C. R. Acad. Sci. Paris, Ser. I 353 (2015) 117–119

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Complex analysis/Analytic geometry

Additivity of the approximation functional of currents induced by Bergman kernels

Additivité de la fonctionnelle d'approximation des courants induite par les noyaux de Bergman

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ARTICLE INFO

Article history:
Received 31 October 2014
Accepted 5 November 2014
Available online 26 November 2014

Presented by Jean-Pierre Demailly

ABSTRACT

In this note, we give a positive answer to a question raised by Jean-Pierre Demailly in 2013, and show the additivity of the approximation functional of closed positive $(1,1)$-currents induced by Bergman kernels.

RÉSUMÉ

Dans cette note, nous apportons une réponse positive à une question soulevée par Jean-Pierre Demailly en 2013, et démontrons l'additivité de la fonctionnelle d'approximation des courants positifs fermés de type $(1,1)$ induite par les noyaux de Bergman.

1. Introduction

Let X be a compact complex n-dimensional manifold. An important positive cone in complex analytic geometry is the pseudoeffective classes $\mathcal{E}(X)$, namely the subset of cohomology classes $H^{1,1}(X)$ containing a closed positive $(1,1)$-current $T = \alpha + dd^c\varphi$, where α is a smooth $(1,1)$-form and φ is a quasi-psh function on X. In various geometric problems (for example, the Nadel vanishing theorem), we need to keep the information on the singularities. To preserve the information about the asymptotic multiplier ideal sheaves $\mathcal{I}(m\varphi)$, Demailly constructed a new cone by using in an essential way a Bergman kernel approximation. Before explaining this new construction, we first recall some elementary notions about quasi-psh functions.

Definition 1.1. Let φ_1, φ_2 be two quasi-psh functions on X.

(1) We say that φ_1 has analytic singularities if locally one can write it as:

$$\varphi_1 = c \ln \sum_i |g_i|^2 + O(1)$$

where the g_i are holomorphic functions and c is a positive constant.

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http://dx.doi.org/10.1016/j.crma.2014.11.004

(2) We say that φ_1 has less singularities than φ_2, and write as $\varphi_1 \preccurlyeq \varphi_2$, if we have $\varphi_1 \geq \varphi_2 + C$ for some constant C.
(3) We say that φ_1 and φ_2 have equivalent singularities, and write $\varphi_1 \sim \varphi_2$, when we have both $\varphi_1 \preccurlyeq \varphi_2$ and $\varphi_2 \preccurlyeq \varphi_1$.

We now recall briefly the constructions in [2, Section 3] and recommend the reader to see [2] for its applications. Let $\mathcal{S}(X)$ be the set of singularity equivalence classes of closed positive $(1,1)$-currents. It is naturally equipped with a cone structure. Continuing his work [1] of the early 1990's on the approximation theorem, Demailly recently defined in [2] another cone that has a more algebraic appearance.

Definition 1.2. For each class $\alpha \in \mathcal{E}(X)$, we define $\widehat{\mathcal{S}}_\alpha(X)$ as a set of equivalence classes of sequences of quasi-positive currents $T_k = \alpha + dd^c\psi_k$ (we suppose from now on that α is a smooth $(1,1)$-form on X) such that:
(a) $T_k = \alpha + dd^c\varphi_k \geq -\epsilon_k\omega$ with $\lim_{k\to+\infty}\epsilon_k = 0$.
(b) The functions φ_k have analytic singularities and $\varphi_k \preccurlyeq \varphi_{k+1}$ for all k. We say that $(T_k) \preccurlyeq_W (T'_k)$ if, for every $\epsilon > 0$ and k, there exists l such that $(1-\epsilon)T_k \preccurlyeq T'_l$. Finally, we write $(T_k) \sim (T'_k)$ when we have both $(T_k) \preccurlyeq_W (T'_k)$ and $(T'_k) \preccurlyeq_W (T_k)$, and define $\widehat{\mathcal{S}}_\alpha(X)$ to be the quotient space by this equivalence relation.
(c) We set $\widehat{\mathcal{S}}(X) := \bigcup_{\alpha\in\mathcal{E}(X)} \widehat{\mathcal{S}}_\alpha(X)$.

Let φ be a quasi-psh function on X, and (φ_k) be a Bergman kernel type approximation of φ, i.e., $\varphi_k \sim \frac{1}{k}\ln(\sum_j |g_j|^2)$ on U_i, where $\{U_i\}$ is a Stein cover of X and $\{g_j\}$ is an orthonormal basis of $H^0(V_i, \mathcal{O}_{V_i})$ for some Stein open set $V_i \Supset U_i$ with respect to the L^2 norm $\int_{V_i} |\cdot|^2 e^{-2k\varphi}$. Let $\alpha + dd^c\varphi$ be a positive current. By using the comparison theorem (cf. for example [4, Theorem 2.2.1, step 3]), Demailly [2, (3.1.10)] proved that the following map is well defined:

$$\mathbf{B} : \mathcal{S}(X) \to \widehat{\mathcal{S}}(X).$$

$$\alpha + dd^c\varphi \to (\alpha + dd^c\varphi_{2^k}).$$

It is called here the Bergman kernel approximation functional.

