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Complex analysis

The weighted log canonical thresholds of toric plurisubharmonic functions

Seuils log canoniques pondérés des fonctions plurisousharmoniques toriques

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ABSTRACT

In this article, we compute the weighted log canonical thresholds of toric plurisubharmonic functions, i.e. convex increasing functions of the logarithms of the absolute values of their complex arguments.

RÉSUMÉ

Dans cet article, nous calculons les seuils log canoniques pondérés des fonctions plurisousharmoniques toriques, c'est-à-dire s'exprimant comme des fonctions convexes croissantes des logarithmes des modules de leurs arguments complexes.

1. Introduction and main results

Let Ω be a domain in $\mathbb{C}^n$ and φ in the set $\mathrm{PSH}(\Omega)$ of plurisubharmonic functions on Ω. Following Demailly and Kollár [6], we introduce the log canonical threshold of φ at a point $z_0 \in \Omega$:

$$c_\varphi(z_0) = \sup\left\{c > 0 : \ \mathrm{e}^{-2c\varphi} \text{ is } L^1(\mathrm{d}V_{2n}) \text{ on a neighborhood of } z_0\right\} \in (0, +\infty],$$

where $\mathrm{d}V_{2n}$ is the Lebesgue measure in $\mathbb{C}^n$. It is an invariant of the singularity of φ at z_0. We refer to [1,3–7,9,10,8,13] for further information about this number. For every non-negative Radon measure μ on Ω, we introduce the *weighted log canonical threshold* of φ with weight μ at z_0:

$$c_{\varphi,\mu}(z_0) = \sup\left\{c > 0 : \ \mathrm{e}^{-2c\varphi} \text{ is } L^1(\mathrm{d}\mu) \text{ on a neighborhood of } z_0\right\} \in [0, +\infty].$$

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For every φ in the set $\mathrm{PSH}^-(\Delta^n)$ of negative plurisubharmonic functions on the polydisc Δ^n, we consider Kiselman's refined Lelong numbers of φ at 0 (see [2,12]):

$$\nu_\varphi(x) = \lim_{t\to-\infty} \frac{\max\{\varphi(z) : |z_1| = e^{tx_1}, \ldots, |z_n| = e^{tx_n}\}}{t}.$$

This function is increasing in each variable x_j and concave on $\mathbb{R}^n_+ = [0, +\infty)^n$. We set

$$\bar{\varphi}(z) = -\nu_\varphi\big(-\ln|z_1|, ..., -\ln|z_n|\big).$$

We have $\varphi \le \bar{\varphi}$ and $\bar{\varphi}$ is a function in the set $\mathrm{TPSH}^-(\Delta^n)$ of toric negative plurisubharmonic functions on Δ^n, it mean that $\bar{\varphi}(z) = \bar{\varphi}(|z_1|, ..., |z_n|)$ depends only on $|z_1|, ..., |z_n|$.

For each function $f(z) = a_{\alpha^1} z^{\alpha^1} + a_{\alpha^2} z^{\alpha^2} + ...$ (with $a_{\alpha^k} \ne 0$) in the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of holomorphic functions at 0, we define $\mathcal{I}_f$ to be the ideal generated by $\{z^{\alpha^1}, z^{\alpha^2}, ...\}$. From Noetherian property of the ring $\mathcal{O}_{\mathbb{C}^n,0}$, $\mathcal{I}_f$ is generated by finite elements $\{z^{\alpha^1}, z^{\alpha^2}, ..., z^{\alpha^m}\}$. The main result is contained in the following theorem, which is a generalization of Theorem 5.8 in [12] (see also [11] for similar results in an algebraic context).

Main theorem. *Let $\varphi \in \mathrm{TPSH}^-(\Delta^n)$ and a non-negative Radon measure μ on Δ^n. Assume that $\mu(\Delta_{r_1} \times ... \times \Delta_{r_n}) = O(1) \times \sum_{k=1}^m r_1^{2s_{k1}} ... r_n^{2s_{kn}}$ ($s_{k1}, ..., s_{kn} > 0,\ \forall 1 \le k \le m$) for all $r_1, ..., r_n > 0$, where $O(1)$ is a positive constant and Δ_r is the disc of center 0 and radius r. Then*

$$c_{\varphi,\mu}(0) = \left(\max\left\{\nu_\varphi(x) \,:\, x \in \mathbb{R}^n_+,\ \exists k = 1, ..., m, \sum_{j=1}^n s_{kj} x_j = 1\right\}\right)^{-1}.$$

