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Stability of electromagnetic cavities perturbed by small perfectly conducting inclusions

Stabilité des cavités électromagnétiques perturbées par des petites inclusions parfaitement conductrices

Xavier Claeys [a,b,c,1]

[a] *Sorbonne Universités, UPMC Université Paris-6, UMR 7598, Laboratoire Jacques-Louis-Lions, 75005 Paris, France*
[b] *CNRS, UMR 7598, Laboratoire Jacques-Louis-Lions, 75005 Paris, France*
[c] *INRIA Paris–Rocquencourt, EPI Alpines, Domaine de Voluceau, BP105, 78153 Le Chesnay cedex, France*

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ABSTRACT

In this note, we consider an electromagnetic wave propagation problem in harmonic regime in a bounded cavity, in the case where the medium of propagation contains small perfectly conducting inclusions. We prove that the solution to this problem depends continuously on the data in a uniform manner with respect to the size of the inclusions.

RÉSUMÉ

Dans cette note, nous considérons un problème de propagation d'ondes électromagnétiques en régime harmonique dans une cavité bornée, dans le cas où la cavité contient de petites inclusions parfaitement conductrices. Nous montrons que la solution de ce problème dépend continuement des données de manière uniforme vis-à-vis de la taille des inclusions.

Version française abrégée

On considère deux ouverts lipschitziens bornés $\Omega, \mathrm{D} \subset \mathbb{R}^3$ tels que $0 \in \Omega$, et on pose $\Omega_\delta := \{\mathbf{x} \in \Omega,\ \mathbf{x}/\delta \notin \mathrm{D}\}$. On considère également $\omega > 0$ ainsi que deux fonctions à valeurs matricielles $\epsilon, \mu : \Omega \mapsto \mathbb{C}^{3\times 3}$ uniformément bornées pour lesquelles il existe $\epsilon_*, \mu_* > 0$ tels que $\epsilon_*|\mathbf{y}|^2 < \Re e\{\mathbf{y}^{\mathrm{T}}\epsilon(\mathbf{x})\mathbf{y}\}$ et $\mu_*|\mathbf{y}|^2 < \Re e\{\mathbf{y}^{\mathrm{T}}\mu(\mathbf{x})\mathbf{y}\}$ pour tout $\mathbf{x} \in \Omega, \mathbf{y} \in \mathbb{R}^3$. Sous l'hypothèse que ω n'est pas une fréquence de résonance du problème de Maxwell dans Ω avec condition de conducteur parfait sur le bord, on démontre (voir Théorème 2.1) qu'il existe deux constantes $C, \delta_0 > 0$ indépendantes de δ telles que, pour tout $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega_\delta) := \{\mathbf{u} \in \mathbf{L}^2(\Omega_\delta),\ \mathbf{curl}(\mathbf{u}) \in \mathbf{L}^2(\Omega_\delta),\ \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega_\delta\}$, on a :

E-mail address: claeys@ann.jussieu.fr.
[1] Correspondence to: Sorbonne Universités, UPMC Université Paris-6, UMR 7598, Laboratoire Jacques-Louis-Lions, 75005 Paris, France.

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$$\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega_\delta)} \leq C \sup_{\mathbf{v}\in\mathbf{H}_0(\mathbf{curl},\Omega_\delta)\setminus\{0\}} \frac{\int_{\Omega_\delta} \mu^{-1}\mathbf{curl}(\mathbf{u})\mathbf{curl}(\mathbf{v}) - \omega^2(\epsilon\mathbf{u})\mathbf{v}\,\mathrm{d}\mathbf{x}}{\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_\delta)}} \quad \forall \delta \in [0,\delta_0].$$

0. Introduction

In this note, $\Omega, \mathrm{D} \subset \mathbb{R}^3$ refer to bounded Lipschitz open sets with $0 \in \Omega$. Set $\Omega_\delta := \{\mathbf{x} \in \Omega,\ \mathbf{x}/\delta \notin \mathrm{D}\}$ and $\Xi = \mathbb{R}^3 \setminus \overline{\mathrm{D}}$. Let $\epsilon, \mu : \Omega \mapsto \mathbb{C}^{3\times 3}$ refer to bounded matrix valued functions such that there exist constants $\epsilon_*, \mu_* > 0$ with $\epsilon_*|\mathbf{y}|^2 < \Re e\{\mathbf{y}^{\mathrm{T}}\epsilon(\mathbf{x})\mathbf{y}\}$ and $\mu_*|\mathbf{y}|^2 < \Re e\{\mathbf{y}^{\mathrm{T}}\mu(\mathbf{x})\mathbf{y}\}$ for all $\mathbf{x} \in \Omega, \mathbf{y} \in \mathbb{R}^3$. The matrices ϵ and μ stand respectively for the electric permittivity and the magnetic permeability of the medium. We also consider a fixed frequency $\omega > 0$, and study the corresponding Maxwell source problem

