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2-Jordan blocks for the eigenvalue $\lambda = 1$ of Yomdin–Lê surface singularities ☆

2-*Blocs de Jordan pour la valeur propre* $\lambda = 1$ *des singularités de surface de type Yomdin–Lê*

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ABSTRACT

The main purpose of this paper is to explicitly calculate the Jordan blocks of size 2 for the eigenvalue $\lambda = 1$ of a Yomdin–Lê surface singularity, in terms of the combinatorial data of its tangent cone. Our method relies on the use of a generalization of Steenbrink's spectral sequence and a certain partial toric resolution of this family of singularities. Both the spectral sequence and the partial resolution have already been developed by the author in previous works.

RÉSUMÉ

L'objet principal de cet article est de calculer explicitement les blocs de Jordan d'ordre 2 pour la valeur propre $\lambda = 1$ d'une singularité de surface de type Yomdin–Lê, en fonction des données combinatoires de son cône tangent. Notre méthode s'appuie sur l'utilisation d'une généralisation de la suite spectrale de Steenbrink et d'une certaine résolution torique partielle de cette famille de singularités. La suite spectrale et la résolution partielle ont déjà été développées par l'auteur dans des travaux précédents.

Version française abrégée

Soit $f = f_m + f_{m+k} + \cdots$ la décomposition de $f \in \mathbb{C}\{x, y, z\}$ en composantes homogènes, $k \geq 1$. Supposons que la courbe projective $V(f_{m+k})$ ne passe pas par les points singuliers du *cône tangent projectivisé* $\mathbf{C} := V(f_m)$, c.-à-d. $\mathrm{Sing}(\mathbf{C}) \cap V(f_{m+k}) = \emptyset$ est vérifié dans $\mathbb{P}^2$. En particulier, $\mathbf{C}$ est nécessairement réduit. Ces singularités sont appelées *Yomdin–Lê* (YLS) d'après [11, 4] et elles ont été largement étudiées par de nombreux auteurs, voir par exemple [2].

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Le cas des singularités *superisolées* ($k = 1$) fut présenté par Luengo et est apparu aussi dans un article de Stevens, où la strate μ-constante est considérée [5,10]. D'après la description de [1], les blocs de Jordan de la monodromie complexe s'expriment en fonction des données combinatoires du cône tangent projectivisé. Cependant, aucune description n'était connue pour $k \geq 2$.

L'objectif principal de cet article est de calculer de façon explicite les blocs de Jordan d'ordre 2 pour la valeur propre $\lambda = 1$ et un $k \geq 1$ quelconque, voir Théorème 0.1 plus bas. Une description complète, voir (2), pour les autres valeurs propres est annoncée aussi comme un travail futur, mais aucune preuve n'est fournie ici.

On introduit quelques notations avant d'énoncer le résultat principal. On fixe une $\mathbb{Q}$-résolution plongée de $(\mathbf{C}, P)$, l'espace final peut contenir des singularités quotient abéliennes [3]. On note par Γ_+^P le graphe dual (arbre) de la résolution contenant seulement les diviseurs exceptionnels, c'est-à-dire que la transformée stricte n'est pas incluse dans ce type d'arbre. Le graphe associé à la réduction semistable [9] est noté $N\Gamma_+^P$. L'application naturelle $\varrho^P : N\Gamma_+^P \to \Gamma_+^P$ de CW-complexes est un revêtement cyclique et sa monodromie est identifiée avec la monodromie de $(\mathbf{C}, P)$. Il existe un unique graphe $N\Gamma_+^P(k)$ et un unique revêtement cyclique $\varrho_k^P : N\Gamma_+^P(k) \to \Gamma_+^P$ tels que, $\forall a \in V(\Gamma_+^P)$ et $\forall \{a, b\} \in E(\Gamma_+^P)$, on ait les relations (1).

