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Differential geometry

There exist no locally symmetric Finsler spaces of positive or negative flag curvature



Il n'existe pas d'espace de Finsler localement symétrique de courbure de drapeau positive ou négative

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ABSTRACT

We show that the results of Foulon [5,6] and Kim [7] (independently, of Deng and Hou [4]) about the nonexistence of locally symmetric Finsler metrics of positive or negative flag curvature are in fact local.

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R É S U M É

Nous montrons que les résultats de Foulon [5,6] et de Kim [7] (et indépendamment, de Deng et Hou [4]) sur l'inexistence de métriques de Finsler localement symétriques, de courbure de drapeau positive ou négative, sont en fait locaux.

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Finsler metric on a smooth manifold M is a continuous function $F : TM \rightarrow [0, \infty)$ such that for every point $p \in M$ the restriction $F_p = F|_{T_p M}$ on the tangent space at p is a Minkowski norm, that is F_p is positively homogeneous and convex and it vanishes only at $y = 0$:

- (a) $F_p(\lambda \cdot y) = \lambda \cdot F_p(y)$ for any $\lambda \geq 0$.
- (b) $F_p(y + \tilde{y}) \leq F_p(y) + F_p(\tilde{y})$.
- (c) $F_p(y) = 0 \Rightarrow y = 0$.

We will also assume that our Finsler metric is of class C^2 , that is the restriction of F to the slit tangent bundle $TM^0 = TM \setminus (\text{the zero section})$ is a function of class C^2 , and that it is *strongly convex*, that is the Hessian of the restriction of $\frac{1}{2}F^2$ to $T_p M \setminus \{0\}$ (which will be denoted by $g = g_{ij}$ later) is positively definite for any $p \in M$ and any nonzero vector $y \in T_p M$.

The Finsler manifold (M, F) is called *locally symmetric* if, for every point $p \in M$, there exists $r = r(p) > 0$ and an isometry $I_p : B_r(p) \rightarrow B_r(p)$ (called the *reflection* at p) such that $I_p(p) = p$ and $d_p(I_p) = -\text{id} : T_p M \rightarrow T_p M$. Here $B_r(p)$ denotes the ball of radius r around p .

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Our main result is:

Theorem 1. *Suppose the flag curvature of a locally symmetric Finsler metric is negative or positive. Then, the metric is actually a Riemannian metric, that is, there exists a Riemannian metric h such that for all $p \in M$, $y \in T_pM$ we have $F_p(y) = \sqrt{h_p(y, y)}$.*

Note that there exist examples of locally (and even globally) symmetric Finsler metrics such that the flag curvature changes the sign, or is nonpositive, or is nonnegative. Actually, the (reversible) Minkowski space is already an example of globally symmetric Finsler space of nonpositive and of nonnegative flag curvature, since its flag curvature is zero. One can also take the direct product of a Minkowski space with the round sphere and/or with the hyperbolic space constructing examples such that the flag curvature is nonpositive and is negative somewhere, or such that the flag curvature is nonnegative and is positive somewhere, or such that it changes the sign. The holonomy group of all these examples is reducible, but one can also construct irreducible examples by perturbing a locally symmetric Riemannian metric of rank ≥ 2 by an arbitrary (smooth, small and homogeneous of degree 1) function of the Chevalley's polynomials. Since the Chevalley's polynomials are preserved by any isometry of the initial Riemannian metric (see, e.g., [9,3]), the obtained Finsler metric is still locally symmetric.

Special cases of [Theorem 1](#) are the following two statements, which were known before in the “global” setting; the initial proof of these statements is also “global”, i.e., it requires the assumption that the manifold M is compact, and is very different from our proof, which is local.

Special Case. (Corollary 1 of Foulon [6]; also follows from theorem (A) of Foulon [5].) *Let (M, F) be a Finsler locally symmetric space of negative flag curvature. If M is compact, then F is actually a Riemannian metric.*

Special Case. (Independently Deng and Hou [4] and Kim [7].) *Let (M, F) be a Finsler locally symmetric space of positive flag curvature. If M is compact, then F is actually a Riemannian metric.*

Remark. Actually, Foulon in [5] has a less restrictive definition of locally symmetric spaces (locally symmetric spaces in the our definition are also locally symmetric in the definition of [5] but not vice versa), so his result is in fact stronger and we cannot repeat it by our methods or prove its local version.

