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The essential spectrum of N-body systems with asymptotically homogeneous order-zero interactions

Le spectre essentiel des systèmes à N-corps avec interactions asymptotiquement homogènes d'ordre zéro

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ABSTRACT

We study the essential spectrum of N-body Hamiltonians with potentials defined by functions that have radial limits at infinity. The results extend the HVZ theorem which describes the essential spectrum of usual N-body Hamiltonians. The proof is based on a careful study of algebras generated by potentials and their cross-products. We also describe the topology on the spectrum of these algebras, thus extending to our setting a result of A. Mageira. Our techniques apply to more general classes of potentials associated with translation invariant algebras of bounded uniformly continuous functions on a finite-dimensional vector space X.

RÉSUMÉ

Nous étudions le spectre essentiel des hamiltoniens des systèmes à N corps avec potentiels définis par des fonctions qui ont des limites radiales à l'infini. Les résultats étendent le théorème HVZ, qui décrit le spectre essentiel des hamiltoniens des systèmes à N corps usuels. La preuve de notre théorème principal est basée sur une étude approfondie des algèbres générées par les potentiels avec des limites radiales à l'infini et de leurs produits croisés. Nous décrivons également la topologie sur le spectre de ces algèbres, étendant ainsi à notre cas un résultat de A. Mageira. Nos techniques s'appliquent à des classes plus générales de potentiels associées à des algèbres de fonctions uniformément continues bornées invariantes par translation.

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Soit X un espace vectoriel réel de dimension finie et $\mathbb{S}_X := (X \setminus \{0\})/\mathbb{R}_+$ *la sphère à l'infini de* X. On dit qu'une fonction $v : X \to \mathbb{C}$ *a des limites radiales uniformes à l'infini* si $v(\hat{a}) := \lim_{r\to\infty} v(ra)$ existe uniformément en $\hat{a} \in \mathbb{S}_X$. Soit $V_Y : X/Y \to \mathbb{R}$ une fonction borélienne ayant des limites radiales uniformes à l'infini, pour chaque sous-espace linéaire $Y \subset X$. Nous supposons $V_Y = 0$, sauf pour un nombre fini d'espaces Y. On note π_Y la surjection canonique $X \to X/Y$ et on garde la notation V_Y pour la fonction $V_Y \circ \pi_Y$. Dans cet article, nous utilisons des produits croisés de C^*-algèbres pour étudier le spectre essentiel des opérateurs de la forme $H := h(P) + \sum_Y V_Y$. Ici, $h : X^* \to [0, \infty[$ est une fonction continue et propre et P est l'observable moment (formellement $P = -\mathrm{i}\nabla$). Soit $v : X \to \mathbb{C}$ et $a \neq 0$ tel que $\lim_{r\to\infty} v(ra + x)$ existe pour tout $x \in X$. Cette limite est une fonction de $x \in X$, qui ne dépend que de la classe $\alpha = \hat{a}$ de a dans $\mathbb{S}_X$, que nous noterons $\tau_\alpha(v)$. Par exemple, si $v = V_Y$ avec V_Y comme plus haut, alors $\tau_\alpha(V_Y) = V_Y$ si $\alpha \subset Y$ et $\tau(V_Y) = V_Y(\pi_Y(\alpha)) \in \mathbb{R}$ si $\alpha \not\subset Y$, où $\pi(\alpha) \in \mathbb{S}_{X/Y}$ est naturellement défini. Plus tard (voir le Théorème 3.1), nous définirons $\tau_\alpha(S)$ pour une classe générale d'opérateurs S, en particulier pour $S = H$, ce qui donnera une nouvelle signification à la définition de τ_α.