Corollary. *Let $\varphi \in \mathrm{TPSH}^-(\Delta^n)$ and $f \in \mathcal{O}_{\mathbb{C}^n,0}$. Assume that $\mathcal{I}_f$ is generated by $\{z^{\alpha^1}, z^{\alpha^2}, ..., z^{\alpha^m}\}$ with $\alpha^k = (\alpha^k_1, ..., \alpha^k_n)$. Then for all $p > 0$ we have:*

$$c_{\varphi,|f|^{2p}dV_{2n}}(0) = \left(\max\left\{\nu_\varphi(x) \,:\, x \in \mathbb{R}^n_+, \exists k = 1, ..., m, \sum_{j=1}^n (p\alpha^k_j + 1) x_j = 1\right\}\right)^{-1}.$$

2. Proof of the main theorem

First, we need the following lemmas.

Lemma 2.1. *i) Let $\varphi \in \mathrm{TPSH}^-(\Delta^n)$. Then for all $\epsilon > 0$, there exists $\delta > 0$ and $C < 0$ such that*

$$\varphi(z) \ge \bar{\varphi}(z) + \epsilon\left(\sum_{j=1}^n \ln|z_j|\right) + C, \quad \forall z \in \Delta^n_\delta.$$

ii) Let $\varphi \in \mathrm{TPSH}^-(\Delta^n)$ and a non-negative Radon measure μ on Δ^n. Assume that $c_{\ln|z_j|,\mu}(0) > 0$ for all $1 \le j \le n$. Then

$$c_{\varphi,\mu}(0) = c_{\bar{\varphi},\mu}(0).$$

Proof. i) Take $0 < \epsilon_1 < \epsilon$. Since $\lim_{r\to0} \frac{\varphi(r,...,r)}{\ln r} = \nu_\varphi(1, ..., 1) = e_1(\varphi)$, we can find $\delta > 0$ such that

$$\varphi(r, ..., r) \ge \big(e_1(\varphi) + \epsilon_1\big)\ln r, \quad \forall r \in (0, \delta).$$

It follows that

$$\varphi(z) \ge \varphi\Big(\min_{1\le j\le n}|z_j|, ..., \min_{1\le j\le n}|z_j|\Big) \ge \big(e_1(\varphi) + \epsilon_1\big)\ln\Big(\min_{1\le j\le n}|z_j|\Big) \ge \big(e_1(\varphi) + \epsilon_1\big)\sum_{j=1}^n \ln|z_j|, \quad \forall z \in \Delta^n_\delta. \tag{1}$$

Set $\Sigma = \{x \in \mathbb{R}^n_+ : \sum_{j=1}^n x_j = 1\}$. Since $\lim_{t\to-\infty} \frac{\varphi(e^{tx_1},...,e^{tx_n})}{t} = \nu_\varphi(x)$, for each $x \in \Sigma$, we can find $t(x) < 0$ such that

$$\varphi\big(e^{tx_1}, ..., e^{tx_n}\big) \ge \big[\nu_\varphi(x) + \epsilon_1\big]t, \quad \forall t \le t(x).$$

Set $C(x) = \varphi(e^{t(x)x_1}, ..., e^{t(x)x_n})$. We have:

$$\varphi\big(e^{tx_1}, ..., e^{tx_n}\big) \ge \big[\nu_\varphi(x) + \epsilon_1\big]t + C(x), \quad \forall t \le 0. \tag{2}$$

As arguments in the proof of Lemma 5.7 in [12], we can choose finitely many points $x^1, \dots, x^m \in \Sigma$ such that, for every $x \in \Sigma$, there exist x^k $(1 \le k \le m)$ and $y \in \Sigma$ such that $x^k = (1-\epsilon_1)x + \epsilon_1 y$ and $|\nu_\varphi(x) - \nu_\varphi(x^k)| < \epsilon_1$. By the log-convexity of φ we get:

$$\varphi\left(e^{tx_1^k}, \dots, e^{tx_n^k}\right) \le (1-\epsilon_1)\varphi\left(e^{tx_1}, \dots, e^{tx_n}\right) + \epsilon_1 \varphi\left(e^{ty_1}, \dots, e^{ty_n}\right) \le (1-\epsilon_1)\varphi\left(e^{tx_1}, \dots, e^{tx_n}\right), \quad \forall t \le 0 \tag{3}$$