Théorème 0.1. *Le nombre de blocs de Jordan d'ordre 2 pour la valeur propre $\lambda = 1$ d'une singularité de surface de type Yomdin–Lê est*

$$\sum_{P \in \mathrm{Sing}(\mathbf{C})} (r_P - 1) - (r - 1) + \sum_{P \in \mathrm{Sing}(\mathbf{C})} b_1\big(N\Gamma_+^P(k)\big),$$

où r est le nombre de composantes irréductibles de $\mathbf{C} \subset \mathbb{P}^2$, r_P est le nombre de branches locales du germe $(\mathbf{C}, P)$, et b_1 est le premier nombre de Betti du graphe correspondant.

On remarque que, pour $k = 1$, le graphe $N\Gamma_+^P(k) = \Gamma_+^P$ est contractile, donc son premier nombre de Betti est nul, et on obtient la description donnée dans [1].

On esquisse les idées clés de la preuve. Soit $V := \{f = 0\} \subset \mathbb{C}^3$ définissant une YLS. D'après la réduction semistable de la $\mathbb{Q}$-résolution plongée calculée dans [7], on obtient les V-variétés compactes $D_+^{[0]}$ et $D_+^{[1]}$, définies dans (3) et calculées dans (4). La suite spectrale généralisée de Steenbrink dans [8] implique que le nombre de 2-blocs de Jordan pour $\lambda = 1$ est $1 - b_0(D_+^{[0]} \cap \widehat{V}) + b_0(D_+^{[1]} \cap \widehat{V})$, voir Section 2 et Section 3 pour plus de détails, en particulier la Proposition 3.1.

Les intersections $D_0 \cap \widehat{V}$ et $D_0 \cap D_a^P \cap \widehat{V}$ sont identifiées avec la normalisation de $\mathbf{C} \subset \mathbb{P}^2$ et $\widehat{\mathbf{C}} \cap \mathcal{E}_a^P$, respectivement. On obtient $b_0(D_0 \cap \widehat{V}) = r$ et $\sum_a b_0(D_0 \cap D_a^P \cap \widehat{V}) = r_P$. Le résultat le plus technique du présent article est le Lemme 4.1, où les nombres des composantes irréductibles de $D_a^P \cap \widehat{V}$ et $D_a^P \cap D_b^P \cap \widehat{V}$ sont calculés. En particulier, on a prouvé que

$$1 - \sum_a b_0\big(D_a^P \cap \widehat{V}\big) + \sum_{a,b} b_0\big(D_a^P \cap D_b^P \cap \widehat{V}\big)$$

est le premier nombre de Betti du graphe $N\Gamma_+^P(k)$. Ce nombre coïncide avec le degré de $\Delta^{(k)}_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t)$ et aussi avec le degré du facteur $t - 1$ dans le polynôme $\Delta^k_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t)$, voir Remarque 1. Finalement, en utilisant tous ces ingrédients, la preuve du Théorème 0.1 devient un calcul simple, voir (5).

1. Introduction

Let $f = f_m + f_{m+k} + \cdots$ be the decomposition of $f \in \mathbb{C}\{x, y, z\}$ into homogeneous parts, $k \geq 1$. Assume that the projective curve $V(f_{m+k})$ does not pass through the singular points of the *projectivized tangent cone* $\mathbf{C} := V(f_m)$, i.e. $\mathrm{Sing}(\mathbf{C}) \cap V(f_{m+k}) = \emptyset$ holds in $\mathbb{P}^2$. In particular, $\mathbf{C}$ is necessarily reduced. These singularities are called *Yomdin–Lê* (YLS) after [11,4] and they have been widely studied by many authors, see for instance the survey [2].

The *superisolated* case ($k = 1$) was introduced by Luengo and also appeared in a paper by Stevens, where the μ-constant stratum was considered [5,10]. Afterward Artal described in his PhD thesis [1] the Jordan blocks of the complex monodromy in terms of the combinatorial data of the tangent cone. However, such a description was not known for $k \geq 2$.

The main goal of this paper is to explicitly compute the Jordan blocks of size 2 for the eigenvalue $\lambda = 1$ and arbitrary $k \geq 1$, see Theorem 1.1 below. The complete description, see (2), for the other eigenvalues is announced as a future work too, but no proof is provided here. Our method consists in applying the generalized Steenbrink's spectral sequence in [8] to the embedded $\mathbb{Q}$-resolution of YLS obtained in [7].