Nous énonçons maintenant un cas particulier de notre résultat principal : *si les fonctions $V_Y : X/Y \to \mathbb{R}$ sont bornées et ont des limites radiales uniformes à l'infini et si, pour chaque $\alpha \in \mathbb{S}_X$, on pose $\tau_\alpha(H) = h(P) + \sum_{Y\supset\alpha} V_Y + \sum_{Y\not\supset\alpha} V_Y(\pi_Y(\alpha))$, alors le spectre essentiel de H est $\sigma_{\mathrm{ess}}(H) = \overline{\bigcup}_{\alpha\in\mathbb{S}_X} \sigma(\tau_\alpha(H))$.*

1. Introduction

Let X be a finite dimensional real vector space and, for each linear subspace Y of X, let $V_Y : X/Y \to \mathbb{R}$ be a Borel function. We assume $V_Y = 0$, except for a finite number of Y. We keep the notation V_Y for the function on X given by $V_Y \circ \pi_Y$, where $\pi_Y : X \to X/Y$ is the natural map. In this paper, we use crossed-products of C^*-algebras to study the essential spectrum of Hamiltonians of the form $H := h(P) + \sum_Y V_Y$, under certain conditions on the potentials V_Y. Here $h : X^* \to [0, \infty[$ is a continuous, proper function and P is the momentum observable (recall that *proper* means that $\lim_{|k|\to\infty} h(k) = +\infty$). More precisely, $h(P) = \mathcal{F}^{-1} M_h \mathcal{F}$, where $\mathcal{F} : L^2(X) \to L^2(X^*)$ is the Fourier transform and M_h is the operator of multiplication by h (formally $P = -\mathrm{i}\nabla$). Operators of this form cover the Hamiltonians that are currently the most interesting (from a physical point of view) Hamiltonians of N-body systems. Here are two main examples. In a generalized version of the non-relativistic case, a scalar product is given on X, so, by taking $h(\xi) = |\xi|^2$, we get $h(P) = \Delta$, the positive Laplacian. In the simplest relativistic case, $X = (\mathbb{R}^3)^N$ and, writing the momentum $P = (P_1, \ldots, P_N)$, we have $h(P) = \sum_{k=1}^N (P_k^2 + m_k^2)^{1/2}$ for some real numbers m_k. We refer to [3] for a thorough introduction to the subject and study of these systems.

Let $\mathbb{S}_X := (X \setminus \{0\})/\mathbb{R}_+$ be *the sphere at infinity of* X, i.e. the set of all half-lines $\hat{a} := \mathbb{R}_+ a$. A function $v : X \to \mathbb{C}$ is said *to have uniform radial limits at infinity* if $v(\hat{a}) := \lim_{r\to\infty} v(ra)$ exists uniformly in $\hat{a} \in \mathbb{S}_X$. From the definition of the topology on $\mathbb{S}_X$, we get $v(\hat{a}) = \lim_{r\to\infty} v(ra + x)$, $\forall x \in X$. More generally, we are interested in functions v such that $\lim_{r\to\infty} v(ra + x)$ exists for all $x \in X$. The limit may depend on x and defines a function $\tau_\alpha(v) : X \to \mathbb{C}$, where $\alpha := \hat{a}$. For example, let us consider $v = V_Y$. Then $\tau_\alpha(V_Y)(x) = \lim_{r\to\infty} V_Y(r\pi_Y(a) + \pi_Y(x))$. In particular, $\tau_\alpha(V_Y) = V_Y$ whenever $\alpha := \hat{a} \subset Y$ (i.e. $a \in Y$). On the other hand, if $V_Y : X/Y \to \mathbb{C}$ has uniform radial limits at infinity and $\hat{a} = \alpha \not\subset Y$, then $\pi_Y(\alpha) := \mathbb{R}_+ \pi_Y(a) \in \mathbb{S}_{X/Y}$ is well defined and $\tau_\alpha(V_Y)(x) = V_Y(\pi_Y(\alpha))$ turns out to be a *constant*.

Theorem 1.1. *Let $V_Y : X/Y \to \mathbb{R}$ be bounded with uniform radial limits at infinity. If $\alpha \in \mathbb{S}_X$ set*

$$\tau_\alpha(H) = h(P) + \sum_Y \tau_\alpha(V_Y) = h(P) + \sum_{Y\supset\alpha} V_Y + \sum_{Y\not\supset\alpha} V_Y\big(\pi_Y(\alpha)\big). \tag{1}$$

Then $\sigma(\tau_\alpha(H)) = [c_\alpha, \infty)$ for some real c_α and $\sigma_{\mathrm{ess}}(H) = \overline{\bigcup}_{\alpha\in\mathbb{S}_X} \sigma(\tau_\alpha(H)) = [\inf_\alpha c_\alpha, \infty)$.

