \infty$) when $\nu^{-1}\|u^0\|_{\dot{H}^{1/2}} < \epsilon$ (see, e.g., [11]).

(4) Using the elementary inequality $\|\hat{\theta}\|_{L^1} \le K(s)\|\theta\|_{H^s}$, $s > 3/2$, it follows that $\hat{u}(t) \in \mathcal{C}([0, T^*_\nu), L^1)$, and, by a simple scaling argument, it also gives:

$$\|\hat{u}(t)\|_{L^1} \le C(s) \|u(t)\|^{1-\frac{3}{2s}}_{L^2} \|u(t)\|^{\frac{3}{2s}}_{\dot{H}^s} \quad (s > 3/2). \tag{A.8}$$

Therefore, (*ii*) is obtained once we show (*iv*). Recalling that $\|\hat{u}(t)\|_{L^1} \ge k\|u(t)\|_{L^\infty}$, we see that, from the celebrated Leray's blow-up result [7, expression (3.9), p. 224]

$$\|u(t)\|_{L^\infty} \ge K\nu^{1/2}(T^*_\nu - t)^{-1/2} \quad \forall 0 \le t < T^*_\nu, \tag{A.9}$$

one gets (*iv*) already, but with a far less explicit constant. In this sense, Benameur's approach achieves slightly more; moreover, with a few minor modifications (indicated in Remark **(5)** below), his derivation is truly very simple.

(5) Using Gronwall's lemma to obtain (*iv*), as in [2], requires the delicate estimate

$$\int_0^{T^*_\nu} \|\hat{u}(t)\|^2_{L^1} \mathrm{d}t = \infty. \tag{A.10}$$

(A.10) is shown in [2] by a careful analysis using Chemin's Lemma (cf. (3.5), p. 723), or we can get it directly from (A.9) above, whose proof involves some nontrivial work. The following alternative does not need (A.10), thus simplifying the analysis. Taking the scalar product $\langle \hat{u}(t), \hat{u}_t(t)\rangle_{\mathbb{C}^3}$ and integrating on $\mathbb{R}^3$, one gets (cf. [2, p. 724]), for $0 < t < T^*_\nu$,

$$\partial_t \|\hat{u}(t)\|_{L^1} + \nu \|\widehat{\Delta u}(t)\|_{L^1} \le (2\pi)^{-3} \sum_{i,j} \|\hat{u}_j(t)\|_{L^1} \|\widehat{D_j u_i}(t)\|_{L^1} \le 3\sqrt{3}(2\pi)^{-3} \|\hat{u}(t)\|^{3/2}_{L^1} \|\widehat{\Delta u}(t)\|^{1/2}_{L^1}$$

($D_j = \partial/\partial x_j$), so that we have the differential inequality:

$$\partial_t \|\hat{u}(t)\|_{L^1} \le K\nu^{-1} \|\hat{u}(t)\|^3_{L^1}, \qquad K = 27(2\pi)^{-6}/4. \tag{A.11}$$

Hence, for each $0 \le t_0 < T^*_\nu$, $\|\hat{u}(t)\|_{L^1}$ is bounded above on $[t_0, T^*_\nu)$ by the solution to the problem $[\mathrm{v}'(t) = K\nu^{-1}\mathrm{v}(t)^3, \mathrm{v}(t_0) = \|\hat{u}(t_0)\|_{L^1}]$, which blows up at the time $t_* = t_0 + \nu K^{-1}\|\hat{u}(t_0)\|^{-2}_{L^1}/2$. As $\|\hat{u}(t)\|_{L^1} \le \mathrm{v}(t)$, it follows that $T^*_\nu \ge t_*$, that is,

$$\|\hat{u}(t_0)\|_{L^1} \ge \frac{\sqrt{2}}{3\sqrt{3}}(2\pi)^3\nu^{1/2}(T^*_\nu - t_0)^{-1/2} \quad \forall 0 \le t_0 < T^*_\nu, \tag{A.12}$$

which is precisely (*iv*), except for the better constant. By (A.8), this shows (*ii*) also.

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