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Differential geometry

A gap theorem for minimal submanifolds in Euclidean space

Un théorème de seuil pour les sous-variétés minimales dans l'espace euclidien

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ABSTRACT

We prove that for a complete minimal submanifold M^n immersed in the Euclidean space $\mathbb{R}^{n+d}$, if the second fundamental form A and the intrinsic distance function r from a fixed point satisfy $r(x)|A|(x) \le \varepsilon$ for all $x \in M$, where ε is a positive constant depending only on n, then M is an affine subspace of $\mathbb{R}^{n+d}$.

RÉSUMÉ

On démontre que, pour une sous-variété minimale complète M^n immergée dans l'espace euclidien $\mathbb{R}^{n+d}$, si la seconde forme fondamentale A et la fonction distance intrinsèque r mesurée à partir d'un point fixe satisfont l'inégalité $r(x)|A|(x) \le \varepsilon$ pour tous $x \in M$, où ε est une constante positive ne dépendant que de n, alors M est un sous-espace affine de $\mathbb{R}^{n+d}$.

1. The result

The minimal submanifold is one of the most striking subjects in the study of differential geometry, which is also very useful in general relativity, see [3,6,13], etc. for instance. Various types of gap phenomenon of minimal submanifolds and their generalizations have been investigated extensively, see [1,5–8,10–12,15,17–20], etc. Let M^n be an n-dimensional minimal submanifold immersed in the $(n+d)$-dimensional Euclidean space $\mathbb{R}^{n+d}$. Denote by A the second fundamental form of M in $\mathbb{R}^{n+d}$ and $r(x) = d(O, x)$ the intrinsic distance function on M from a fixed point $O \in M$. In this note, we prove the following gap theorem for minimal submanifolds.

Theorem 1.1. *Let M^n be an n-dimensional connected complete minimal submanifold immersed in $\mathbb{R}^{n+d}$. There is a positive constant ε depending only on n such that if $r(x)|A|(x) \le \varepsilon$ for all $x \in M$, then M is an affine subspace of $\mathbb{R}^{n+d}$.*

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We could compare Theorem 1.1 with the results of Kasue and Sugahara [8,10], where several gap theorems are proved under similar conditions about the second fundamental form and the extrinsic distance function and additionally on the number of ends. Notice that in Theorem 1.1 there is no assumption on the number of ends. We should mention another theorem proved by Kasue and Sugahara [9] that for a connected complete noncompact submanifold immersed in $\mathbb{R}^{n+d}$, if the Ricci curvature is nonnegative and $r(x)|A|(x) \to 0$ as x goes to infinity, then it is an affine subspace of $\mathbb{R}^{n+d}$.

Let S denote the scalar curvature of M. By the Gauss equation, we can restate Theorem 1.1 as a gap theorem under an intrinsic curvature pinching condition.

Theorem 1.2. *Let M^n be an n-dimensional connected complete minimal submanifold immersed in $\mathbb{R}^{n+d}$. There is a positive constant ε' depending only on n such that if $S(x)r^2(x) \geq -\varepsilon'$ for all $x \in M$, then M is an affine subspace of $\mathbb{R}^{n+d}$.*

Our theorem is motivated by the ϵ-rigidity result for critical Riemannian metrics proved in [4]. To prove our theorem, we first give a derivative estimate for the second fundamental form on intrinsic balls, then argue by contradiction to get the conclusion.

2. The proof

We need the following proposition.

Proposition 2.1. *For a complete minimal submanifold M^n immersed in $\mathbb{R}^{n+d}$, if $|A|(x) \leq \frac{1}{R}$ for $x \in B(x_0, R)$, where $B(x_0, R) \subset M$ is an intrinsic geodesic ball in M centered at x_0 with radius R, then $|\nabla A|(x) \leq \frac{c(n)}{R^2}$ for $x \in B(x_0, \frac{R}{2})$, with $c(n)$ a positive constant depending only on n.*

Proof. Firstly, we have the following inequalities.

$$\Delta|A|^2 \geq 2|\nabla A|^2 - 3|A|^4,$$
$$\Delta|\nabla A|^2 \geq 2\left|\nabla^2 A\right|^2 - C(n)|A|^2|\nabla A|^2.$$

Here and in the following proof $C(n)$ denotes a positive constant depending only on n, which may be different in different places. The first inequality can be found, for example, in [18]. The second inequality can be derived using fundamental equations for submanifolds, or see [2], since minimal submanifolds are steady solutions to the mean curvature flow.