Here $\overline{\bigcup}_\alpha$ is the closure of the union. Sometimes the union is already closed [11]. Unbounded potentials are considered in Theorem 3.2. If all the radial limits are zero, which is the case of the usual N-body potentials, then the terms corresponding to $\alpha \not\subset Y$ are dropped in Eq. (1). Consequently, if $h(P) = \Delta$ is the non-relativistic kinetic energy, we recover the Hunziker, van Winter, Zhislin (HVZ) theorem. Descriptions of the essential spectrum of various classes of self-adjoint operators in terms of limits at infinity of translates of the operators have already been obtained before, see for example [7,12,8] (in historical order). Our approach is based on the "localization at infinity" technique developed in [5,6] in the context of crossed-product C^*-algebras.

Let us sketch the main idea of this approach. Let $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ be the algebra of bounded uniformly continuous functions, $\mathcal{C}_0(X)$ the ideal of functions vanishing at infinity, and $\mathcal{C}(X^+) = \mathbb{C} + \mathcal{C}_0(X)$. Consider a translation invariant C^*-subalgebra $\mathcal{A} \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ containing $\mathcal{C}(X^+)$ and let $\hat{\mathcal{A}}$ be its character space. Note that $\hat{\mathcal{A}}$ is a compact topological space that naturally contains X as an open dense subset and $\delta(\mathcal{A}) = \hat{\mathcal{A}} \setminus X$ can be thought of as a boundary of X at infinity. Recall that a self-adjoint operator H on a Hilbert space $\mathcal{H}$ is said to be *affiliated* to a C^*-algebra $\mathscr{A}$ of operators on $\mathcal{H}$ if one has $(H + \imath)^{-1} \in \mathscr{A}$. Then with each self-adjoint operator H affiliated to the crossed product $\mathcal{A} \rtimes X$ of $\mathcal{A}$ by the action of X, one may associate a family of self-adjoint operators $H_\varkappa$ affiliated to $\mathcal{A} \rtimes X$ indexed by the characters $\varkappa \in \delta(\mathcal{A})$. This family completely describes the image of H (in the sense of affiliated operators) in the quotient of $\mathcal{A} \rtimes X$ with respect to the ideal of compact operators. In particular, the essential spectrum of H is the closure of the union of the spectra of the operators $H_\varkappa$. These operators are the *localizations at infinity of* H, more precisely, $H_\varkappa$ *is the localization of* H *at point* $\varkappa$.

Once chosen the algebra $\mathcal{A}$, in order to use these techniques of this paper, we also need: (1) to have a good description of the character space of the Abelian algebra $\mathcal{A}$, and (2) to have an efficient criterion for affiliation to the crossed product $\mathcal{A} \rtimes X$. We also indicate how to achieve (1) and (2).

2. Crossed products and localizations at infinity

For $p \in X^*$ and $q \in X$ let $(S_p f)(x) = \mathrm{e}^{\mathrm{i}\langle x|p\rangle} f(x)$ and $(T_q f)(q) = f(x+q)$. We say that $A \in \mathcal{B}(L^2(X))$ *has the position-momentum limit property* if $\lim_{p\to 0} \|[S_p, A]\| = 0$ and $\lim_{q\to 0} \|(T_q - 1)A^{(*)}\| = 0$ (where $A^{(*)}$ means that the relation holds for A and A^*). The set of such operators is a C^*-algebra equal to the crossed product $\mathcal{C}_\mathrm{b}^\mathrm{u}(X) \rtimes X$ [5]. Note that if $\mathcal{A}$ is a translation invariant C^*-subalgebra of $\mathcal{C}_\mathrm{b}^\mathrm{u}(X)$, then there is a natural realization of the abstract crossed product $\mathcal{A} \rtimes X$ as a C^*-algebra of operators on $L^2(X)$ and we do not distinguish the two algebras. We describe this concrete version of $\mathcal{A} \rtimes X$ below.