Proof of Theorem 1. Our proof of [Theorem 1](#) is based on the following recent result:

Fact. ([8], Remark (A) in Section 8 + Theorem 9.2.) *Let (M, F) be a C^2 -smooth Finsler manifold. If (M, F) is locally symmetric, then F is Berwald. Moreover, the associated connection is the Levi–Civita connection of a locally symmetric Riemannian metric. (We denote this Riemannian metric by h , in paper [8] it is called the Binet–Legendre metric associated with the Finsler metric.)*

Recall that a Finsler manifold is *Berwald*, if there exists a torsion free linear connection ∇ called *associated connection* such that the parallel transport preserves the Finsler metric.

It is well known (see for example [1,2]) that the flag curvature $K_p(y, V)$ for the Berwald metrics can be calculated by the following procedure. Compute the curvature tensor $R^J_{ik\ell}$ of the associated connection; for every p we view the curvature tensor as the mapping

$$R : T_pM \times T_pM \times T_pM \rightarrow T_pM, R(a, b)c = c^i R^J_{ik\ell} a^k b^\ell. \quad (1)$$

Then, for every two linearly independent vectors $y^i, V^i \in T_pM$ we have:

$$K_p(y, V) = \frac{g_y(V, R(V, y)y)}{g_y(y, y)g_y(V, V) - g_y(V, y)^2}. \quad (2)$$

Here g_y is the second differential of the restriction of the function $\frac{1}{2}F^2$ to T_pM ; $g_y = g_{ij}$ is a $(0, 2)$ -tensor whose components depend on the point p in M and on the tangent vector $y \in T_pM$.

Let us now consider the Riemannian metric h whose existence we recalled in fact above: it is locally symmetric and its Levi–Civita connection is the associated connection of F . Let us now show that if the flag curvature of F is positive (for all linearly independent y and $V \in T_pM$) then the sectional curvature of h is also positive, and if the flag curvature of F is negative, then the sectional curvature of h is also negative.

In order to do this, for each vector $y \in T_xM$, let us consider the endomorphism:

$$A_y : T_pM \rightarrow T_pM, \quad V \mapsto R(V, y)y.$$

In the tensor notation, $(A_y)^j_k = R^j_{ik\ell} y^i y^\ell$. Since the bilinear form $(\xi, \nu) \mapsto h(\xi, R(\nu, y)y)$ is symmetric with respect to ξ and ν because of the symmetries of the curvature tensor, for each y the endomorphism A_y is diagonalizable. Clearly, y is an eigenvector of A_y with eigenvalue 0. Comparing the formula for A_y with (2), we see that the flag curvature is given by

$$K_p(y, V) = \frac{g_y(V, A_y(V))}{g_y(y, y)g_y(V, V) - g_y(V, y)^2}.$$

If $K_p(y, V)$ is positive for all linearly independent y and V , then for each $y \neq 0$, all eigenvalues of A_y except of the eigenvalue 0 corresponding to the eigenvector y are positive, otherwise the pair (y, V) , where V is an eigenvector with nonpositive eigenvalue has $K_p(y, V) \leq 0$. Then, the sectional curvature of h (which is given by $\frac{h(V, A_y(V))}{h(y, y)h(V, V) - h(V, y)^2}$) is also positive for all linearly independent y and V as we claim.

The case when $K_p(y, V)$ is negative for all linearly independent y and V is virtually the same—one needs to replace “positive” by “negative” in the previous arguments.

Finally, if the flag curvature is positive, or if it is negative, the sectional curvature of h is positive resp. negative, i.e., is never equal to zero. But then the holonomy group of h is transitive by [10, Theorem 9]. Since both the Finsler function F and the metric h are preserved by the holonomy group, the function $\frac{F_p(y)^2}{h_p(y, y)}$ does not depend on y , so the metric F is a Riemannian metric as we claimed. Theorem 1 is proved.

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