To state precisely our main result, some notations need to be introduced. Let us fix an embedded $\mathbb{Q}$-resolution of $(\mathbf{C}, P)$, that is, the final space may contain Abelian quotient singularities [3]. Denote by Γ_+^P the dual graph (tree) of the resolution corresponding to the exceptional divisors, that is, the strict transform is not included in such a tree. The graph associated with the semistable reduction [9] is denoted by $N\Gamma_+^P$. The natural map $\varrho^P : N\Gamma_+^P \to \Gamma_+^P$ of CW-complexes is a cyclic covering and its monodromy is identified with the monodromy of $(\mathbf{C}, P)$. There is a unique graph $N\Gamma_+^P(k)$ and a unique cyclic covering $\varrho_k^P : N\Gamma_+^P(k) \to \Gamma_+^P$ such that, $\forall a \in V(\Gamma_+^P)$ and $\forall \{a, b\} \in E(\Gamma_+^P)$, one has

$$\#\big(\varrho_k^P\big)^{-1}(a) = \gcd\big(k, \#\big(\varrho^P\big)^{-1}(a)\big), \qquad \#\big(\varrho_k^P\big)^{-1}\big(\{a, b\}\big) = \gcd\big(k, \#\big(\varrho^P\big)^{-1}\big(\{a, b\}\big)\big). \tag{1}$$

Theorem 1.1. *The number of Jordan blocks of size 2 for the eigenvalue $\lambda = 1$ of a Yomdin–Lê surface singularity is*

$$\sum_{P\in\mathrm{Sing}(\mathbf{C})} (r_P - 1) - (r-1) + \sum_{P\in\mathrm{Sing}(\mathbf{C})} b_1\big(N\Gamma_+^P(k)\big),$$

where r is the number of irreducible components of $\mathbf{C}\subset\mathbb{P}^2$, r_P is the number of local branches of the germ $(\mathbf{C},P)$, and b_1 stands for the first Betti number of the corresponding graph.

Let $H := H^2(F,\mathbb{C})$ be the cohomology of the Milnor fiber. Studying the spectral sequence converging to H, one obtains a decomposition, the so-called mixed Hodge structure $H = \bigoplus_{i=0}^4 \mathrm{Gr}_i^W H$, and the monodromy acts on each of the preceding vector spaces. Denote by $\Delta_M(t)$ the characteristic polynomial of the monodromy acting on a vector space M. Then the 3-Jordan blocks for the eigenvalues $\lambda\neq 1$ are encoded in $\Delta_{\mathrm{Gr}_0^W H}(t) = \prod_{P\in\mathrm{Sing}(\mathbf{C})} \Delta^{(m)}_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t)$, and those of size 2 in

$$\Delta_{\mathrm{Gr}_1^W H}(t) = \frac{1}{\Delta_{H^1(D_0)}(t)} \prod_{P\in\mathrm{Sing}(\mathbf{C})} \frac{\widetilde{\Delta}^{(m)}_{H_{(\mathbf{C},P)}}(t)\cdot \Delta^k_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t^{m+k})}{\Delta^{(m)}_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t)^3\cdot (t-1)^{b_1(N\Gamma_+^P(k))}}, \tag{2}$$

where if $\Delta(t) = \prod_i (t^{m_i}-1)^{a_i}$ then $\Delta^\ell(t) = \prod_i (t^{\frac{m_i}{\gcd(\ell,m_i)}}-1)^{a_i \gcd(\ell,m_i)}$, $\Delta^{(\ell)}(t) = \prod_i (t^{\gcd(\ell,m_i)}-1)^{a_i}$, and $\widetilde{\Delta}^{(\ell)}$ stands for the polynomial resulting from deleting the factor $t-1$ in $\Delta^{(\ell)}$. The action of the monodromy on $H^1(D_0)$ and the cohomology itself are completely determined by the pair $(\mathbb{P}^2,\mathbf{C})$.

