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Algebraic geometry

Geometric construction of generators of CoHA of doubled quiver

Construction géométrique des générateurs de l'algèbre cohomologique de Hall du double d'un carquois

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ABSTRACT

Let Q be the double of a quiver. According to Efimov, Kontsevich and Soibelman, the cohomological Hall algebra (CoHA) associated with Q is a free super-commutative algebra. In this short note, we confirm a conjecture of Hausel, which gives a geometric realisation of the generators of the CoHA.

RÉSUMÉ

Soit Q le double d'un carquois. Selon Efimov, Kontsevich et Soibelman, l'algèbre cohomologique de Hall (CoHA) associée à Q est une algèbre libre super-commutative. Dans cette note, nous démontrons la conjecture de Hausel, donnant une réalisation géométrique des générateurs de cette algèbre.

1. Introduction

Let $Q=(I,\Omega)$ be a quiver with a set of vertices I and with a_{ij} arrows from $i\in I$ to $j\in I$. For each dimension vector $\gamma=(\gamma^i)_{i\in I}\in\mathbf{Z}^I_{\geq 0}$, we have the affine $\mathbf{C}$-variety M_γ of representations of Q in complex coordinate space $\bigoplus_{i\in I}\mathbf{C}^{\gamma^i}$. It is acted on by the complex algebraic group $G_\gamma=\prod_{i\in I}\mathrm{GL}_{\gamma^i}(\mathbf{C})$ and the action factors through $PG_\gamma:=G_\gamma/\mathbb{G}_m$, where $\mathbb{G}_m$ is embedded diagonally in G_γ. We denote by $[M_\gamma/G_\gamma]$ the moduli stack of representations of Q of dimension γ.

For $\gamma_1,\gamma_2\in\mathbf{Z}^I_{\geq 0}$, let M_{γ_1,γ_2} be the subvariety of $M_{\gamma_1+\gamma_2}$ consisting of the representations of Q such that the subspace $\bigoplus_{i\in I}\mathbf{C}^{\gamma_1^i}\subset\bigoplus_{i\in I}(\mathbf{C}^{\gamma_1^i}\oplus\mathbf{C}^{\gamma_2^i})$ forms a sub-representation. Let $G_{\gamma_1,\gamma_2}\subset G_{\gamma_1+\gamma_2}$ be the subgroup preserving $\bigoplus_{i\in I}\mathbf{C}^{\gamma_1^i}$. Then $[M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}]$ is the moduli stack classifying the extensions of representations of Q of dimension vector γ_2 by that of dimension vector γ_1. We have the correspondence:

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$$\begin{array}{ccc} & [M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}] & \\ {}^{p}\swarrow & & \searrow^{q} \\ [M_{\gamma_1}/G_{\gamma_1}]\times[M_{\gamma_2}/G_{\gamma_2}] & & [M_{\gamma_1+\gamma_2}/G_{\gamma_1+\gamma_2}], \end{array}$$

from which Kontsevich and Soibelman [8] have constructed an associative algebra structure on

$$\mathcal{H} := \bigoplus_{\gamma\in\mathbf{Z}^I_{\geq 0}} \mathcal{H}_\gamma, \quad \mathcal{H}_\gamma := H^*\big([M_\gamma/G_\gamma]\big) = H^*_{G_\gamma}(M_\gamma),$$

which is called the *cohomological Hall algebra* (CoHA) of the quiver Q. Here and after, all the cohomological groups take coefficients in $\mathbf{Q}$. The resulting product has a shift in cohomological degree:

$$H^*_{G_{\gamma_1}}(M_{\gamma_1}) \times H^*_{G_{\gamma_2}}(M_{\gamma_2}) \to H^{*-2\chi_Q(\gamma_1,\gamma_2)}_{G_{\gamma_1+\gamma_2}}(M_{\gamma_1+\gamma_2}),$$

where

$$\chi_Q(\gamma_1,\gamma_2) = \sum_{i\in I}\gamma_1^i\gamma_2^i - \sum_{i,j\in I} a_{i,j}\gamma_1^i\gamma_2^j.$$