$$\begin{cases} \mathbf{u}_\delta \in \mathbf{H}_0(\mathrm{curl},\Omega_\delta) \quad \text{such that} \\ \mathbf{curl}^2_\mu(\mathbf{u}_\delta) - \omega^2\epsilon\mathbf{u}_\delta = \mathbf{f} \quad \text{in } \Omega_\delta, \end{cases} \tag{1}$$

where $\mathbf{H}_0(\mathrm{curl},\Omega_\delta) := \{\mathbf{u} \in \mathbf{L}^2(\Omega_\delta),\ \mathbf{curl}(\mathbf{u}) \in \mathbf{L}^2(\Omega_\delta),\ \mathbf{u}\times\mathbf{n} = 0 \text{ on } \partial\Omega_\delta\}$, and $\mathbf{curl}^2_\mu := \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\cdot)$. Here $\mathbf{f} \in \mathbf{H}_0(\mathrm{curl},\Omega_\delta)^*$ is an arbitrarily chosen right hand side. It is well established that, for any fixed $\delta > 0$, the problem above is of Fredholm type with index 0, i.e. it admits a unique solution except for a discrete set of eigenfrequencies, see, e.g., [7]. In the present note, we will assume that ω is not an eigenfrequency of the limit problem associated with $\delta = 0$ (where $\Omega_0 = \Omega$), and we wish to show that, for $\mathbf{f}$ fixed, the solution remains uniformly bounded as $\delta \to 0$.

Although this kind of asymptotic stability result is well known for many different situations in the case of scalar elliptic problems (see [8–14] for example), Maxwell's equations have received much less attention.

A series of works [2,1,3,6] deals with electromagnetic scattering in homogeneous ambient media containing small penetrable heterogeneities. In such situations, the propagation medium is fixed, i.e. the perturbed problem is posed in the same domain as the limit problem. On the other hand, the contributions [4] and [5, Chap. 3] deal with the asymptotics of perfectly conducting (impenetrable) objects embedded in a homogeneous medium by means of boundary integral equation techniques. It provides a stability estimate for a second kind integral equation (the so-called MFIE), and deduces asymptotic formulas for the electromagnetic fields in a region of the domain located at a fixed positive distance from the small scatterers. The analysis in [4,5] strongly relies on boundary integral representation formulas, adopting a very different approach compared to [2] as regards stability.

Adapting the proof of stability contained in [2] to the case of perfectly conducting inclusions seems difficult at first sight because, in this case, the medium of propagation varies as $\delta \to 0$, which prevents the use of compact embedding theorems that play a key role in the approach of [2]. This is the goal of the present note to show how to circumvent this difficulty by means of an asymptotic version of Hardy's inequality.

1. Asymptotic Hardy inequality

In the sequel, for any bounded Lipschitz open set $\mathcal{O} \subset \mathbb{R}^3$, the space $\mathrm{L}^2(\mathcal{O})$, resp. $\mathbf{L}^2(\mathcal{O}) := \mathrm{L}^2(\mathcal{O})^3$, will refer to square integrable functions, resp. fields, equipped with the norm $\|v\|^2_{\mathrm{L}^2(\mathcal{O})} := \int_{\mathcal{O}} |v|^2\,\mathrm{d}\mathbf{x}$. Moreover we consider the spaces $\mathbf{H}(\mathrm{curl},\mathcal{O}) := \{\mathbf{u} \in \mathbf{L}^2(\mathcal{O}),\ \mathbf{curl}(\mathbf{u}) \in \mathbf{L}^2(\mathcal{O})\}$, and $\mathbf{H}_0(\mathrm{curl},\mathcal{O}) := \{\mathbf{u} \in \mathbf{H}(\mathrm{curl},\mathcal{O}),\ \mathbf{u}\times\mathbf{n} = 0\}$ equipped with the norm $\|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{curl},\mathcal{O})} = \|\mathbf{v}\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{curl}(\mathbf{v})\|^2_{\mathbf{L}^2(\mathcal{O})}$, where $\mathbf{n}$ refers to the normal vector to $\partial\mathcal{O}$. In addition, $\mathbf{X}(\mathcal{O})$ will refer to the fields $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl},\mathcal{O})$ such that $\mathrm{div}(\epsilon\mathbf{v}) \in \mathrm{L}^2(\mathcal{O})$ and $\int_\Sigma \epsilon\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\sigma = 0$ for each connected component Σ of $\partial\mathcal{O}$, equipped with the norm