Note that for $k=1$ the graph $N\Gamma_+^P(k) = \Gamma_+^P$ is contractible, thus its first Betti number is zero, and one obtains the description in [1].

2. An embedded $\mathbb{Q}$-resolution of YLS

As mentioned in the introduction, the first step in our calculation is the explicit computation of an embedded $\mathbb{Q}$-resolution of this family of singularities. This was already performed in [7]; its main result is a collection of several results that can be summarized as follows.

Theorem 2.1. *Let $\varpi^P : Y^P \to (\mathbf{C}^2,P)$ be an embedded $\mathbb{Q}$-resolution of the projectivized tangent cone $(\mathbf{C},P)$ for each $P\in\mathrm{Sing}(\mathbf{C})$. Assume that $(\varpi^P)^*(\mathbf{C},P) = \widehat{\mathbf{C}} + \sum_{a\in V(\Gamma_+^P)} m_a^P \mathcal{E}_a^P$ is the total transform of $(\mathbf{C},P)$, where $\mathcal{E}_a^P$ is the exceptional divisor of the (p_a^P,q_a^P)-blow-up at a point P_a belonging to the locus of non-transversality. Denote by ν_a^P the (p_a^P,q_a^P)-multiplicity of $\mathbf{C}$ at P_a.*

Then, one can construct an embedded $\mathbb{Q}$-resolution $\pi : X \to (\mathbb{C}^3,0)$ of the Yomdin–Lê singularity $(V,0)$ such that the total transform is

$$\pi^*(V,0) = \widehat{V} + mE_0 + \sum_{P\in\mathrm{Sing}(\mathbf{C})} \sum_{a\in V(\Gamma_+^P)} \frac{(m+k)\cdot m_a^P}{\gcd(k,m_a^P)} E_a^P,$$

and E_a^P appears after the $\left(\frac{kp_a^P}{\gcd(k,\nu_a^P)}, \frac{kq_a^P}{\gcd(k,\nu_a^P)}, \frac{\nu_a^P}{\gcd(k,\nu_a^P)}\right)$-blow-up at the point P_a (the loci of non-transversality in dimensions 2 and 3 are identified).

Using this resolution and the generalized A'Campo formula of [6], the characteristic polynomial of the complex monodromy of $(V,0)$ can be obtained as

$$\Delta_{(V,0)}(t) = \frac{(t^m-1)^{\chi(\mathbb{P}^2\setminus\mathbf{C})}}{t-1} \prod_{P\in\mathrm{Sing}(\mathbf{C})} \Delta^k_{(\mathbf{C},P)}\big(t^{m+k}\big),$$

where $\Delta_{(\mathbf{C},P)}(t)$ denotes the characteristic polynomial of the local complex monodromy of $(\mathbf{C},P)$ and the notation Δ^k is explained right after (2). Recall that $\chi(\mathbb{P}^2\setminus\mathbf{C}) = m^2-3m+3-\sum_P \mu_P$.

3. Generalized Steenbrink's spectral sequence

Typically, after the characteristic polynomial, one studies the spectral sequence converging to the cohomology of the Milnor fiber $H := H^2(F,\mathbb{C})$ in order to understand the Jordan blocks for each eigenvalue. In particular, one obtains the mixed Hodge structure $H = \bigoplus_{i=0}^4 \mathrm{Gr}_i^W H$, and the monodromy acts on each one of the preceding vector spaces.

The Jordan blocks of size 2 for $\lambda = 1$ appears in the 2nd and 4th W-graded parts of H. This is a consequence of the fact that $i=3$ is the central index of the monodromy filtration for $\lambda = 1$, see [8] for a more careful explanation.