Suppose that the quiver Q is symmetric, i.e. $a_{ij}=a_{ji}$. In this case, $\mathcal{H}$ has more structures. First of all, one can make $\mathcal{H}$ into a $(\mathbf{Z}^I_{\geq 0},\mathbf{Z})$-graded algebra, by requiring elements in $H^k_{G_\gamma}(M_\gamma)$ to be of bidegree $(\gamma, k+\chi_Q(\gamma,\gamma))$. Secondly, following Efimov [5], we can twist the multiplication by a sign such that $(\mathcal{H},*)$ is a super-commutative algebra with respect to the $\mathbf{Z}$-grading. In fact, for $a_{\gamma,k}\in\mathcal{H}_{\gamma,k}, a_{\gamma',k'}\in\mathcal{H}_{\gamma',k'}$, we have:

$$a_{\gamma,k}a_{\gamma',k'} = (-1)^{\chi_Q(\gamma,\gamma')}a_{\gamma',k'}a_{\gamma,k}.$$

We can find a bilinear form $\psi:(\mathbf{Z}/2)^I\times(\mathbf{Z}/2)^I\to\mathbf{Z}/2$ such that

$$\psi(\gamma_1,\gamma_2)+\psi(\gamma_2,\gamma_1) \equiv \chi_Q(\gamma_1,\gamma_2)+\chi_Q(\gamma_1,\gamma_1)\chi_Q(\gamma_2,\gamma_2) \mod 2.$$

Then the twisted product on $\mathcal{H}$ is defined to be

$$a_{\gamma,k} * a_{\gamma',k'} = (-1)^{\psi(\bar\gamma,\bar\gamma')}a_{\gamma,k}\cdot a_{\gamma',k'},$$

where $\bar\gamma$ is the image of γ in $(\mathbf{Z}/2)^I$.

For the symmetric quiver Q, it is conjectured by Kontsevich and Soibelman [8] and proved by Efimov [5] that the $(\mathbf{Z}^I_{\geq 0},\mathbf{Z})$-graded algebra $(\mathcal{H},*)$ is a free super-commutative algebra generated by a $(\mathbf{Z}^I_{\geq 0},\mathbf{Z})$-graded vector space V of the form $V=V^{\text{prim}}\otimes\mathbf{Q}[x]$, with x an element of degree $(0,2)$, and for all $\gamma\in\mathbf{Z}^I_{\geq 0}$ the vector space $V^{\text{prim}}_{\gamma,k}$ is non-zero for only finitely many k. Geometrically, via the isogeny $G_\gamma\to PG_\gamma\times\mathbb{G}_m$, we have

$$\mathcal{H}_\gamma = H^*_{G_\gamma}(M_\gamma) \cong H^*_{PG_\gamma}(M_\gamma)\otimes H^*_{\mathbb{G}_m}(\text{pt}),$$

and it gives the above factorisation $V=V^{\text{prim}}\otimes\mathbf{Q}[x]$. Let M^{st}_γ be the stable part of M_γ with respect to the 0-stability condition, i.e. M^{st}_γ consists of all the simple Q-modules. Then as we will see in Theorem 2.2, $V^{\text{prim}}_{\gamma,*+\chi_Q(\gamma,\gamma)}$ is in fact the pure part of $H^*([M^{\text{st}}_\gamma/PG_\gamma])$, where the word "pure" refers to the mixed Hodge structure on the cohomological group. Let $c_{\gamma,k}=\dim_{\mathbf{C}}V^{\text{prim}}_{\gamma,k}$, the above result implies that the quantum Donaldson–Thomas invariants of the quiver Q without potential and with 0-stability condition is:

$$\Omega(\gamma)(q) = \sum_{k\in\mathbf{Z}} c_{\gamma,k}q^{k/2} \in \mathbf{Z}\big[q^{\pm\frac{1}{2}}\big].$$

In particular, the coefficients are positive.

On the other hand, in the work of Hausel, Letellier and Rodriquez-Villegas [6], they found another expression for the quantum Donaldson–Thomas invariants. From now on, we work with quivers $Q=(I,\Omega)$ that are the double of another quiver, i.e. $\Omega=\Omega_0\sqcup\Omega_0^{\text{op}}$, where Ω_0^{op} is obtained by reversing all the arrows in Ω_0. In this case, M_γ is endowed with a G_γ-invariant holomorphic symplectic form ω. Let $\mu: M_\gamma\to\mathfrak{g}^0_\gamma$ be the corresponding moment map, here $\mathfrak{g}^0_\gamma$ is the trace 0 part of $\mathfrak{g}_\gamma := \text{Lie}(G_\gamma)$. Let $\mathcal{O}$ be the G_γ-orbit of a *generic* (to be explained below) regular semisimple element in $\mathfrak{g}^0_\gamma$. The group PG_γ acts freely on $\mu^{-1}(\mathcal{O})$ and we have the geometric quotient $\mu^{-1}(\mathcal{O})/PG_\gamma$, which is a smooth quasi-projective algebraic variety. Furthermore, the Weyl group W_γ of G_γ acts on the cohomological groups $H^*(\mu^{-1}(\mathcal{O})/PG_\gamma)$. One of the main results of [6] states that:

$$\Omega(\gamma)(q) = q^{\frac{1}{2}\chi_Q(\gamma,\gamma)}\sum_i \dim\big(H^{2i}\big(\mu^{-1}(\mathcal{O})/PG_\gamma\big)^{W_\gamma}\big)q^i.$$

Based on this result, Hausel conjectured that the cohomological groups

$$H^k\big(\mu^{-1}(\mathcal{O})/PG_\gamma\big)^{W_\gamma}$$

are geometric realisations of the generating set $V^{\mathrm{prim}}_{\gamma,k+\chi_Q(\gamma,\gamma)}$. (Of course, the conjecture is meaningful only when $\Omega(\gamma)(q)$ is non-zero. We will always impose this condition.) In this article, we confirm this conjecture. Our construction goes as follows:

Let $\chi:\mathfrak{g}^0_\gamma\to\mathfrak{g}^0_\gamma /\!\!/ G_\gamma\cong\mathfrak{t}^0_\gamma /\!\!/ W_\gamma$ be the characteristic morphism. Following Ginzburg [7], we consider the composition $f:M_\gamma\xrightarrow{\mu}\mathfrak{g}^0_\gamma\xrightarrow{\chi}\mathfrak{t}^0_\gamma /\!\!/ W_\gamma$. Let

$$\mathfrak{t}^{0,\mathrm{gen}}_\gamma:=\mathfrak{t}^{0,\mathrm{reg}}_\gamma\setminus\bigcup_{\substack{\gamma_1,\gamma_2\in\mathbf{Z}^I_{\geq 0},\\ \gamma_1+\gamma_2=\gamma}}W_\gamma\cdot\big(\mathfrak{t}^0_{\gamma_1}\oplus\mathfrak{t}^0_{\gamma_2}\big),$$

we call conjugates of elements in it *generic* regular semisimple elements. Let

$$U_\gamma:=f^{-1}\big(\mathfrak{t}^{0,\mathrm{gen}}_\gamma/W_\gamma\big),$$

then the group PG_γ acts freely on U_γ and the quotient U_γ/PG_γ is a quasi-projective algebraic variety. Furthermore, the restriction of the morphism f to U_γ descends to a morphism $\bar{f}:U_\gamma/PG_\gamma\to\mathfrak{t}^{0,\mathrm{gen}}_\gamma/W_\gamma$. We will prove that it makes U_γ/PG_γ a fiber bundle on $\mathfrak{t}^{0,\mathrm{gen}}_\gamma/W_\gamma$ with fibers isomorphic to $\mu^{-1}(\mathcal{O})/PG_\gamma$. Our main result is the following:

Theorem 1.1. *The pure part of $H^*(U_\gamma/PG_\gamma)$ is equal to $H^*(\mu^{-1}(\mathcal{O})/PG_\gamma)^{W_\gamma}$. The restriction $H^*_{PG_\gamma}(M_\gamma)\to H^*(U_\gamma/PG_\gamma)$ factors through and is surjective onto the pure part of the latter, and its restriction to $V^{\mathrm{prim}}_{\gamma,*+\chi_Q(\gamma,\gamma)}$ induces an isomorphism*

$$V^{\mathrm{prim}}_{\gamma,*+\chi_Q(\gamma,\gamma)}\cong H^*\big(\mu^{-1}(\mathcal{O})/PG_\gamma\big)^{W_\gamma}.$$