Define a function $f = |\nabla A|^2(15R^{-2} + |A|^2)$. Then we have:

$$\begin{aligned}
\Delta f &= \Delta\left(|\nabla A|^2\left(15R^{-2} + |A|^2\right)\right) \\
&= 15R^{-2}\Delta|\nabla A|^2 + \Delta\left(|\nabla A|^2|A|^2\right) \\
&\geq 15R^{-2}\left(2\left|\nabla^2 A\right|^2 - C(n)|A|^2|\nabla A|^2\right) \\
&\quad + |\nabla A|^2\left(2|\nabla A|^2 - 3|A|^4\right) + |A|^2\left(2\left|\nabla^2 A\right|^2 - C(n)|A|^2|\nabla A|^2\right) \\
&\quad + 2\left\langle \nabla|\nabla A|^2, \nabla|A|^2\right\rangle.
\end{aligned}$$

For the last term, we get by the Schwartz inequality,

$$\begin{aligned}
2\left\langle \nabla|\nabla A|^2, \nabla|A|^2\right\rangle &\leq 8|A||\nabla A|^2\left|\nabla^2 A\right| \\
&\leq 2\left|\nabla^2 A\right|^2\left(15R^{-2} + |A|^2\right) + \frac{8|A|^2}{15R^{-2} + |A|^2}|\nabla A|^4.
\end{aligned}$$

So we obtain:

$$\begin{aligned}
\Delta f &\geq 15R^{-2}\left(2\left|\nabla^2 A\right|^2 - C(n)|A|^2|\nabla A|^2\right) \\
&\quad + |\nabla A|^2\left(2|\nabla A|^2 - 3|A|^4\right) + |A|^2\left(2\left|\nabla^2 A\right|^2 - C(n)|A|^2|\nabla A|^2\right) \\
&\quad - 2\left|\nabla^2 A\right|^2\left(15R^{-2} + |A|^2\right) - \frac{8|A|^2}{15R^{-2} + |A|^2}|\nabla A|^4 \\
&\geq \frac{3}{2}|\nabla A|^4 - \left(3 + 16C(n)\right)R^{-4}|\nabla A|^2 \\
&\geq |\nabla A|^4 - \frac{1}{2}\left(3 + 16C(n)\right)^2 R^{-8} \\
&\geq \frac{R^4}{256}f^2 - \frac{1}{2}\left(3 + 16C(n)\right)^2 R^{-8}.
\end{aligned}$$

Define $\phi(x) = (R^2 - r(x)^2)^2$ for $x \in B(x_0, R)$ and let $F = \phi f$. Then $F \geq 0$ and $F = 0$ on $\partial B(x_0, R)$. Hence F attains its maximum in the interior of $B(x_0, R)$. Suppose that x_1 is the maximum point of F. We first assume that x_1 is not a conjugate point of x_0, hence F is smooth in a neighborhood of x_1. So we have $\nabla F(x_1) = 0$ and $\Delta F(x_1) \leq 0$.

Firstly, we have:

$$|\nabla\phi|^2 = \left|\nabla\left(R^2 - r^2\right)^2\right|^2 = \left|4\left(R^2 - r^2\right)r\nabla r\right|^2 = 16\phi r^2.$$

By [16], the Ricci curvature Ric of M satisfies $\mathrm{Ric}(x) \geq -\frac{n-1}{n}|A|^2(x) \geq -\frac{n-1}{n}R^{-2}$ for all $x \in B(x_0, R)$. Let $\gamma(t)$ be the unique minimal geodesic parameterized by arc-length connecting x_0 and x_1. Then by the Laplacian comparison theorem (see [14] for example), we have at x_1:

$$\Delta r \leq \frac{n-1}{r}\left(1 + n^{1/2}R^{-1}r\right) \leq (n-1)\left(1 + n^{1/2}\right)\frac{1}{r}.$$