By (2) and (3) we get

$$(1-\epsilon_1)\varphi\left(e^{tx_1}, \dots, e^{tx_n}\right) \ge \left[\nu_\varphi\left(x^k\right) + \epsilon_1\right]t + C\left(x^k\right) \ge \left[\nu_\varphi(x) + 2\epsilon_1\right]t + C', \quad \forall t \le 0$$

where $C' = \min_{1\le k\le m} C(x^k)$. Hence

$$\varphi\left(e^{tx_1}, \dots, e^{tx_n}\right) \ge \frac{-\nu_\varphi(-tx)}{1-\epsilon_1} + 4\epsilon_1 t + 2C' = \frac{\bar{\varphi}(e^{tx_1}, \dots, e^{tx_n})}{1-\epsilon_1} + 4\epsilon_1 t + 2C', \quad \forall x \in \Sigma,\ t \le 0,$$

and we get:

$$\varphi(z) \ge \frac{\bar{\varphi}(z)}{1-\epsilon_1} + 4\epsilon_1 \sum_{j=1}^n \ln|z_j| + 2C', \quad \forall z \in \Delta^n. \tag{4}$$

Thanks to (1) and (4), we obtain:

$$\varphi(z) \ge \bar{\varphi}(z) + \epsilon \sum_{j=1}^n \ln|z_j| + C, \quad \forall z \in \Delta_\delta^n.$$

ii) By the Hölder inequality, we have $\frac{1}{c_{\varphi+\psi,\mu}(0)} \le \frac{1}{c_{\varphi,\mu}(0)} + \frac{1}{c_{\psi,\mu}(0)}$ for all $\varphi, \psi \in PSH^-(\Delta^n)$. Moreover, by part i) of Lemma 2.1, we get

$$\frac{1}{c_{\bar{\varphi},\mu}(0)} \le \frac{1}{c_{\varphi,\mu}(0)} \le \frac{1}{c_{\bar{\varphi}+\epsilon(\sum_{j=1}^n \ln|z_j|),\mu}(0)} \le \frac{1}{c_{\bar{\varphi},\mu}(0)} + \epsilon \sum_{j=1}^n \frac{1}{c_{\ln|z_j|,\mu}(0)}$$

for all $\epsilon > 0$. Therefore

$$c_{\varphi,\mu}(0) = c_{\bar{\varphi},\mu}(0). \qquad \square$$

Lemma 2.2. *i) Let μ be a non-negative Radon measure on Δ^n such that $\mu(\Delta_{r_1} \times \dots \times \Delta_{r_n}) \ge O(1) r_1^{2s_1} \dots r_n^{2s_n}$ $(s_1, \dots, s_n > 0)$ for all $r_1, \dots, r_n > 0$, where $O(1)$ is a positive constant. Then*

$$c_{\max(p_1 \ln|z_1|, \dots, p_n \ln|z_n|),\mu}(0) \le \sum_{j=1}^n \frac{s_j}{p_j}.$$

ii) Let μ be a non-negative Radon measure on Δ^n such that $\mu(\Delta_{r_1} \times \dots \times \Delta_{r_n}) \le O(1) \sum_{k=1}^m r_1^{2s_{k1}} \dots r_n^{2s_{kn}}$ $(s_{k1}, \dots, s_{kn} > 0,\ \forall 1 \le k \le m)$ for all $r_1, \dots, r_n > 0$, where $O(1)$ is a positive constant. Then

$$c_{\max\{s_{k1}\ln|z_1| + \dots + s_{kn}\ln|z_n|:\ 1\le k\le m\},\mu}(0) \ge 1.$$

Proof. i) We have:

$$\mu\left(\left\{z \in \Delta^n :\ \max\left(p_1 \ln|z_1|, \dots, p_n \ln|z_n|\right) < \ln r\right\}\right) = \mu(\Delta_{r^{\frac{1}{p_1}}} \times \dots \times \Delta_{r^{\frac{1}{p_n}}}) \ge O(1) r^{2\sum_{j=1}^n \frac{s_j}{p_j}}.$$