2. Proof of the main theorem

We begin by recalling the construction of Kontsevich and Soibelman of the cohomological Hall algebra. Given two vectors $\gamma_1,\gamma_2\in\mathbf{Z}^I_{\geq 0}$, let $\gamma=\gamma_1+\gamma_2$. The product $\mathcal{H}_{\gamma_1}\times\mathcal{H}_{\gamma_2}\to\mathcal{H}_\gamma$ is defined to be the composition of the Künneth isomorphism:

$$H^*_{G_{\gamma_1}}(M_{\gamma_1})\otimes H^*_{G_{\gamma_2}}(M_{\gamma_2})\cong H^*_{G_{\gamma_1}\times G_{\gamma_2}}(M_{\gamma_1}\times M_{\gamma_2}),$$

and of the following morphisms:

$$H^*_{G_{\gamma_1}\times G_{\gamma_2}}(M_{\gamma_1}\times M_{\gamma_2})\cong H^*_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2})\xrightarrow{\phi_1}H^{*+2c_1}_{G_{\gamma_1,\gamma_2}}(M_\gamma)\xrightarrow{\phi_2}H^{*+2c_1+2c_2}_{G_\gamma}(M_\gamma),\tag{1}$$

where $c_1=\dim_{\mathbf{C}}M_\gamma-\dim_{\mathbf{C}}M_{\gamma_1,\gamma_2}$ and $c_2=\dim_{\mathbf{C}}G_{\gamma_1,\gamma_2}-\dim_{\mathbf{C}}G_\gamma$, and the first isomorphism is induced by the fibrations in affine spaces:

$$M_{\gamma_1,\gamma_2}\to M_{\gamma_1}\times M_{\gamma_2},\qquad G_{\gamma_1,\gamma_2}\to G_{\gamma_1}\times G_{\gamma_2},$$

and the other morphisms ϕ_1,ϕ_2 are natural push forwards.

Lemma 2.1. *Under the restriction $H^*([M_\gamma/G_\gamma])\to H^*([M^{\mathrm{st}}_\gamma/G_\gamma])$, the image of*

$$\bigoplus_{\substack{\gamma_1,\gamma_2\in\mathbf{Z}^I_{\geq 0}\\ \gamma_1+\gamma_2=\gamma}}\mathcal{H}_{\gamma_1}\times\mathcal{H}_{\gamma_2}$$

in $\mathcal{H}_\gamma=H^*([M_\gamma/G_\gamma])$ goes to 0.

Proof. By the definition of Gysin map, the morphism ϕ_1 in composition (1) factorises as

$$H^*\big([M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}]\big)\to H^{*+2c_1}_{[M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}]}\big([M_\gamma/G_{\gamma_1,\gamma_2}]\big)\to H^{*+2c_1}\big([M_\gamma/G_{\gamma_1,\gamma_2}]\big).$$

Using the long exact sequence

$$\cdots\to H^*_{[M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}]}\big([M_\gamma/G_{\gamma_1,\gamma_2}]\big)\to H^*\big([M_\gamma/G_{\gamma_1,\gamma_2}]\big)\to H^*\big([M_\gamma/G_{\gamma_1,\gamma_2}]\setminus[M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}]\big)\to\cdots,$$

we see that $\mathrm{Im}(\phi_1)$ goes to 0 when we restrict it to

$$H^{*+2c_1}\big([M_\gamma/G_{\gamma_1,\gamma_2}]\backslash[M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}]\big).$$

Since $[M_\gamma^{\mathrm{st}}/G_{\gamma_1,\gamma_2}]$ is contained in $[M_\gamma/G_{\gamma_1,\gamma_2}]\backslash[M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}]$, $\mathrm{Im}(\phi_1)$ vanishes when we restrict it further to

$$H^{*+2c_1}\big(\big[M_\gamma^{\mathrm{st}}/G_{\gamma_1,\gamma_2}\big]\big).$$

Now applying ϕ_2, we see that $\mathrm{Im}(\phi_2\circ\phi_1)$ vanishes when we restrict it to $H^{*+2c_1+2c_2}([M_\gamma^{\mathrm{st}}/G_\gamma])$. $\square$