$$\|\mathbf{v}\|^2_{\mathbf{X}(\mathcal{O})} = \big\|\mathbf{curl}(\mathbf{v})\big\|^2_{\mathbf{L}^2(\mathcal{O})} + \big\|\mathrm{div}(\epsilon\mathbf{v})\big\|^2_{\mathrm{L}^2(\mathcal{O})} + \|\mathbf{v}\|^2_{\mathbf{L}^2(\mathcal{O})}.$$

The space $\mathbf{X}(\mathcal{O})$ is compactly embedded into $\mathbf{L}^2(\mathcal{O})$, see [17].

Lemma 1.1. *There exist constants $C, \delta_0 > 0$ independent of δ such that*

$$C\int_{\Omega_\delta} \frac{|\mathbf{v}(\mathbf{x})|^2}{\delta^2 + |\mathbf{x}|^2}\,\mathrm{d}\mathbf{x} \leq \|\mathbf{v}\|^2_{\mathbf{X}(\Omega_\delta)} \quad \forall \mathbf{v} \in \mathbf{X}(\Omega_\delta),\ \forall \delta \in (0,\delta_0).$$

Proof. Let $\mathbf{W}(\Xi)$ refer to the closure of $\mathcal{C}^\infty_{\mathrm{comp}}(\overline{\Xi}) = \{\mathbf{v}|_\Xi,\ \mathbf{v} \in \mathcal{C}^\infty(\mathbb{R}^3)^3,\ \mathrm{supp}(\mathbf{v}) \text{ bounded}\}$ with respect to the norm $\|\mathbf{v}\|^2_{\mathbf{W}(\Xi)} := \int_\Xi (1+|\xi|^2)^{-1}|\mathbf{v}(\xi)|^2\,\mathrm{d}\xi + \|\mathbf{curl}(\mathbf{v})\|^2_{\mathbf{L}^2(\Xi)} + \|\mathrm{div}(\mathbf{v})\|^2_{\mathbf{L}^2(\Xi)}$. Set in addition $\mathbf{W}_0(\Xi) = \{\mathbf{v} \in \mathbf{W}(\Xi),\ \mathbf{v}\times\mathbf{n} = 0 \text{ on } \partial\Xi\}$, which is a closed subspace of $\mathbf{W}(\Xi)$. According to Lemma 6 in [15], if Γ_j, $j = 0, 1, \ldots, J$ refer to the connected components of $\partial\Xi$, there exists $C > 0$ such that

$$C\int_\Xi \frac{|\mathbf{w}(\xi)|^2\,\mathrm{d}\xi}{1+|\xi|^2} \leq \big\|\mathbf{curl}(\mathbf{w})\big\|_{\mathbf{L}^2(\Xi)} + \big\|\mathrm{div}(\mathbf{w})\big\|_{\mathbf{L}^2(\Xi)} + \sum_{j=1}^{J}\left|\int_{\Gamma_j} \mathbf{w}\cdot\mathbf{n}\,\mathrm{d}\sigma\right| \quad \forall \mathbf{w} \in \mathbf{W}_0(\Xi). \tag{2}$$

Choose a fixed $t>0$ small enough to guarantee $\mathrm{B}(0,2t)\subset\Omega$, and let $\chi:\mathbb{R}^3\to\mathbb{R}_+$ refer to a $\mathcal{C}^\infty$ cut-off function such that $\chi(\mathbf{x})=\chi(|\mathbf{x}|)=1$ for $|\mathbf{x}|\leq t$ and $\chi(\mathbf{x})=0$ for $|\mathbf{x}|\geq 2t$. Also set $\psi:=1-\chi$. Clearly,

$$\int_{\Omega_\delta}\left(|\mathbf{x}|^2+\delta^2\right)^{-1}\left|\psi(\mathbf{x})\mathbf{v}(\mathbf{x})\right|^2\mathrm{d}\mathbf{x}\leq t^{-2}\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega_\delta)}\leq t^{-2}\|\mathbf{v}\|^2_{\mathbf{X}(\Omega_\delta)}\quad\forall\delta>0,\ \forall\mathbf{v}\in\mathbf{X}(\Omega_\delta).\tag{3}$$