Let D be the divisor associated with the semistable reduction of an embedded $\mathbb{Q}$-resolution. Let us decompose $D = D_0 \cup D_1 \cup \cdots \cup D_s$ so that $D_0 := \widehat{V}$ corresponds to the strict transform of the singularity and the other D_i's correspond to the exceptional components. Consider the following V-manifolds:

$$D^{[k]} := \bigsqcup_{0 \le i_0 < \cdots < i_k \le s} D_{i_0} \cap \cdots \cap D_{i_k}, \qquad D_+^{[k]} := \bigsqcup_{1 \le i_0 < \cdots < i_k \le s} D_{i_0} \cap \cdots \cap D_{i_k}. \tag{3}$$

Proposition 3.1. *Let $(V, 0) \subset (\mathbb{C}^3, 0)$ be a surface defining an isolated singularity. Then the number of Jordan blocks of size 2 for the eigenvalue $\lambda = 1$ is $1 - b_0(D_+^{[0]} \cap \widehat{V}) + b_0(D_+^{[1]} \cap \widehat{V})$.*

Proof. On the one hand, the generalized Steenbrink spectral sequence of [8] gives rise the following exact sequences: $0 \longrightarrow \mathrm{Gr}_4^W H \longrightarrow H^0(D^{[2]}) \longrightarrow H^2(D^{[1]}) \longrightarrow H^4(D_+^{[0]}) \longrightarrow 0$. On the other hand, by definition, $D^{[2]} = D_+^{[2]} \sqcup (D_+^{[1]} \cap \widehat{V})$ and $D^{[1]} = D_+^{[1]} \sqcup (D_+^{[0]} \cap \widehat{V})$. The additivity of the functor H^i and the fact that Δ is multiplicative imply that:

$$\Delta_{\mathrm{Gr}_4^W H}(t) = \frac{\Delta_{H^0(D^{[2]})}(t) \cdot \Delta_{H^4(D_+^{[0]})}(t)}{\Delta_{H^2(D^{[1]})}(t)} = \frac{\Delta_{H^0(D_+^{[2]})}(t) \cdot \Delta_{H^0(D_+^{[1]} \cap \widehat{V})}(t) \cdot \Delta_{H^4(D_+^{[0]})}(t)}{\Delta_{H^2(D_+^{[1]})}(t) \cdot \Delta_{H^2(D_+^{[0]} \cap \widehat{V})}(t)}$$
$$= \frac{\Delta_{H^0(D_+^{[2]})}(t) \cdot \Delta_{H^4(D_+^{[0]})}(t)}{(t-1) \cdot \Delta_{H^2(D_+^{[1]})}(t)} \cdot \frac{(t-1) \cdot \Delta_{H^0(D_+^{[1]} \cap \widehat{V})}(t)}{\Delta_{H^2(D_+^{[0]} \cap \widehat{V})}(t)}.$$

The monodromy acts trivially on $D_+^{[0]} \cap \widehat{V}$ and $D_+^{[1]} \cap \widehat{V}$. Besides, the Poincaré duality with complex coefficient still holds for V-manifolds. Finally, the exact sequence coming again from the spectral sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0\big(D_+^{[0]}\big) \longrightarrow H^0\big(D_+^{[1]}\big) \longrightarrow H^0\big(D_+^{[2]}\big) \longrightarrow \mathrm{Gr}_0^W H \longrightarrow 0$$

tells us that the latter polynomial equals $\Delta_{\mathrm{Gr}_0^W H}(t) \cdot (t-1)^{1 - b_0(D_+^{[0]} \cap \widehat{V}) + b_0(D_+^{[1]} \cap \widehat{V})}$. The claim follows because $\mathrm{Gr}_0^W H$ only contains information about the Jordan blocks of size 3 for the eigenvalues $\lambda \neq 1$. □

4. Proof of the main result

Let $V := \{f = 0\} \subset \mathbb{C}^3$ defining a YLS. After the semistable reduction $\rho : \widehat{X} \to X$ of the embedded $\mathbb{Q}$-resolution presented in Theorem 2.1, one obtains the compact V-manifolds

$$D_+^{[0]} = D_0 \sqcup \bigsqcup_{P,a} D_a^P, \qquad D_+^{[1]} = \bigsqcup_{P,a} \big(D_0 \cap D_a^P\big) \sqcup \bigsqcup_{P,a,b} \big(D_a^P \cap D_b^P\big), \tag{4}$$

where $D_0 := \rho^{-1}(E_0)$, $D_a^P := \rho^{-1}(E_a^P)$, and $P \in \mathrm{Sing}(\mathbf{C})$, $a, b \in V(\Gamma_+^P)$ with $a \neq b$. All these divisors are irreducible because $\widehat{V}$ passes through E_0 and E_a^P as it is explained in [7].