Hence we get:

$$\begin{aligned}\Delta\phi &= \Delta\left(R^2 - r^2\right)^2\\ &= -2\left(R^2 - r^2\right)\Delta r^2 + 2\left|\nabla\left(R^2 - r^2\right)\right|^2\\ &\geq -4\phi^{1/2}r\Delta r - 4\phi^{1/2}\\ &\geq -4\left(n + n^{3/2} - n^{1/2}\right)\phi^{1/2}.\end{aligned}$$

In the following, we compute $\phi\Delta F$ at x_1.

$$\begin{aligned}0 &\geq \phi\Delta F\\ &= \phi\Delta(\phi f)\\ &= \phi^2\Delta f + \phi f\Delta\phi + 2\phi\langle\nabla\phi, \nabla f\rangle\\ &\geq \frac{R^4}{256}(\phi f)^2 - \frac{1}{2}\left(3 + 16C(n)\right)^2\phi^2 R^{-8}\\ &\quad - 4\left(n + n^{3/2} - n^{1/2}\right)\phi^{3/2} f + 2\left\langle\nabla\phi, \nabla(\phi f)\right\rangle - 2|\nabla\phi|^2 f\\ &\geq \frac{R^4}{512}(\phi f)^2 - 32R^2\phi f\\ &\quad - \left(\frac{1}{2}\left(3 + 16C(n)\right)^2 + 2048\left(n + n^{3/2} - n^{1/2}\right)^2\right)\\ &= \frac{R^4}{512}F^2 - 32R^2 F - C(n).\end{aligned}$$

Hence we have at x_1,

$$F \leq \frac{C(n)}{R^2}.$$

Since x_1 is the maximum point, we have $F(x) \leq \frac{C(n)}{R^2}$ for all $x \in B(x_0, \frac{R}{2})$. Also we have $\phi(x) \geq \frac{9}{16}R^4$ for $x \in B(x_0, \frac{R}{2})$. So we get for $x \in B(x_0, \frac{R}{2})$

$$|\nabla A|^2 \leq \frac{16C(n)}{135R^4}.$$

If x_1 is a conjugate point of x_0, we can define a support function $\tilde{F}$ of F at x_1 as in [14]. Let $\hat{\gamma}(t)$ be a minimal geodesic parameterized by arc-length connecting x_0 and x_1 and $P \in \hat{\gamma}$ a point with $d(x_0, P) = \epsilon$ for $\epsilon > 0$ small such that P is not a conjugate of x_1. Let $\tilde{F} = \tilde{\phi} f$, where $\tilde{\phi}(x) = (R^2 - (r_P(x) + \epsilon)^2)^2$ and $r_P(x) = d(P, x)$. There is a neighborhood N_P of $\tilde{\gamma} = \hat{\gamma}|_{[P,x_1]}$ such that $\tilde{F}$ is smooth in N_P. By the triangular inequality we have $\tilde{F}(x_1) = F(x_1)$ and $\tilde{F}(x) \leq F(x)$ for $x \in N_P$. Hence x_1 is a local maximum point of $\tilde{F}$.

Firstly, we have at x_1:

$$
\begin{aligned}
|\nabla\tilde{\phi}|^2 &= \big|\nabla\big(R^2-\big(r_P(x_1)+\epsilon\big)^2\big)^2\big|^2\\
&= \big|2\big(R^2-\big(r_P(x_1)+\epsilon\big)^2\big)\cdot\nabla\big(r_P(x_1)+\epsilon\big)^2\big|^2\\
&= \big|4\big(R^2-\big(r_P(x_1)+\epsilon\big)^2\big)\big(r_P(x_1)+\epsilon\big)\nabla r_P\big|^2\\
&= 16\tilde{\phi}r^2.
\end{aligned}
$$

Here we have used the fact that $r_P(x_1)+\epsilon = r(x_1)$. By the Laplacian comparison theorem, we have at x_1:

$$
\Delta r_P \le \frac{n-1}{r_P}\big(1+n^{1/2}R^{-1}r_P\big) \le (n-1)\big(1+n^{1/2}\big)\frac{1}{r_P}.
$$