Take an arbitrary $\mathbf{v}\in\mathbf{X}(\Omega_\delta)$ and denote $\mathbf{w}_\delta(\mathbf{y}):=\chi(\delta\mathbf{y})\mathbf{v}(\delta\mathbf{y})$. By definition of $\mathbf{X}(\Omega_\delta)$, we have $\mathbf{v}\times\mathbf{n}=0$ on $\partial\Omega_\delta$, which implies $\mathbf{w}_\delta\times\mathbf{n}=0$ on $\partial\Xi$. Moreover, we have $\int_\Sigma\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\sigma=0$ for any connected component Σ of $\partial\Omega_\delta$, which implies $\int_{\Gamma_j}\mathbf{w}_\delta\cdot\mathbf{n}\mathrm{d}\sigma=0$ for all $j=0\ldots n$. Since $\mathbf{w}_\delta$ has bounded support, $\mathbf{w}_\delta\in\mathbf{W}_0(\Xi)$, so we can apply (2). Using the change of variable $\mathbf{y}=\mathbf{x}/\delta$, we deduce that there exist constants $C,C'>0$ independent of δ such that

$$\begin{aligned}\int_{\Omega_\delta}\frac{|\chi(\mathbf{x})\mathbf{v}(\mathbf{x})|^2}{\delta^2+|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}&=\delta\int_\Xi\frac{|\mathbf{w}_\delta(\mathbf{y})|^2}{1+|\mathbf{y}|^2}\,\mathrm{d}\mathbf{y}\leq\delta C\left(\left\|\mathbf{curl}(\mathbf{w}_\delta)\right\|^2_{\mathbf{L}^2(\Xi)}+\left\|\mathrm{div}(\mathbf{w}_\delta)\right\|^2_{\mathbf{L}^2(\Xi)}\right)\\&\leq C\left(\left\|\mathbf{curl}(\chi\mathbf{v})\right\|^2_{\mathbf{L}^2(\Omega_\delta)}+\left\|\mathrm{div}(\chi\mathbf{v})\right\|^2_{\mathbf{L}^2(\Omega_\delta)}\right)\leq C'\|\mathbf{v}\|^2_{\mathbf{X}(\Omega_\delta)}.\end{aligned}\tag{4}$$

Since $\mathbf{v}$ was chosen arbitrarily and $\psi+\chi=1$, to conclude the proof, there only remains to gather inequalities (3) and (4). □

2. Stability theorem

In the sequel, $\mathbf{H}_0(\mathbf{curl},\mathcal{O})^*$ will refer to the topological dual to $\mathbf{H}_0(\mathbf{curl},\mathcal{O})$. The duality pairing will be denoted $\langle\,,\rangle_{\mathcal{O}}$, and we equip this space with the natural dual norm $\|f\|_{\mathbf{H}_0(\mathbf{curl},\mathcal{O})^*}:=\sup_{v\in\mathbf{H}_0(\mathbf{curl},\mathcal{O})\setminus\{0\}}|\langle f,v\rangle_{\mathcal{O}}|/\|v\|_{\mathbf{H}(\mathbf{curl},\mathcal{O})}$.

Theorem 2.1. *Assume that $\mathbf{curl}^2_\mu-\omega^2\epsilon:\mathbf{H}_0(\mathbf{curl},\Omega)\to\mathbf{H}_0(\mathbf{curl},\Omega)^*$ is an isomorphism (i.e. ω is not a resonant frequency of the limit problem). Then there exist constants $C,\delta_0>0$ independent of δ such that $\mathbf{curl}^2_\mu-\omega^2\epsilon:\mathbf{H}_0(\mathbf{curl},\Omega_\delta)\to\mathbf{H}_0(\mathbf{curl},\Omega_\delta)^*$ is invertible for all $\delta\in[0,\delta_0]$, with the uniform bound*

$$0<C\leq\inf_{\delta\in[0,\delta_0]}\ \inf_{\mathbf{u}\in\mathbf{H}_0(\mathbf{curl},\Omega_\delta)\setminus\{0\}}\frac{\|\mathbf{curl}^2_\mu(\mathbf{u})-\omega^2\epsilon\mathbf{u}\|_{\mathbf{H}_0(\mathbf{curl},\Omega_\delta)^*}}{\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega_\delta)}}.$$