The intersections $D_0 \cap \widehat{V}$ and $D_0 \cap D_a^P \cap \widehat{V}$ are identified with the normalization of $\mathbf{C}$ and $\widehat{\mathbf{C}} \cap \mathcal{E}_a^P$, respectively. Hence $b_0(D_0 \cap \widehat{V}) = r$, the number of irreducible components of $\mathbf{C}$, and $\sum_a b_0(D_0 \cap D_a^P \cap \widehat{V}) = r_P$, the number of local branches of $(\mathbf{C}, P)$. We plan to apply Proposition 3.1 to our particular case. Before that, a technical but essential result is needed.

Lemma 4.1. $b_0(D_a^P \cap \widehat{V}) = \gcd(k, m(\mathcal{E}_a^P)) =: \#(\varrho_k^P)^{-1}(a)$, $\forall a \in V(\Gamma_+^P)$, *and* $b_0(D_a^P \cap D_b^P \cap \widehat{V}) = \gcd(k, m(\mathcal{E}_a^P \cap \mathcal{E}_b^P)) =: \#(\varrho_k^P)^{-1}(\{a,b\})$, $\forall \{a,b\} \in E(\Gamma_+^P)$, *where $m(S)$ denotes the number of irreducible components of S, which is either a divisor or a point, in the semistable reduction, see [9, Def. 2.4].*

Proof. The proof of Theorem 2.1 relies on computations with local equations, see [7, Prop. 6.4]. In particular, it was proven that $m(\mathcal{E}_b^P)$ is the number of irreducible components of $\{t^L - x^{n_1} y^{n_2} H_\nu(x,y) = 0\}$ regarded as a subset of $\mathbb{C} \times X(\mathbf{d}; \mathbf{a}, \mathbf{b})$, where L is a multiple of all multiplicities appearing in the resolution of $(\mathbf{C}, P)$, $H_\nu(x,y) := x^\alpha y^\beta \prod_i (x^{\frac{q}{\gcd(p,q)}} + \gamma_i y^{\frac{p}{\gcd(p,q)}})^{e_i}$ is the (p,q)-homogeneous part of $\widehat{\mathbf{C}}$ at certain point, both n_1 and n_2 are multiple of k, and $X(\mathbf{d}; \mathbf{a}, \mathbf{b}) := \mathbb{C}^2 / G_{\mathbf{d}}$ is a quotient space (not necessarily cyclic), the action defined by $\xi_{\mathbf{d}} \cdot (x, y) = (\xi_{\mathbf{d}}^{\mathbf{a}} x, \xi_{\mathbf{d}}^{\mathbf{b}} y)$ using a multi-index notation.

Let $\ell := \gcd(n_1 + \alpha, n_2 + \beta, \{e_i\}_i)$ denote the number of irreducible factors of $t^L - x^{n_1} y^{n_2} H_\nu(x,y)$ as an element in $\mathbb{C}^3$. The action of $G_{\mathbf{d}}$ on one of these factors is as follows:

$$t^{\frac{L}{\ell}} - x^{\frac{n_1+\alpha}{\ell}} y^{\frac{n_2+\beta}{\ell}} \prod \big(x^{\frac{q}{\gcd(p,q)}} + \gamma_i y^{\frac{p}{\gcd(p,q)}}\big)^{\frac{e_i}{\ell}} \stackrel{\xi_{\mathbf{d}}}{\longmapsto} t^{\frac{L}{\ell}} - \xi_{\mathbf{d}}^{\frac{\mathbf{v}}{\ell}} x^{\frac{n_1+\alpha}{\ell}} y^{\frac{n_2+\beta}{\ell}} \prod \big(x^{\frac{q}{\gcd(p,q)}} + \gamma_i y^{\frac{p}{\gcd(p,q)}}\big)^{\frac{e_i}{\ell}},$$

where $\mathbf{v} := \frac{n_1+\alpha}{\ell} \cdot \mathbf{a} + \frac{n_2+\beta}{\ell} \cdot \mathbf{b} + \sum_i \frac{q}{\gcd(p,q)} \frac{e_i}{\ell} \cdot \mathbf{a} \in \mathbb{Z}^n$. Hence $m(\mathcal{E}_b) = \frac{\ell}{|H|}$, being $H = \{\xi^{\frac{\mathbf{v}}{\ell}} \mid \xi \in G_{\mathbf{d}}\}$, which is a subgroup of the cyclic group of the ℓ-th roots of unity.