We need some preliminary results before proceeding to the proof of the main theorem. Given an element $t=(t_i)_{i\in I}\in \mathfrak{t}_\gamma^{0,\mathrm{gen}}$, let $\mathcal{O}$ be its orbit under the action of PG_γ by conjugation. Recall that Crawley-Boevey [3] has identified the geometric quotient $\mu^{-1}(\mathcal{O})/PG_\gamma$ with a quiver variety: let $\widetilde{Q}$ be the quiver obtained from Q by attaching to each vertex $i\in I$ a leg of length γ_i-1. More precisely, vertices of $\widetilde{Q}$ are labeled $[i,j], i\in I, j=0,\cdots,\gamma_i-1$, and we identify $[i,0]$ with i. Besides the arrows in Q, the new arrows in $\widetilde{Q}$ are $[i,j]\rightleftharpoons[i,j+1]$ for each $i\in I$, $j=0,\cdots,\gamma_i-2$. The new dimension vector of $\widetilde{Q}$ is defined to be $\widetilde{\gamma}_{[i,j]}=\gamma_i-j$. Again, we have the moment map $\widetilde{\mu}:M_{\widetilde{\gamma}}^{\widetilde{Q}}\to\mathfrak{g}_{\widetilde{\gamma}}^0$, where $M_{\widetilde{\gamma}}^{\widetilde{Q}}$ is the space of representations of $\widetilde{Q}$ of dimension vector $\widetilde{\gamma}$. For each $i\in I$, let $t_{i,1},\cdots,t_{i,\gamma_i}$ be the eigenvalues of t_i. Define $\lambda=(\lambda_{[i,j]})\in\mathfrak{g}_{\widetilde{\gamma}}^0$ to be

$$\lambda_{[i,0]}=-t_{i,1},$$
$$\lambda_{[i,j]}=t_{i,j}-t_{i,j+1},\quad j=1,\cdots,\gamma_i-1.$$

Notice that $\widetilde{\gamma}\cdot\lambda=0$. Now the result of Crawley-Boevey [3] states that

$$\mu^{-1}(\mathcal{O})/PG_\gamma\cong\widetilde{\mu}^{-1}(\lambda)/PG_{\widetilde{\gamma}}. \tag{2}$$

Moreover, according to [6], corollary 1.6 (iv), $\Omega(\gamma)(q)$ is non-zero if and only if $\widetilde{\gamma}$ is a positive root of Q'; here we write $\widetilde{Q}$ as the double of another quiver Q'.

Lemma 2.2. *The morphism $\bar{f}:U_\gamma/PG_\gamma\to\mathfrak{t}_\gamma^{0,\mathrm{gen}}/W_\gamma$ makes U_γ/PG_γ a fiber bundle over $\mathfrak{t}_\gamma^{0,\mathrm{gen}}/W_\gamma$. Moreover, the sheaf $R^i\bar{f}_*\mathbf{Q}$ is constant on the étale neighbourhood $\mathfrak{t}_\gamma^{0,\mathrm{gen}}\to\mathfrak{t}_\gamma^{0,\mathrm{gen}}/W_\gamma$ of $\mathfrak{t}_\gamma^{0,\mathrm{gen}}/W_\gamma$.*

Proof. This is basically Lemma 48 of [9], with one difference. As in the proof of [6] Theorem 2.3, Maffei works with quiver without loops, but his proof carries over in our case. In his proof, the important point is the surjectivity of $\bar{f}$ (or rather the hyper-Kähler moment map on the generic locus, but this can be reduced to $\bar{f}$ by hyper-Kähler rotation). According to [2], Theorem 4.4, this is fulfilled in our situation, since $\mu^{-1}(\mathcal{O})/PG_\gamma\cong\widetilde{\mu}^{-1}(\lambda)/PG_{\widetilde{\gamma}}$ and $\widetilde{\gamma}\cdot\lambda=0$, taking into account that $\widetilde{\gamma}$ is a positive root of Q'. $\square$

Lemma 2.3 *(Crawley-Boevey–van den Bergh). The smooth quasi-projective algebraic variety $\mu^{-1}(\mathcal{O})/PG_\gamma$ has pure mixed Hodge structure.*

Proof. This is a corollary of [4], §2.4, taking into account isomorphism (2). $\square$

Now we can prove the first part of Theorem 1.1.