Proof. For the sake of brevity, we define the continuous operator $\mathrm{A}_\delta:\mathbf{H}_0(\mathbf{curl},\Omega_\delta)\to\mathbf{H}_0(\mathbf{curl},\Omega_\delta)^*$ by $\langle\mathrm{A}_\delta(\mathbf{u}),\mathbf{v}\rangle_{\Omega_\delta}:=\int_{\Omega_\delta}\mu^{-1}\mathbf{curl}(\mathbf{u})\mathbf{curl}(\mathbf{v})-\omega^2(\epsilon\mathbf{u})\mathbf{v}\,\mathrm{d}\mathbf{x}$ for all $\mathbf{u},\mathbf{v}\in\mathbf{H}_0(\mathbf{curl},\Omega_\delta)$. We will proceed by contradiction, assuming that there exist sequences $\delta_n\to0$ and $\mathbf{u}_n\in\mathbf{H}_0(\mathbf{curl},\Omega_{\delta_n})$ such that $\|\mathbf{u}_n\|_{\mathbf{H}(\mathbf{curl},\Omega_{\delta_n})}=1$ and $\lim_{n\to\infty}\|\mathrm{A}_{\delta_n}(\mathbf{u}_n)\|_{\mathbf{H}_0(\mathbf{curl},\Omega_{\delta_n})^*}=0$. We set $\mathbf{f}_n:=\mathrm{A}_{\delta_n}(\mathbf{u}_n)$.

Consider $\mathbf{H}_0(\mathbf{curl}=0,\Omega_\delta):=\{\mathbf{v}\in\mathbf{H}_0(\mathbf{curl},\Omega_\delta)\mid\mathbf{curl}(\mathbf{v})=0\}$, which is obviously a closed subspace of $\mathbf{H}_0(\mathbf{curl},\Omega_\delta)$. Let $\mathring{\mathbf{u}}_n$ refer to the unique element of $\mathbf{H}_0(\mathbf{curl}=0,\Omega_{\delta_n})$ satisfying $\int_{\Omega_{\delta_n}}\epsilon(\mathbf{u}_n-\mathring{\mathbf{u}}_n)\mathbf{v}\,\mathrm{d}\mathbf{x}=0$ for all $\mathbf{v}\in\mathbf{H}_0(\mathbf{curl}=0,\Omega_{\delta_n})$. We have in particular

$$\epsilon_*\|\mathring{\mathbf{u}}_n\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\delta_n})}\leq\Re e\left\{\int_{\Omega_{\delta_n}}(\epsilon\mathbf{u}_n)\mathring{\mathbf{u}}_n\mathrm{d}\mathbf{x}\right\}\leq|\omega|^{-2}\left|\langle\mathbf{f}_n,\mathring{\mathbf{u}}_n\rangle_{\Omega_{\delta_n}}\right|,$$

which implies $\lim_{n\to\infty}\|\mathring{\mathbf{u}}_n\|_{\mathbf{H}(\mathbf{curl},\Omega_{\delta_n})}=0$. Set $\tilde{\mathbf{u}}_n:=\mathbf{u}_n-\mathring{\mathbf{u}}_n$ and $\tilde{\mathbf{f}}_n:=\mathrm{A}_{\delta_n}(\tilde{\mathbf{u}}_n)$. We will obtain a contradiction if we prove that $\lim_{n\to\infty}\|\tilde{\mathbf{u}}_n\|_{\mathbf{H}(\mathbf{curl},\Omega_{\delta_n})}=0$. Note that $\|\tilde{\mathbf{f}}_n\|_{\mathbf{H}_0(\mathbf{curl},\Omega_{\delta_n})^*}\leq\|\mathbf{f}_n\|_{\mathbf{H}_0(\mathbf{curl},\Omega_{\delta_n})^*}+C\|\mathring{\mathbf{u}}_n\|_{\mathbf{H}(\mathbf{curl},\Omega_{\delta_n})}\to0$ for $C=\omega^2\sup_\Omega|\epsilon|$. Besides, since $\nabla\varphi\in\mathbf{H}_0(\mathbf{curl}=0,\Omega_{\delta_n})$ for any $\varphi\in\mathrm{H}^1_0(\Omega_{\delta_n})$, we obtain:

$$\int_{\Omega_{\delta_n}}\mathrm{div}(\epsilon\tilde{\mathbf{u}}_n)\varphi\,\mathrm{d}\mathbf{x}=\int_{\Omega_{\delta_n}}\epsilon(\mathbf{u}_n-\mathring{\mathbf{u}}_n)\nabla\varphi\,\mathrm{d}\mathbf{x}=0,\tag{5}$$

which yields $\mathrm{div}(\epsilon\tilde{\mathbf{u}}_n)=0$ in Ω_{δ_n}. Observe in addition that any $\varphi\in\mathrm{H}^1(\Omega)$ that is constant in each connected component of $\Omega\setminus\Omega_{\delta_n}$ also satisfies $\nabla\varphi\in\mathbf{H}_0(\mathbf{curl}=0,\Omega_{\delta_n})$. Proceeding like in (5), we conclude that $\int_\Gamma(\epsilon\tilde{\mathbf{u}}_n)\cdot\mathbf{n}\,\mathrm{d}\sigma=0$ for any Γ that is a connected component of $\partial\Omega_\delta$. This shows that $\tilde{\mathbf{u}}_n\in\mathbf{X}(\Omega_{\delta_n})$. Moreover, since $\|\tilde{\mathbf{u}}_n\|_{\mathbf{X}(\Omega_{\delta_n})}\leq\|\mathbf{u}_n\|_{\mathbf{H}(\mathbf{curl},\Omega_{\delta_n})}+\|\mathring{\mathbf{u}}_n\|_{\mathbf{H}(\mathbf{curl},\Omega_{\delta_n})}$, we have:

$$\limsup_{n\to\infty}\|\tilde{\mathbf{u}}_n\|_{\mathbf{X}(\Omega_{\delta_n})}<+\infty.$$

Consider now a $\mathcal{C}^\infty$ cut-off function $\psi:\mathbb{R}^3\to\mathbb{R}_+$ such that $\psi(\mathbf{x})=\psi(|\mathbf{x}|)=1$ for $|\mathbf{x}|\geq2$ and $\psi(\mathbf{x})=0$ for $|\mathbf{x}|\leq1$, set $\psi_n(\mathbf{x}):=\psi(\mathbf{x}/\delta_n)$, and denote $\mathrm{Q}_n:=\{\mathbf{x}\in\Omega\mid\delta_n\leq|\mathbf{x}|\leq2\delta_n\}$. Using extension by 0 inside $\Omega\setminus\Omega_{\delta_n}$, we can consider that $\psi_n\tilde{\mathbf{u}}_n$ belongs to $\mathbf{X}(\Omega)$. This is actually a bounded sequence of $\mathbf{X}(\Omega)$. Indeed $\delta_n^{-2}\leq5/(\delta_n^2+|\mathbf{x}|^2)$ for $\mathbf{x}\in\mathrm{Q}_n$. As a consequence, applying Lemma 1.1, we obtain the existence of two constants $C,C'>0$ independent of n, such that

$$\begin{aligned}\left\|\mathbf{curl}(\psi_n\tilde{\mathbf{u}}_n)\right\|^2_{\mathbf{L}^2(\Omega)} &\leq C\left\|\mathbf{curl}(\tilde{\mathbf{u}}_n)\right\|^2_{\mathbf{L}^2(\Omega_{\delta_n})} + C\delta_n^{-2}\|\tilde{\mathbf{u}}_n\|^2_{\mathbf{L}^2(\mathrm{Q}_n)}\\ &\leq C\left\|\mathbf{curl}(\tilde{\mathbf{u}}_n)\right\|^2_{\mathbf{L}^2(\Omega_{\delta_n})} + C\int_{\mathrm{Q}_n}\frac{|\tilde{\mathbf{u}}_n(\mathbf{x})|^2}{\delta_n^2+|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}\\ &\leq C\left\|\mathbf{curl}(\tilde{\mathbf{u}}_n)\right\|^2_{\mathbf{L}^2(\Omega_{\delta_n})} + C\int_{\Omega_{\delta_n}}\frac{|\tilde{\mathbf{u}}_n(\mathbf{x})|^2}{\delta_n^2+|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x} \leq C'\|\tilde{\mathbf{u}}_n\|^2_{\mathbf{X}(\Omega_{\delta_n})}\end{aligned}$$