On the other hand, $b_0(D_b^P \cap \widehat{V}) = b_0(E_b^P \cap \widehat{V})$ equals the number of irreducible components of $\{z^k + H_\nu(x,y) = 0\}$ in $\mathbb{P}^2_{(p,q,\frac{\nu}{k})}/G_{(\mathbf{d};\mathbf{a},\mathbf{b},\mathbf{c})}$ with $\frac{n_1}{k}\cdot\mathbf{a} + \frac{n_2}{k}\cdot\mathbf{b} + \mathbf{c} \equiv 0 \bmod \mathbf{d}$. The latter number is exactly the number of irreducible components of the same polynomial in $X(\mathbf{d};\mathbf{a},\mathbf{b},\mathbf{c}) := \mathbb{C}^3/G_{\mathbf{d}}$. Since k divides n_1 and n_2, the previous polynomial has $\gcd(k,\alpha,\beta,\{e_i\}_i) = \gcd(k,\ell)$ factors in $\mathbb{C}^3$. Working as above, $b_0(D_b^P \cap \widehat{V}) = \frac{\gcd(k,l)}{|H'|}$, where $H' := \{\xi^{\frac{\mathbf{v}}{\gcd(k,\ell)}} \mid \xi \in G_{\mathbf{d}}\} = \{\eta^{\frac{\ell}{\gcd(k,\ell)}} \mid \eta \in H\}$.

The fact that H and H' are subgroup of a cyclic group implies that $|H'| = \frac{|H|}{\gcd(|H|,\frac{\ell}{\gcd(k,\ell)})}$, which is equal to $\frac{\gcd(k,\ell)}{\gcd(\gcd(k,\ell),\frac{\ell}{|H|})}$ by a general fact. Then $b_0(D_b^P \cap \widehat{V}) = \frac{\gcd(k,\ell)}{|H'|} = \gcd(k,\ell,\frac{\ell}{|H|}) = \gcd(k, m(\mathcal{E}_b))$.

The 2nd part of the statement is a particular case of the 1st one assuming $H_\nu(x,y) = 1$. □

Now using all these ingredients, the proof of Theorem 1.1 is a simple calculation. In fact, using Lemma 4.1 and considering the decomposition of $D_+^{[0]}$ and $D_+^{[1]}$ given in (4), the number of 2-Jordan blocks for the eigenvalue $\lambda = 1$, which by Theorem 3.1 equals $1 - b_0(D_+^{[0]} \cap \widehat{V}) + b_0(D_+^{[1]} \cap \widehat{V})$, is

$$
\begin{aligned}
&1 - b_0(D_0 \cap \widehat{V}) - \sum_{P,a} b_0\big(D_a^P \cap \widehat{V}\big) + \sum_{P,a} b_0\big(D_0 \cap D_a^P \cap \widehat{V}\big) + \sum_{P,a,b} b_0\big(D_a^P \cap D_b^P \cap \widehat{V}\big) \\
&= 1 - r - \sum_{P,a} \gcd\big(k, m(\mathcal{E}_a^P)\big) + \sum_P r_P + \sum_{P,a,b} \gcd\big(k, m(\mathcal{E}_a^P \cap \mathcal{E}_b^P)\big) \\
&= \sum_P (r_P - 1) - (r-1) + \sum_P \Big(1 - \sum_a \#\big(\varrho_k^P\big)^{-1}(a) + \sum_{a,b} \#\big(\varrho_k^P\big)^{-1}\big(\{a,b\}\big)\Big),
\end{aligned} \tag{5}
$$

where $P \in \mathrm{Sing}(\mathbf{C})$ and $a, b \in V(\Gamma_+^P)$ with $a \neq b$. The latter number in brackets is exactly the first Betti number of the graph $N\Gamma_+^P(k)$, since it is connected and the number of vertices and the number of edges are, by definition, $\sum_a \#(\varrho_k^P)^{-1}(a)$ and $\sum_{a,b} \#(\varrho_k^P)^{-1}(\{a,b\})$, respectively.