If $\varphi : X \to \mathbb{C}$ and $\psi : X^* \to \mathbb{C}$ are measurable functions, then $\varphi(Q)$ and $\psi(P)$ are the operators on $L^2(X)$ defined as follows: $\varphi(Q) := M_\phi$ acts as multiplication by φ and $\psi(P) = \mathcal{F}^{-1} M_\psi \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform $L^2(X) \to L^2(X^*)$ and M_ψ is the operator of multiplication by ψ. Then $\psi \mapsto \psi(P)$ is an isomorphism between $\mathcal{C}_0(X^*)$ and the group C^*-algebra $C^*(X)$ and $\mathcal{A} \rtimes X$ is the norm closed linear space of bounded operators on $L^2(X)$ generated by the products $\varphi(Q)\psi(P)$ with $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{C}_0(X^*)$. In particular, $\mathcal{A} \rtimes X$ consists of operators that have the position-momentum limit property.

We recall the definition of localizations at infinity for such operators. Assume $\mathcal{C}(X^+) \subset \mathcal{A}$, so $\hat{\mathcal{A}}$ is a compactification of X and $\delta(\mathcal{A}) = \hat{\mathcal{A}} \setminus X$ is a compact. If $q \in X$ and φ is a function on X then $T_q\varphi$ is its translation by q. We extend this definition of T_q by replacing $q \in X$ with $\varkappa \in \hat{\mathcal{A}}$: $(T_\varkappa\varphi)(x) = \varkappa(T_x\varphi)$, for any $\varphi \in \mathcal{A}$, $\varkappa \in \hat{\mathcal{A}}$, and, $x \in X$. It is clear that $T_\varkappa\varphi \in \mathcal{C}_\mathrm{b}^\mathrm{u}(X)$ and that its definition coincides with the previous one if $\varkappa = q \in X$. Moreover, we also get "translations at infinity" of $\varphi \in \mathcal{A}$ by elements $\varkappa \in \delta(\mathcal{A})$; note however that such a translation does not belong to $\mathcal{A}$ in general. Also, the function $\varkappa \mapsto T_\varkappa\varphi \in \mathcal{C}_\mathrm{b}^\mathrm{u}(X)$ defined on $\hat{\mathcal{A}}$ is continuous if $\mathcal{C}_\mathrm{b}^\mathrm{u}(X)$ is equipped with the topology of local uniform convergence, hence $T_\varkappa\varphi = \lim_{q\to\varkappa} T_q\varphi$ in this topology for any $\varkappa \in \delta(\mathcal{A})$. If A is an operator on $L^2(X)$, let $\tau_q(A) = T_q A T_q^*$ be its translation by $q \in X$. Clearly $\tau_q(\varphi(Q)) = (T_q\varphi)(Q)$. If $A \in \mathcal{A} \rtimes X$, then we may also consider "translations at infinity" by elements of the boundary $\delta(\mathcal{A})$ of X in $\hat{\mathcal{A}}$ and we get a useful characterization of the compact operators. The following are mainly consequences of [6, Theorem 1.15]:

(i) For each $\varkappa \in \hat{\mathcal{A}}$, there is a unique morphism $\tau_\varkappa : \mathcal{A} \rtimes X \to \mathcal{C}_\mathrm{b}^\mathrm{u}(X) \rtimes X$ such that $\tau_\varkappa(\varphi(Q)\psi(P)) = (T_\varkappa\varphi)(Q)\psi(P)$, $\varphi \in \mathcal{C}_\mathrm{b}^\mathrm{u}(X)$, $\psi \in \mathcal{C}_0(X)$. (ii) $\bigcap_{\varkappa\in\delta(\mathcal{A})} \ker\tau_\varkappa = \mathcal{C}_0(X) \rtimes X \equiv \mathscr{K}(X) =$ ideal of compact operators on $L^2(X)$. (iii) If H is a self-adjoint operator on $L^2(X)$ affiliated to $\mathcal{A}$ then for each $\varkappa \in \delta(\mathcal{A})$ the limit $\tau_\varkappa(H) := \text{s-lim}_{q\to\varkappa} T_q H T_q^*$ exists and $\sigma_{\mathrm{ess}}(H) = \overline{\bigcup}_{\varkappa\in\delta(\mathcal{A})} \sigma(\tau_\varkappa(H))$.