Remark 1.3. Although we will not use it here, we should mention the following important property of the map $\mathbf{B}$ (cf. [4, Thm. 2.2.1] or [2, Cor. 1.12]): for every pair of positive numbers $\lambda' > \lambda > 0$, there exists an integer $k_0(\lambda,\lambda') \in \mathbb{N}$ such that

$$\mathcal{I}(\lambda'\varphi_{2^k}) \subset \mathcal{I}(\lambda\varphi) \quad \text{for } k \geq k_0(\lambda,\lambda').$$

Evidently, both $\mathcal{S}(X)$ and $\widehat{\mathcal{S}}(X)$ admit an additive structure. [2, Section 3] asked whether $\mathbf{B}$ is a morphism for addition. In this short note, we will give a positive answer to this question. More precisely, we have the following theorem.

Theorem 1.4 *(Main theorem). Let $T_1 = \alpha_1 + dd^c\varphi$, $T_2 = \alpha_2 + dd^c\psi$ be two elements in $\mathcal{S}(X)$. The we have $\mathbf{B}(T_1 + T_2) = \mathbf{B}(T_1) + \mathbf{B}(T_2)$.*

2. Proof of the Main theorem

Proof. In the setting of Theorem 1.4, let τ_k (respectively φ_k, ψ_k) be a Bergman kernel type approximation of $\varphi + \psi$ (respectively φ, ψ). By the subadditive property of ideal sheaves $\mathcal{I}(k\varphi + k\psi) \subset \mathcal{I}(k\varphi)\mathcal{I}(k\psi)$ [3, Thm. 2.6], we have $\varphi_k + \psi_k \preccurlyeq \tau_k$. By Definition 1.2, to prove our main theorem, it is sufficient to prove that for every $k \in \mathbb{N}$ fixed, there exists a positive sequence $\lim_{p\to+\infty}\epsilon_p = 0$, such that

$$(1-\epsilon_p)\tau_k \preccurlyeq \varphi_p + \psi_p \quad \text{for every } p \gg 1. \tag{1}$$

For every $k \in \mathbb{N}$ fixed, there exists a bimeromorphic map $\pi : \widetilde{X} \to X$, such that

$$\tau_k \circ \pi = \sum_i c_i \ln|s_i| + C^\infty \quad \text{for some } c_i > 0 \tag{2}$$

and the effective divisor $\sum_i \mathrm{div}(s_i)$ is normal crossing. By the construction of τ_k, we have $\tau_k \preccurlyeq (\varphi + \psi)$. Therefore

$$\tau_k \circ \pi \preccurlyeq (\varphi + \psi) \circ \pi. \tag{3}$$

Applying Siu's decomposition of closed positive current theorem to $dd^c(\varphi \circ \pi)$, $dd^c(\psi \circ \pi)$ respectively, (3) and (2) imply the existence of numbers $a_i, b_i \geq 0$ satisfying:
(i) $a_i + b_i = c_i$ for every i.
(ii) $\sum_i a_i \ln|s_i| \preccurlyeq \varphi \circ \pi$ and $\sum_i b_i \ln|s_i| \preccurlyeq \psi \circ \pi$.
Let $p \in \mathbb{N}$ be an arbitrary integer, J be the Jacobian of π, $x \in X$, $f \in \mathcal{I}(p\varphi)$ and $g \in \mathcal{I}(p\psi)$. (ii) implies that

$$\int_{\pi^{-1}(U_x)} \frac{|f\circ\pi|^2|J|^2}{\prod_i |s_i|^{2pa_i}} < +\infty \quad \text{and} \quad \int_{\pi^{-1}(U_x)} \frac{|g\circ\pi|^2|J|^2}{\prod_i |s_i|^{2pb_i}} < +\infty \tag{4}$$

for some small open neighborhood U_x of x. Since $\sum_i \mathrm{div}(s_i)$ is normal crossing, (4) implies that

$$\sum_i (pa_i - 1)\ln|s_i| \preccurlyeq \ln\big(|f\circ\pi|\big) + \ln|J| \quad \text{and} \quad \sum_i (pb_i - 1)\ln|s_i| \preccurlyeq \ln\big(|g\circ\pi|\big) + \ln|J|.$$

Combining this with (i), we have

$$\sum_i (pc_i - 2)\ln|s_i| \preccurlyeq \ln\big(\big|(f\cdot g)\circ\pi\big|\big) + 2\ln|J|. \tag{5}$$

Note that J is independent of p, and $c_i > 0$. (5) implies thus that, when $p \to +\infty$, we can find a sequence $\epsilon_p \to 0^+$, such that

$$\sum_i pc_i(1-\epsilon_p)\ln|s_i| \preccurlyeq \ln\big|(f\cdot g)\circ\pi\big|. \tag{6}$$

Since f (respectively g) is an arbitrary element in $\mathcal{I}(p\varphi)$ (respectively $\mathcal{I}(p\psi)$), by the constructions of φ_p and ψ_p, (6) implies that

$$\sum_i c_i(1-\epsilon_p)\ln|s_i| \preccurlyeq (\varphi_p + \psi_p)\circ\pi\,.$$

Combining this with the fact that $(1-\epsilon_p)\tau_k\circ\pi \sim \sum_i c_i(1-\epsilon_p)\ln|s_i|$, we get

$$(1-\epsilon_p)\tau_k\circ\pi \preccurlyeq (\varphi_p + \psi_p)\circ\pi\,.$$

Therefore $(1-\epsilon_p)\tau_k \preccurlyeq (\varphi_p + \psi_p)$ and (1) is proved. $\square$

Acknowledgements

I would like to thank Professor S. Boucksom for calling my attention to this problem and for further helpful discussions. I would also like to thank Professor J.-P. Demailly for helpful remarks and encouragement.

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