This implies that $c_{\max(p_1 \ln|z_1|, \dots, p_n \ln|z_n|),\mu}(0) \le \sum_{j=1}^n \frac{s_j}{p_j}$.

ii) To simplify the notation, we assume that $n = 2$. We choose $0 < t_{k1} < s_{k1}$, $0 < t_{k2} < s_{k2}$ such that $\frac{s_{k1}}{t_{k1}} < \frac{s_{k2}}{t_{k2}}$, $\forall 1 \le k \le m$. Take $0 < \delta < 1$. We have:

$$\begin{aligned}&\left\{\max\{t_{k1}\ln|z_1| + t_{k2}\ln|z_2| :\ 1 \le k \le m\} < \ln r\right\} \\ &= \bigcap_{k=1}^m \left\{|z_1|^{t_{k1}}|z_2|^{t_{k2}} < r\right\} \subset \bigcup_{l=1}^\infty \left(\bigcap_{k=1}^m \left\{|z_1| < \left(\frac{r}{\delta^{(l+1)t_{k2}}}\right)^{\frac{1}{t_{k1}}}\right\}\right) \times \left\{\delta^{l+1} < |z_2| < \delta^l\right\}.\end{aligned}$$

Hence

$$\mu\Big(\big\{\max\{t_{k1}\ln|z_1|+t_{k2}\ln|z_2|:\ 1\le k\le m\}<\ln r\big\}\Big)$$
$$\le\sum_{l=1}^{\infty}\mu\left(\bigcap_{k=1}^{m}\left\{|z_1|<\left(\frac{r}{\delta^{(l+1)t_{k2}}}\right)^{\frac{1}{t_{k1}}}\right\}\times\left\{\delta^{l+1}<|z_2|<\delta^{l}\right\}\right)$$
$$\le O(1)\sum_{k=1}^{m}r^{\frac{s_{k1}}{t_{k1}}}\sum_{l=1}^{\infty}\delta^{l(s_{k2}-\frac{(l+1)s_{k1}t_{k2}}{lt_{k1}})}$$
$$=O(1)\sum_{k=1}^{m}r^{\frac{s_{k1}}{t_{k1}}}.$$

This implies that $e^{-2(\max\{t_{k1}\ln|z_1|+t_{k2}\ln|z_2|:\ 1\le k\le m\})}$ is $L^1(d\mu)$ on a neighborhood of 0. Letting $t_{k1}\to s_{k1}$ and $t_{k2}\to s_{k2}$, we obtain:

$$c_{\max\{s_{k1}\ln|z_1|+s_{k2}\ln|z_2|:1\le k\le m\},\mu}(0)\ge\lim_{t_{k1}\to s_{k1},t_{k2}\to s_{k2}}\min\left(\frac{t_{k1}}{s_{k1}},\frac{t_{k2}}{s_{k2}}\right)c_{\max\{t_{k1}\ln|z_1|+t_{k2}\ln|z_2|:\ 1\le k\le m\},\mu}(0)\ge1.$$

For each function h on Δ^n we set

$$h_{\rm sym}(z)=\frac{1}{(2\pi)^n|z_1|\dots|z_n|}\int_{\{|\xi_1|=|z_1|\}}\dots\int_{\{|\xi_n|=|z_n|\}}h(\xi_1,\dots,\xi_n)d\xi_1\dots d\xi_n,$$

$$h_{\rm int}(z)=\frac{1}{\pi^n|z_1|^2\dots|z_n|^2}\int_{\{|\xi_1|<|z_1|\}}\dots\int_{\{|\xi_n|<|z_n|\}}h(\xi_1,\dots,\xi_n)dV_{2n}(\xi),$$

$$h_{\max}(z)=\max\big\{h(\xi_1,\dots,\xi_n):\ |\xi_1|=|z_1|,\dots,|\xi_n|=|z_n|\big\}.$$

If $h\in PSH(\Delta^n)$ and $h\ge0$, for each $a>1$ from Poisson inequality we have

$$h_{\rm int}(z)\le h_{\rm sym}(z)\le h_{\max}(z)\le\left(\frac{a+1}{a-1}\right)^n h_{\rm sym}(az),\quad\forall z\in\Delta^n.\qquad\square$$