To be precise, the last strong limit means: $\tau_\varkappa(H)$ is a self-adjoint operator (not necessarily densely defined) on $L^2(X)$ and $\text{s-lim}_{q\to\varkappa} \theta(T_q H T_q^*) = \theta(\tau_\varkappa(H))$ for all $\theta \in \mathcal{C}_0(\mathbb{R})$. It is clear that in the last three statements above one may replace $\delta(\mathcal{A})$ by a subset π if for each $A \in \mathcal{A} \rtimes X$ we have: $\tau_\varkappa(A) = 0\ \forall\varkappa \in \pi \Rightarrow \tau_\varkappa(A) = 0\ \forall\varkappa \in \delta(\mathcal{A})$. In the case of groupoid (pseudo)differential algebras (that is, when $\hat{\mathcal{A}}$ is a manifold with corners), the morphisms $\tau_\varkappa$ can be defined using restrictions to fibers, as in [9], and the last three statements above (i)–(iii) remain valid.

3. Main results

As a warm-up and in order to introduce some general notation, we treat first the two-body case, where complete results may be obtained by direct arguments. The algebra of interactions in the standard two-body case is $\mathcal{C}(X^+)$, and hence the Hamiltonian algebra is

$$\mathcal{C}\big(X^+\big) \rtimes X = \mathbb{C} \rtimes X + \mathcal{C}_0(X) \rtimes X = C^*(X) + \mathscr{K}(X) \tag{2}$$

where the sums are direct. Thus $\mathcal{C}(X^+) \rtimes X/\mathscr{K}(X) = C^*(X)$, which finishes the theory. Another elementary case, which has been considered as an example in [5], is $X = \mathbb{R}$ with $\mathcal{C}(\mathbb{R}^+)$ replaced by the algebra $\mathcal{C}(\overline{\mathbb{R}})$ of continuous functions that have limits (distinct in general) at $\pm\infty$. Then there is no natural direct sum decomposition of $\mathcal{C}(\overline{\mathbb{R}}) \rtimes \mathbb{R}$ as in (2), but one has, by standard arguments, $\mathcal{C}(\overline{\mathbb{R}}) \rtimes \mathbb{R}/\mathscr{K}(\mathbb{R}) \simeq C^*(\mathbb{R}) \oplus C^*(\mathbb{R})$. Our purpose in this section is to extend this equation to arbitrary X.

Let $\mathcal{C}(\overline{X})$ be the closure in $\mathcal{C}_\mathrm{b}(X)$ of the subalgebra of functions homogeneous of degree zero outside a compact set. Then $\mathcal{C}(\overline{X}) = \{u \in \mathcal{C}(X) \mid \lim_{\lambda\to+\infty} u(\lambda a)$ exists uniformly in $\hat{a} \in \mathbb{S}_X\}$, where, we recall, $\hat{a} := \mathbb{R}_+ a$ and $\mathbb{S}_X := (X \setminus \{0\})/\mathbb{R}_+$, so $\hat{a} \in \mathbb{S}_X$. As a set, the character space of $\mathcal{C}(\overline{X})$ can be identified with the disjoint union $\overline{X} = X \cup \mathbb{S}_X$. The topology induced by the character space on X is the usual one and the intersections with X of the neighborhoods of some $\alpha \in \mathbb{S}_X$ are the sets that contain a truncated cone C such that there is $a \in \alpha$ such $\lambda a \in C$ if $\lambda \geq 1$. The set of such subsets is a filter $\tilde{\alpha}$ on X and, if Y is a Hausdorff space and $u : X \to Y$, then $\lim_{\tilde{\alpha}} u = y$ means that $u^{-1}(V) \in \tilde{\alpha}$ for any neighborhood V of u. We shall write $\lim_{x\to\alpha} u(x)$ instead of $\lim_{\tilde{\alpha}} u$. We have that $\mathcal{C}(\overline{X})$ is a translation invariant C^*-subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ and so the crossed product $\mathcal{C}(\overline{X}) \rtimes X$ is well defined. We have the following explicit description of this algebra.