Theorem 2.1. *The pure part of $H^*(U_\gamma/PG_\gamma)$ is equal to $H^*(\mu^{-1}(\mathcal{O})/PG_\gamma)^{W_\gamma}$.*

Proof. Consider the fiber bundle $\bar{f}:U_\gamma/PG_\gamma\to\mathfrak{t}_\gamma^{0,\mathrm{gen}}/W_\gamma$. By Lemma 2.2, the sheaf $R^i\bar{f}_*\mathbf{Q}$ is constant on the étale neighbourhood $\mathfrak{t}_\gamma^{0,\mathrm{gen}}\to\mathfrak{t}_\gamma^{0,\mathrm{gen}}/W_\gamma$ of $\mathfrak{t}_\gamma^{0,\mathrm{gen}}/W_\gamma$, so we get the Hochschild–Serre spectral sequence,

$$E_2^{p,q}=H^p\big(W_\gamma,H^q\big(\mathfrak{t}_\gamma^{0,\mathrm{gen}},R\bar{f}_*\mathbf{Q}\big)\big)\quad\Longrightarrow\quad H^{p+q}(U_\gamma/PG_\gamma).$$

Since W_γ is a finite group, we have $E_2^{p,q}=0$ for $p\neq0$. So the spectral sequence degenerates, and we get

$$\begin{aligned}H^q(U_\gamma/PG_\gamma)&=\big(H^q\big(\mathfrak{t}_\gamma^{0,\mathrm{gen}},R\bar{f}_*\mathbf{Q}\big)\big)^{W_\gamma}\\&=\Big(\bigoplus_{q_1+q_2=q}H^{q_1}\big(\mathfrak{t}_\gamma^{0,\mathrm{gen}}\big)\otimes H^{q_2}\big(\mu^{-1}(\mathcal{O})/PG_\gamma\big)\Big)^{W_\gamma}.\end{aligned}$$

Since $\mathfrak{t}_\gamma^{0,\mathrm{gen}}$ is the complement of unions of sufficiently many hyperplanes in the vector space $\mathfrak{t}_\gamma^0$, one proves easily by induction on the number of hyperplanes that the mixed Hodge structure of $H^i(\mathfrak{t}_\gamma^{0,\mathrm{gen}})$ is not pure if $i\neq0$. So the pure part of $H^q(U_\gamma/PG_\gamma)$ is exactly $H^q(\mu^{-1}(\mathcal{O})/PG_\gamma)^{W_\gamma}$. $\square$

To prove the second result in the main theorem, we need some facts from algebraic stacks. We refer the reader to Olsson–Laszlo [10,11] and Sun [12] for the proofs. Although they work over the finite fields, their results apply in our situation, since the quiver varieties are in fact $\mathbf{Z}$-schemes. The moduli stack $[M_\gamma/PG_\gamma]$ is a smooth Artin stack over $\mathbf{C}$ with dimension $d_\gamma = \dim(M_\gamma) - \dim(PG_\gamma)$. It has dualizing complex $\mathbf{Q}(d_\gamma)[2d_\gamma]$, and we have the Poincaré duality, which is a perfect non-degenerate bilinear pairing

$$H^i\big([M_\gamma/PG_\gamma]\big) \times H_c^{2d_\gamma - i}\big([M_\gamma/PG_\gamma]\big) \to \mathbf{Q}(d_\gamma).$$

Using the fibration $[M_\gamma/PG_\gamma] \to [\mathrm{pt}/PG_\gamma]$, we have $H^i([M_\gamma/PG_\gamma]) = H^i_{PG_\gamma}(\mathrm{pt})$ is pure of weight i, the groups $H_c^{2d_\gamma - i}([M_\gamma/PG_\gamma])$ are all pure of weight $2d_\gamma - i$.

Furthermore, let $Z_\gamma = M_\gamma \backslash U_\gamma$, then $H_c^i([Z_\gamma/PG_\gamma])$ is of weight less than or equal to i. This is essentially [13]. More precisely, as in [1], let $\{E_n \to B_n\}_{n\in\mathbf{N}}$ be an injective system of finite dimensional n-acyclic approximation to the universal PG_γ-torsor $E \to B$, then

$$H_c^i\big([Z_\gamma/PG_\gamma]\big) = \lim_{n\to\infty} H_c^{i+2\dim(E_n)}(Z_\gamma \times_{PG_\gamma} E_n)\big(-\dim(E_n)\big).$$

Now it suffices to apply [13] to the right-hand side.