for all $n>0$. We prove in a similar manner that $\|\mathrm{div}(\psi_{\delta_n}\tilde{\mathbf{u}}_n)\|_{\mathrm{L}^2(\Omega)}$ remains bounded as $n\to\infty$. From this, we finally conclude that $\limsup_{n\to\infty}\|\psi_n\tilde{\mathbf{u}}_n\|_{\mathbf{X}(\Omega)}<+\infty$. So according to the compact embedding of $\mathbf{X}(\Omega)$ into $\mathbf{L}^2(\Omega)$, see [17], extracting a subsequence if necessary, there exists $\tilde{\mathbf{u}}_\infty\in\mathbf{X}(\Omega)$ such that $\lim_{n\to\infty}\|\tilde{\mathbf{u}}_n-\tilde{\mathbf{u}}_\infty\|_{\mathbf{L}^2(\Omega)}=0$ and $\tilde{\mathbf{u}}_n$ converges weakly towards $\tilde{\mathbf{u}}_\infty$ in $\mathbf{X}(\Omega)$.

Take any test function $\mathbf{w}\in\mathbf{X}_\star(\Omega):=\{\mathbf{v}\in\mathbf{X}(\Omega),\ \mathbf{v}=0$ in a neighborhood of $0\}$. Since $\tilde{\mathbf{u}}_n$ coincides with $\psi_n\tilde{\mathbf{u}}_n$ on $\mathrm{supp}(\mathbf{w})$ for n large enough, weak convergence implies that $\lim_{n\to\infty}\langle \mathrm{A}_{\delta_n}(\tilde{\mathbf{u}}_n),\mathbf{w}\rangle_{\Omega_{\delta_n}}=\lim_{n\to\infty}\langle \mathrm{A}_0(\psi_n\tilde{\mathbf{u}}_n),\mathbf{w}\rangle_\Omega=\langle \mathrm{A}_0(\tilde{\mathbf{u}}_\infty),\mathbf{w}\rangle_\Omega=0$. Since $\mathbf{X}_\star(\Omega)$ is dense into $\mathbf{X}(\Omega)$ (see [16, Prop. 5.14]), this leads to the conclusion that $\mathrm{A}_0\tilde{\mathbf{u}}_\infty=0$ and, as we assumed that A_0 is an isomorphism, we finally conclude that $\tilde{\mathbf{u}}_\infty=0$. Observe that $\mathrm{supp}(1-\psi_n)\subset\mathrm{B}(0,2\delta_n)$. From this, and Lemma 1.1, we conclude that there exist constants $C,C'>0$ independent of n such that

$$\begin{aligned}\|\tilde{\mathbf{u}}_n\|^2_{\mathbf{L}^2(\Omega_{\delta_n})} &\leq 2\|\psi_n\tilde{\mathbf{u}}_n\|^2_{\mathbf{L}^2(\Omega)} + C\delta_n^2\int_{\mathrm{B}(0,2\delta_n)}\frac{|\tilde{\mathbf{u}}_n(\mathbf{x})|^2}{\delta_n^2+|\mathbf{x}|^2}\mathrm{d}\mathbf{x}\\ &\leq 2\|\psi_n\tilde{\mathbf{u}}_n\|^2_{\mathbf{L}^2(\Omega)} + C'\delta_n^2\|\tilde{\mathbf{u}}_n\|^2_{\mathbf{X}(\Omega_{\delta_n})}.\end{aligned}$$

Since $\limsup_{n\to\infty}\|\tilde{\mathbf{u}}_n\|_{\mathbf{X}(\Omega_{\delta_n})}<+\infty$ and $\|\psi_n\tilde{\mathbf{u}}_n\|_{\mathbf{L}^2(\Omega)}\to\|\tilde{\mathbf{u}}_\infty\|_{\mathbf{L}^2(\Omega)}=0$ we conclude that $\lim_{n\to\infty}\|\tilde{\mathbf{u}}_n\|_{\mathbf{L}^2(\Omega_{\delta_n})}=0$. This finally leads to a contradiction, since we assumed that $\|\mathbf{u}_n\|_{\mathbf{H}(\mathbf{curl},\Omega_{\delta_n})}=1$ and, at the same time, we have:

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}_n\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\delta_n})} &\leq \|\hat{\mathbf{u}}_n\|^2_{\mathbf{L}^2(\Omega_{\delta_n})} + \|\tilde{\mathbf{u}}_n\|^2_{\mathbf{X}(\Omega_{\delta_n})}\\ &\leq \|\hat{\mathbf{u}}_n\|^2_{\mathbf{L}^2(\Omega_{\delta_n})} + \left(1+\omega^2\sup_\Omega|\epsilon|\right)\|\tilde{\mathbf{u}}_n\|^2_{\mathbf{L}^2(\Omega_{\delta_n})} + \sup_\Omega|\mu|\left|\langle\tilde{\mathbf{f}}_n,\bar{\tilde{\mathbf{u}}}_n\rangle_{\Omega_{\delta_n}}\right| \underset{n\to\infty}{\longrightarrow} 0. \qquad \square\end{aligned}$$

References

[1] H. Ammari, D. Volkov, Asymptotic formulas for perturbations in the eigenfrequencies of the full Maxwell equations due to the presence of imperfections of small diameter, Asymptot. Anal. 30 (3–4) (2002) 331–350.
[2] H. Ammari, M.S. Vogelius, D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter. II. The full Maxwell equations, J. Math. Pures Appl. (9) 80 (8) (2001) 769–814.
[3] C. Daveau, A. Khelifi, Asymptotic behaviour of the energy for electromagnetic systems in the presence of small inhomogeneities, Appl. Anal. 91 (5) (2012) 857–877.
[4] R. Griesmaier, An asymptotic factorization method for inverse electromagnetic scattering in layered media, SIAM J. Appl. Math. 68 (5) (2008) 1378–1403.
[5] R. Griesmaier, Detection of small buried objects: asymptotic factorization and MUSIC, PhD thesis, Johannes Gutenberg-Universität, Mainz, Germany, 2008.
[6] R. Griesmaier, A general perturbation formula for electromagnetic fields in presence of low volume scatterers, ESAIM: Math. Model. Numer. Anal. 45 (6) (2011) 1193–1218.
[7] R. Hiptmair, Finite elements in computational electromagnetism, Acta Numer. 11 (2002) 237–339.
[8] A.M. Il'in, Study of the asymptotic behavior of the solution of an elliptic boundary value problem in a domain with a small hole, Tr. Semin. Im. I.G. Petrovskogo (6) (1981) 57–82.
[9] A.M. Il'in, Matching of Asymptotic Expansions of Solutions of Boundary Value Problems, Translations of Mathematical Monographs, vol. 102, AMS, Providence, RI, 1992.
[10] V.G. Maz'ya, S.A. Nazarov, Asymptotic behavior of energy integrals under small perturbations of the boundary near corner and conic points, Tr. Mosk. Mat. Obŝ. 50 (1987) 79–129, 261.
[11] V.G. Maz'ya, S.A. Nazarov, B.A. Plamenevskiĭ, Asymptotic expansions of eigenvalues of boundary value problems for the Laplace operator in domains with small openings, Izv. Akad. Nauk SSSR, Ser. Mat. 48 (2) (1984) 347–371.
[12] V. Maz'ya, S. Nazarov, B. Plamenevskij, Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains. Vol. I, Operator Theory: Advances and Applications, vol. 111, Birkhäuser Verlag, Basel, 2000.
[13] S.A. Nazarov, Asymptotic conditions at a point, selfadjoint extensions of operators, and the method of matched asymptotic expansions, in: Proceedings of the St. Petersburg Mathematical Society, Vol. V, in: Amer. Math. Soc. Transl. Ser. 2, vol. 193, Amer. Math. Soc., Providence, RI, 1999, pp. 77–125.
[14] S.A. Nazarov, J. Sokołowski, Self-adjoint extensions for the Neumann Laplacian and applications, Acta Math. Sin. Engl. Ser. 22 (3) (2006) 879–906.
[15] R. Picard, Randwertaufgaben in der verallgemeinerten Potentialtheorie, Math. Methods Appl. Sci. 3 (2) (1981) 218–228.
[16] M.E. Taylor, Partial Differential Equations I. Basic Theory, second edition, Applied Mathematical Sciences, vol. 115, Springer, New York, 2011.
[17] Ch. Weber, A local compactness theorem for Maxwell's equations, Math. Methods Appl. Sci. 2 (1) (1980) 12–25.