Proposition 3.1. *The algebra $\mathcal{C}(\overline{X}) \rtimes X$ acting on $L^2(X)$ consists of bounded operators A that have the position-momentum limit property and are such that the limit $\tau_\alpha(A) = \text{s-lim}_{a\to\alpha} T_a A T_a^*$ exists for each $\alpha = \hat{a} \in \mathbb{S}_X$. If $A \in \mathcal{C}(\overline{X}) \rtimes X$ and $\alpha \in \mathbb{S}_X$, then $\tau_\alpha(A) \in C^*(X)$ and $\tau(A) : \alpha \mapsto \tau_\alpha(A)$ is norm continuous. The map $\tau : \mathcal{C}(\overline{X}) \rtimes X \to C(\mathbb{S}_X) \otimes C^*(X)$ is a surjective morphism whose kernel is the set of compact operators on $L^2(X)$, which gives $\mathcal{C}(\overline{X}) \rtimes X/\mathscr{K}(X) \cong C(\mathbb{S}_X) \otimes C^*(X)$. If H is a self-adjoint operator affiliated to $\mathcal{C}(\overline{X}) \rtimes X$ then $\tau_\alpha(H) = \text{s-lim}_{a\to\alpha} T_a H T_a^*$ exists for all $\alpha \in \mathbb{S}_X$ and $\sigma_{\text{ess}}(H) = \bigcup_\alpha \sigma(\tau_\alpha(H))$.*

In the next two examples $H = h(P) + V$ with $h : X^* \to [0, \infty[$ continuous and proper. We denote by $|\cdot|$ a fixed norm on X^* and by $\mathcal{H}^s$ we denote the usual Sobolev spaces on X ($s \in \mathbb{R}$).

Example 1. Let V be a bounded symmetric operator satisfying: (1) $\lim_{p\to 0} \|[S_p, V]\| = 0$ and (2) the limit $\tau_\alpha(V) = \text{s-lim}_{a\to\alpha} T_a V T_a^*$ exists for each $\alpha \in \mathbb{S}_X$. Then H is affiliated to $\mathcal{C}(\overline{X}) \rtimes X$ and $\tau_\alpha(H) = h(P) + \tau_\alpha(V)$. Moreover, if V is a function, then $\tau_\alpha(V)$ is a number, but in general we have $\tau_\alpha(V) = v_\alpha(P)$ for some function $v_\alpha \in \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X^*)$.

Example 2. Assume that h is locally Lipschitz and that there exist $c, s > 0$ such that, for all p with $|p| > 1$, $|\nabla h(p)| \le c(1 + h(p))$ and $c^{-1}|p|^s \le (1 + h(p))^{1/2} \le c|p|^s$. Let $V : \mathcal{H}^s \to \mathcal{H}^{-s}$ such that $\pm V \le \mu h(P) + \nu$ for some numbers μ, ν with $\mu < 1$ and satisfying the next two conditions: (1) $\lim_{p\to 0} \|[S_p, V]\|_{\mathcal{H}^s\to\mathcal{H}^{-s}} = 0$, (2) $\forall \alpha \in \mathbb{S}_X$ the limit $\tau_\alpha(V) = \text{s-lim}_{a\to\alpha} T_a V T_a^*$ exists strongly in $\mathcal{B}(\mathcal{H}^s, \mathcal{H}^{-s})$. Then $h(P) + V$ and $h(P) + \tau_\alpha(V)$ are symmetric operators $\mathcal{H}^s \to \mathcal{H}^{-s}$ that induce self-adjoint operators H and $\tau_\alpha(H)$ in $L^2(X)$ affiliated to $\mathcal{C}(\overline{X}) \rtimes X$ and $\sigma_{\text{ess}}(H) = \bigcup_\alpha \sigma(\tau_\alpha(H))$.