We choose ϵ sufficiently small such that $rr_P^{-1}\le 2$, then

$$
\begin{aligned}
\Delta\tilde{\phi} &= \Delta\big(R^2-\big(r_P(x_1)+\epsilon\big)^2\big)^2\\
&\ge 2\big(R^2-\big(r_P(x_1)+\epsilon\big)^2\big)\Delta\big(R^2-\big(r_P(x_1)+\epsilon\big)^2\big)\\
&\ge -4\tilde{\phi}^{1/2}\big(\big(r_P(x_1)+\epsilon\big)\Delta r_p+\big|\nabla\big(r_P(x_1)+\epsilon\big)\big|^2\big)\\
&\ge -4\big((n-1)\big(1+n^{-1}\big)rr_P^{-1}+1\big)\tilde{\phi}^{1/2}\\
&\ge -4\big(2(n-1)\big(1+n^{-1}\big)+1\big)\tilde{\phi}^{1/2}.
\end{aligned}
$$

We compute $\tilde{\phi}\Delta\tilde{F}$ at x_1.

$$
\begin{aligned}
0 &\ge \tilde{\phi}\Delta\tilde{F}\\
&= \tilde{\phi}\Delta(\tilde{\phi}f)\\
&= \tilde{\phi}^2\Delta f+\tilde{\phi}f\Delta\tilde{\phi}+2\tilde{\phi}\langle\nabla\tilde{\phi},\nabla f\rangle\\
&\ge \frac{R^4}{256}(\tilde{\phi}f)^2-\frac{1}{2}\big(3+16C(n)\big)^2\tilde{\phi}^2R^{-8}\\
&\quad -4\big(2(n-1)\big(1+n^{-1}\big)+1\big)\tilde{\phi}^{3/2}f+2\big\langle\nabla\tilde{\phi},\nabla(\tilde{\phi}f)\big\rangle-2|\nabla\tilde{\phi}|^2f\\
&\ge \frac{R^4}{512}(\tilde{\phi}f)^2-32R^2\tilde{\phi}f\\
&\quad -\left(\frac{1}{2}\big(3+16C(n)\big)^2+2048\big(2(n-1)\big(1+n^{-1}\big)+1\big)^2\right)\\
&= \frac{R^4}{512}\tilde{F}^2-32R^2\tilde{F}-C(n).
\end{aligned}
$$

Hence we have $\tilde{F}(x_1)\le\frac{C(n)}{R^2}$, which implies $F(x_1)\le\frac{C(n)}{R^2}$. So we can use similar argument to get that for $x\in B(x_0,\frac{R}{2})$

$$
|\nabla A|^2 \le \frac{C(n)}{R^4}.
$$

This completes the proof. □

Now we give the proof of Theorem 1.1.

Proof. Suppose A is not identically zero. By the assumption, there is a maximum point x_0 of $|A|$; suppose $|A|(x_0)=\frac{1}{R}$. Then $|A|(x)\le\frac{1}{R}$ for $x\in B(x_0,R)$. By Proposition 2.1, there is $c(n)\ge 1$ such that $|\nabla A|(x)\le\frac{c(n)}{R^2}$ for $x\in B(x_0,\frac{R}{2})$. Take $\delta=\frac{1}{2c(n)}$. For $y\in\partial B(x_0,\delta R)$, one has:

$$
|A|(y)\ge|A|(x_0)-\delta R\cdot\frac{c(n)}{R^2}=\frac{1}{2R}.
$$

From the assumption, we have $|A|(x_0)\le\frac{\varepsilon}{d(O,x_0)}$, hence $d(O,x_0)\le\varepsilon R$. By the triangular inequality, we have:

$$
d(O,y)\ge d(y,x_0)-d(O,x_0)\ge(\delta-\varepsilon)R.
$$

Hence

$$\frac{1}{2R} \le |A|(y) \le \frac{\varepsilon}{d(O, y)} \le \frac{\varepsilon}{(\delta - \varepsilon)R}.$$

Now we pick $\varepsilon = \frac{1}{5}\delta = \frac{1}{10c(n)}$. Then the inequality above leads to a contradiction. This completes the proof of Theorem 1.1. □

We end the paper by a remark that Theorem 1.1 could be used to give another slightly different proof of the L^n-gap theorem for minimal submanifolds in Euclidean space that has been proved in [12,18], etc. In fact, by carrying out the well-known De Giorgi–Nash–Moser iteration procedure to the Simons equation for minimal submanifolds, we can show that if the L^n-norm of the second fundamental form is small enough, then the assumption in Theorem 1.1 is satisfied. We leave the details to readers.

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