Proof of the second part of Theorem 1.1. We have the long exact sequence:

$$\cdots \to H_c^{i-1}\big([Z_\gamma/PG_\gamma]\big) \to H_c^i\big([U_\gamma/PG_\gamma]\big) \to H_c^i\big([M_\gamma/PG_\gamma]\big) \to H_c^i\big([Z_\gamma/PG_\gamma]\big) \to \cdots. \tag{3}$$

Since $H_c^{i-1}([Z_\gamma/PG_\gamma])$ is of weight less than or equal to $i-1$, the pure part of $H_c^i([U_\gamma/PG_\gamma])$ injects into $H_c^i([M_\gamma/PG_\gamma])$. Taking Poincaré duality, we have that $H^{2d_\gamma - i}([M_\gamma/PG_\gamma])$ maps onto the pure part of $H^{2d_\gamma - i}([U_\gamma/PG_\gamma])$. By Theorem 2.1, we have surjective morphism

$$H^j\big([M_\gamma/PG_\gamma]\big) \twoheadrightarrow H^j\big(\mu^{-1}(\mathcal{O})/PG_\gamma\big)^{W_\gamma}, \quad \forall j.$$

By the definition of $\mathfrak{t}_\gamma^{0,\mathrm{gen}}$, we find easily that $U_\gamma \subset M_\gamma^{\mathrm{st}}$. So the above map factorise by

$$H^j\big([M_\gamma/PG_\gamma]\big) \to H^j\big(M_\gamma^{\mathrm{st}}/PG_\gamma\big) \twoheadrightarrow H^j\big(\mu^{-1}(\mathcal{O})/PG_\gamma\big)^{W_\gamma}. \tag{4}$$

By Lemma 2.1, the first arrow has the same image as its restriction to $V^{\mathrm{prim}}_{\gamma, j+\chi_Q(\gamma,\gamma)}$, so we get a surjective morphism

$$V^{\mathrm{prim}}_{\gamma, j+\chi_Q(\gamma,\gamma)} \twoheadrightarrow H^j\big(\mu^{-1}(\mathcal{O})/PG_\gamma\big)^{W_\gamma}.$$

By the result of [6] recalled in the introduction, they have the same dimension, so they are isomorphic. □

Similar arguments can be used to show the following variant of the geometric construction.

Theorem 2.2. *The restriction*

$$H^*\big([M_\gamma/PG_\gamma]\big) \to H^*\big(\big[M_\gamma^{\mathrm{st}}/PG_\gamma\big]\big)$$

induces an isomorphism

$$V^{\mathrm{prim}}_{\gamma, *+\chi_Q(\gamma,\gamma)} \cong \mathrm{P}H^*\big(\big[M_\gamma^{\mathrm{st}}/PG_\gamma\big]\big)$$

where $\mathrm{P}H^*([M_\gamma^{\mathrm{st}}/PG_\gamma])$ *is the pure part of* $H^*([M_\gamma^{\mathrm{st}}/PG_\gamma])$.

Proof. The proof is almost the same as that of the main theorem, we indicate only the differences. Using an exact sequence as (3), with the pair (U_γ, Z_γ) replaced by $(M_\gamma^{\mathrm{st}}, M_\gamma \backslash M_\gamma^{\mathrm{st}})$, we can show that the restriction $H^*([M_\gamma/PG_\gamma]) \to H^*([M_\gamma^{\mathrm{st}}/PG_\gamma])$ factors through and is surjective onto $\mathrm{P}H^*([M_\gamma^{\mathrm{st}}/PG_\gamma])$. Again by Lemma 2.1, we get the surjection

$$V^{\mathrm{prim}}_{\gamma, *+\chi_Q(\gamma,\gamma)} \twoheadrightarrow \mathrm{P}H^*\big(\big[M_\gamma^{\mathrm{st}}/PG_\gamma\big]\big).$$

Now observe that the second morphism in factorisation (4) has the same image as that of

$$\mathrm{P}H^j\big(\big[M_\gamma^{\mathrm{st}}/PG_\gamma\big]\big) \to H^j\big(\mu^{-1}(\mathcal{O})/PG_\gamma\big)^{W_\gamma},$$

since the morphism preserves the weights of the cohomological groups and $H^j(\mu^{-1}(\mathcal{O})/PG_\gamma)^{W_\gamma}$ is pure of weight j. So factorisation (4) becomes:

$$V^{\mathrm{prim}}_{\gamma, j+\chi_Q(\gamma,\gamma)} \twoheadrightarrow \mathrm{P}H^j\big(\big[M_\gamma^{\mathrm{st}}/PG_\gamma\big]\big) \twoheadrightarrow H^j\big(\mu^{-1}(\mathcal{O})/PG_\gamma\big)^{W_\gamma}. \tag{5}$$

Now that the composition is an isomorphism by our main theorem, all the arrows in (5) are isomorphisms. □

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