We now treat the N-body case. We first indicate a general way of constructing N-body Hamiltonians. For each linear subspace $Y \subset X$, let $\mathcal{A}(X/Y) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X/Y)$ be a translation invariant C^*-subalgebra containing $\mathcal{C}_0(X/Y)$ with $\mathcal{A}(X/X) = \mathcal{A}(0) = \mathbb{C}$. We embed $\mathcal{A}(X/Y) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ as usual by identifying v with $v \circ \pi_Y$. Then the C^*-algebra $\mathcal{A}$ generated by these algebras is a translation invariant C^*-subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ containing $\mathcal{C}(X^+)$ and thus we may consider the crossed product $\mathcal{A} \rtimes X$ which is equal to the C^*-algebra generated by the crossed products $\mathcal{A}(X/Y) \rtimes X$. The operators affiliated to $\mathcal{A} \rtimes X$ are N-body Hamiltonians. The standard N-body algebra corresponds to the minimal choice $\mathcal{A}(X/Y) = \mathcal{C}_0(X/Y)$ and has remarkable properties, which makes its study relatively easy (it is graded by the lattice of subspaces of X). Our purpose in this paper is to study what could arguably be considered to be the simplest extension of the classical N-body obtained by choosing $\mathcal{A}(X/Y) = \mathcal{C}(\overline{X/Y})$ for all Y. The next more general case would correspond to the choice $\mathcal{A}(X/Y) = \mathcal{V}(X/Y)$ (slowly oscillating functions, i.e. the closure in sup norm of the set of bounded functions of class C^1 with derivatives tending to zero at infinity).

Definition 3.2. Let $\mathcal{E}(X)$ be the C^*-subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ generated by $\bigcup_Y \mathcal{C}(\overline{X/Y})$.

Clearly $\mathcal{E}(X)$ is a translation invariant C^*-subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ containing $\mathcal{C}(X^+) := \mathcal{C}_0(X) + \mathbb{C}$. If Y is a linear subspace of X then the C^*-algebra $\mathcal{E}(X/Y) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X/Y)$ is well defined and naturally embedded in $\mathcal{E}(X)$: it is the C^*-algebra generated by $\bigcup_{Z\supset Y} C(\overline{X/Z})$. We have $\mathbb{C} = \mathcal{E}(0) = \mathcal{E}(X/X) \subset \mathcal{E}(X/Y) \subset \mathcal{E}(X/Z) \subset \mathcal{E}(X)$. If $\alpha \in \mathbb{S}_X$, we shall denote by abuse of notation X/α be the quotient $X/[\alpha]$ of X by the subspace $[\alpha] := \mathbb{R}\alpha$ generated by α and let us set $\pi_\alpha = \pi_{[\alpha]}$. It is clear that $\tau_\alpha(u)(x) = \lim_{r\to+\infty} u(r a + x)$ exists $\forall u \in \mathcal{E}(X)$ and that the resulting function $\tau_\alpha(u)$ belongs to $\mathcal{E}(X)$. The map τ_α is an endomorphism of $\mathcal{E}(X)$ and a linear projection of $\mathcal{E}(X)$ onto the C^*-subalgebra $\mathcal{E}(X/\alpha)$.

If $\alpha \in \mathbb{S}_X$ and $\beta \in \mathbb{S}_{X/\alpha}$, then β generates a one-dimensional linear subspace $[\beta] := \mathbb{R}\beta \subset X/\alpha$, as above, and hence $\pi_\alpha^{-1}([\beta])$ is a two-dimensional subspace of X that we shall denote $[\alpha, \beta]$. We shall identify $(X/\alpha)/\beta$ with $X/[\alpha, \beta]$. Then we have two idempotent morphisms $\tau_\alpha : \mathcal{E}(X) \to \mathcal{E}(X/\alpha)$ and $\tau_\beta : \mathcal{E}(X/\alpha) \to \mathcal{E}(X/[\alpha, \beta])$. Thus $\tau_\beta\tau_\alpha : \mathcal{E}(X) \to \mathcal{E}(X/[\alpha, \beta])$ is an idempotent morphism. This construction extends in an obvious way to families $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $n \le \dim X$ and $\alpha_1 \in \mathbb{S}_X, \alpha_2 \in \mathbb{S}_{X/\alpha_1}, \alpha_3 \in \mathbb{S}_{X/[\alpha_1,\alpha_2]}, \ldots$ (we allow $n = 0$ and denote A the set of all such families). The endomorphism $\tau_{\vec{\alpha}}$ of $\mathcal{E}(X)$ is defined by induction: $\tau_{\vec{\alpha}} = \tau_{\alpha_n} \ldots \tau_{\alpha_1}$. We also define $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ by induction, so this is an n-dimensional subspace of X associated with $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and we denote $X/\vec{\alpha}$ the quotient of X with respect to it. Thus $\tau_{\vec{\alpha}}$ is an endomorphism of $\mathcal{E}(X)$ and a projection of $\mathcal{E}(X)$ onto $\mathcal{E}(X/\vec{\alpha})$.