Lemma 2.3. *Let $f\in\mathcal{O}_{\mathbb{C}^n,0}$. Assume that $\mathcal{I}_f$ is generated by $\{z^{\alpha^1},z^{\alpha^2},\dots,z^{\alpha^m}\}$ with $\alpha^k=(\alpha_1^k,\dots,\alpha_n^k)$. Then there exist $C_1,C_2>0$ such that*

$$C_1\max\big\{|z_1|^{\alpha_1^k}\dots|z_n|^{\alpha_n^k}:1\le k\le m\big\}$$
$$\le\big(|f|^{2p}\big)_{\rm int}(z)\le\big(|f|^{2p}\big)_{\rm sym}(z)\le\big(|f|^{2p}\big)_{\max}(z)\le C_2\max\big\{|z_1|^{\alpha_1^k}\dots|z_n|^{\alpha_n^k}:1\le k\le m\big\}.$$

Proof of the main theorem. Set

$$d=\max\left\{\nu_\varphi(x):x\in\mathbb{R}_+^n,\exists k=1,\dots,m,\sum_{j=1}^n s_{kj}x_j=1\right\}.$$

Since $\nu_\varphi(tx)=t\nu_\varphi(x)$, $\forall x\in\mathbb{R}_+^n, t\ge0$, we have:

$$\nu_\varphi(x)\le d\sum_{j=1}^n s_{kj}x_j,\quad\forall x\in\mathbb{R}_+^n,\ \forall1\le k\le m.$$

Hence

$$\bar\varphi(z)=-\nu_\varphi\big(-\ln|z_1|,\dots,-\ln|z_n|\big)\ge d\max\left\{\sum_{j=1}^n s_{kj}\ln|z_j|:\ 1\le k\le m\right\}.$$

By Lemma 2.1 and Lemma 2.2, we get:

$$c_{\varphi,\mu}(0)=c_{\bar\varphi,\mu}(0)\ge c_{d\max\{\sum_{j=1}^n s_{kj}\ln|z_j|:\ 1\le k\le m\},\mu}(0)\ge\frac{1}{d}.$$

We choose $y\in\mathbb{R}_+^n$ with $\sum_{j=1}^n s_{kj}y_j=1$ for some $1\le k\le m$ and $\nu_\varphi(y)=d$. For each $z\in\Delta^n$, we choose $t=\max(\frac{\ln|z_1|}{y_1},\dots,\frac{\ln|z_n|}{y_n})$. We have:

$$\bar\varphi(z)\le\bar\varphi\big(e^{ty_1},\dots,e^{ty_n}\big)=-\nu_\varphi(-ty_1,\dots,-ty_n)=dt=d\max\left(\frac{\ln|z_1|}{y_1},\dots,\frac{\ln|z_n|}{y_n}\right).$$

By Lemma 2.1 and Lemma 2.2, we get

$$c_{\varphi,\mu}(0) = c_{\bar{\varphi},\mu}(0) \le c_{d\max(\frac{\ln|z_1|}{y_1},\dots,\frac{\ln|z_n|}{y_n}),\mu}(0) = \frac{\sum_{j=1}^{n} s_{kj}y_j}{d} = \frac{1}{d}. \qquad \square$$

Proof of the corollary. By Lemma 2.3, we have:

$$\begin{aligned}\int\limits_{\Delta_{r_1}\times\ldots\times\Delta_{r_n}} |f|^{2p}\mathrm{d}V_{2n} &= \int\limits_{\Delta_{r_1}\times\ldots\times\Delta_{r_n}} \left(|f|^{2p}\right)_{\mathrm{sym}}\mathrm{d}V_{2n} \\ &= O(1)\int\limits_{\Delta_{r_1}\times\ldots\times\Delta_{r_n}} \sum_{k=1}^{m} |z_1|^{2p\alpha_1^k}\ldots|z_n|^{2p\alpha_n^k}\mathrm{d}V_{2n} \\ &= O(1)\sum_{k=1}^{m} r_1^{2p\alpha_1^k+2}\ldots r_n^{2p\alpha_n^k+2}.\end{aligned}$$

Thanks to the Main Theorem, we get:

$$c_{\varphi,|f|^{2p}\mathrm{d}V_{2n}}(0) = \left(\max\left\{\nu_\varphi(x) : x\in\mathbb{R}_+^n,\ \exists k=1,\dots,m,\ \sum_{j=1}^{n}(p\alpha_j^k+1)x_j = 1\right\}\right)^{-1} \qquad \square$$

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