Remark 1. Note that $b_1(N\Gamma_+^P(k)) = \deg \Delta^{(k)}_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t)$. This number can also be interpreted as the degree of the factor $t-1$ in the polynomial $\Delta^{k}_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t)$, see notation right after (2). This is a consequence of the fact that $\Delta_{\mathrm{Gr}_0^W H_{(\mathbf{C},P)}}(t) = (t-1)\prod_a (t^{m(\mathcal{E}_a)} - 1)^{-1} \prod_{a,b} (t^{m(\mathcal{E}_a \cap \mathcal{E}_b)} - 1)$.

Example 1. Let V be the YLS defined by $f = f_m(x,y,z) + z^{m+k}$, where $\mathbf{C} = \{f_m = 0\} \subset \mathbb{P}^2$ has only one singular point $P = [0:0:1]$ that is locally isomorphic to $(x^p + y^q)(x^r + y^s)$ with $\gcd(p,q) = \gcd(r,s) = 1$. The dual graph $N\Gamma_+^P$ associated with the tangent cone was computed in [8, Fig. 4]. In particular, $r_P = 2$, $r = 1$, and the number of cycles in the graph $N\Gamma_+^P(k)$ is $\gcd(k,p,s) - 1$. Theorem 1.1 tells us that the number of 2-Jordan blocks for $\lambda = 1$ is $\gcd(k,p,s)$.

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References

[1] E. Artal Bartolo, Forme de Jordan de la monodromie des singularités superisolées de surfaces, Mem. Amer. Math. Soc. 109 (525) (1994), x+84.
[2] E. Artal Bartolo, I. Luengo, A. Melle-Hernández, Superisolated surface singularities, in: Singularities and Computer Algebra, in: Lond. Math. Soc. Lect. Note Ser., vol. 324, Cambridge Univ. Press, Cambridge, UK, 2006, pp. 13–39.
[3] E. Artal Bartolo, J. Martín-Morales, J. Ortigas-Galindo, Intersection theory on abelian-quotient V-surfaces and $\mathbb{Q}$-resolutions, J. Singul. 8 (2014) 11–30.
[4] D.T. Lê, Ensembles analytiques complexes avec lieu singulier de dimension un (d'après I.N. Iomdine), in: Seminar on Singularities, Paris, 1976/1977, in: Publ. Math. Univ. Paris VII, vol. 7, Univ. Paris-VII, Paris, 1980, pp. 87–95.
[5] I. Luengo, The μ-constant stratum is not smooth, Invent. Math. 90 (1) (1987) 139–152.
[6] J. Martín-Morales, Monodromy zeta function formula for embedded $\mathbb{Q}$-resolutions, Rev. Mat. Iberoam. 29 (3) (2013) 939–967.
[7] J. Martín-Morales, Embedded $\mathbb{Q}$-resolutions for Yomdin–Lê surface singularities, Isr. J. Math. 204 (1) (2014) 97–143, MR 3273453.
[8] J. Martín-Morales, Semistable reduction of a normal crossing $\mathbb{Q}$-divisor, ArXiv e-prints, 2014.
[9] J.H.M. Steenbrink, Mixed Hodge structure on the vanishing cohomology, in: Real and Complex Singularities, Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976, Sijthoff and Noordhoff, 1977, pp. 525–563.
[10] J. Stevens, On the μ-constant stratum and the V-filtration: an example, Math. Z. 201 (1) (1989) 139–144.
[11] Y. Yomdin, Complex surfaces with a one-dimensional set of singularities, Sib. Mat. Zh. 15 (1974) 1061–1082, 1181.