Proposition 3.3. *If $\vec{\alpha} \in A$ and $a \in X/\vec{\alpha}$, then $\varkappa(u) = (\tau_{\vec{\alpha}} u)(a)$ defines a character of $\mathcal{E}(X)$. Conversely, each character of $\mathcal{E}(X)$ is of this form.*

Remark 1. A natural Abelian C^*-algebra in the present context is the set $\mathcal{R}(X)$ of all bounded uniformly continuous functions $v : X \to \mathbb{C}$ such that $\lim_{r\to\infty} v(ra + x)$ exists locally uniformly in $x \in X$ for each $a \in X$. It would be interesting to find an explicit description of its spectrum.

This description of the spectrum of $\mathcal{E}(X)$ extends [10]. We now state our main results.

Theorem 3.1. *Let H be a self-adjoint operator on $L^2(X)$ affiliated to $\mathcal{E}(X) \rtimes X$. Then for any $a \in X \setminus \{0\}$ the limit* $\text{s-lim}_{r\to+\infty} T_{ra} H T_{ra}^* =: \tau_{\hat{a}}(H)$ *exists and* $\sigma_{\text{ess}}(H) = \overline{\bigcup_{\alpha\in\mathbb{S}_X} \sigma(\tau_\alpha(H))}$.

Theorem 3.2. *Let h be as in Example 2 and $V = \sum V_Y$ with $V_Y : \mathcal{H}^s \to \mathcal{H}^{-s}$ symmetric operators such that $V_Y = 0$ but for a finite number of Y and satisfying: (i) $\exists \mu_Y, \nu_Y \geq 0$ with $\sum_Y \mu_Y < 1$ such that $\pm V_Y \leq \mu_Y h(P) + \nu_Y$, (ii) $\lim_{p\to 0} \|[S_p, V_Y]\|_{\mathcal{H}^s\to\mathcal{H}^{-s}} = 0$, (iii) $[T_y, V_Y] = 0$ for all $y \in Y$, (iv) $\tau_\alpha(V_Y) :=$ s-$\lim_{a\to\alpha} T_a V_Y T_a^*$ exists in $B(\mathcal{H}^s, \mathcal{H}^{-s})$ for all $\alpha \in \mathbb{S}_{X/Y}$. Then the maps $\mathcal{H}^s \to \mathcal{H}^{-s}$ given by $h(P) + V$ and $h(P) + \sum_Y \tau_\alpha(V_Y)$ induce self-adjoint operators H and $\tau_\alpha(H)$ in $L^2(X)$ affiliated to $\mathscr{E}(X)$ and* $\sigma_{\text{ess}}(H) = \overline{\bigcup_{\alpha\in\mathbb{S}_X} \sigma(\tau_\alpha(H))}$.

Example 3. Using [2], we also obtain that Theorem 3.2 covers uniformly elliptic operators of the form $H = \sum_{|k|,|\ell|\leq s} P^k a_{k\ell} P^\ell$, where $a_{k\ell}$ are finite sums of functions of the form $v_Y \circ \pi_Y$ with $v_Y : X/Y \to \mathbb{R}$ bounded measurable such that $\lim_{z\to\alpha} v_Y(z)$ exists uniformly in $\alpha \in \mathbb{S}_{X/Y}$. The fact that we allow $a_{k\ell}$ to be only bounded measurable for $|k| = |\ell| = s$ is not trivial.

In addition to the above-mentioned results, we also use general results on cross-product C^*-algebras, their ideals, and their representations [4,13]. The maximal ideal spectrum of the algebra $\mathcal{E}(X)$ is of independent interest and can be used to study the regularity properties of the eigenvalues of the N-body Hamiltonian [1]. Its relation to the constructions of Vasy in [14] will be studied elsewhere.

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