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On a conjecture of Lionel Schwartz about the eigenvalues of Lannes' T-functor ☆

À propos d'une conjecture de Lionel Schwartz sur les valeurs propres du foncteur T *de Lannes*

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ABSTRACT

Given a prime p, let $K^{\mathrm{red}}(\mathcal{U})$ denote the Grothendieck group generated by the isomorphism classes of indecomposable injective *reduced* modules in the category of unstable modules over the mod p Steenrod algebra. Let $K_n^{\mathrm{red}}(\mathcal{U})$, $n \in \mathbb{N}$, denote the subgroup of $K^{\mathrm{red}}(\mathcal{U})$ generated by the indecomposable summands of $\mathrm{H}^*\mathrm{B}(\mathbb{Z}/p)^n$. We describe in this note a strategy for the proof of the following conjecture of Lionel Schwartz: *the operator induced by Lannes'* T*-functor on the rational vector space* $\mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathcal{U})$ *is diagonalizable and has eigenvalues* $1, p, \ldots, p^{n-1}, p^n$ *with multiplicities* $p^n - p^{n-1}, p^{n-1} - p^{n-2}, \ldots, p-1, 1$, *respectively*.

RÉSUMÉ

Étant donné un nombre premier p, on note $K^{\mathrm{red}}(\mathcal{U})$ le groupe de Grothendieck engendré par les classes d'isomorphisme de modules *réduits* injectifs indécompsables de la catégorie des modules instable sur l'algèbre de Steenrod modulo p. On note $K_n^{\mathrm{red}}(\mathcal{U})$, $n \in \mathbb{N}$, le sous-groupe de $K^{\mathrm{red}}(\mathcal{U})$ engendré par les facteurs indécomposables de $\mathrm{H}^*\mathrm{B}(\mathbb{Z}/p)^n$. On décrit dans cette note une stratégie pour démontrer la conjecture suivante due à Lionel Schwartz : *l'opérateur induit par le foncteur* T *de Lannes sur l'espace vectoriel rationnel* $\mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathcal{U})$ *est diagonalisable et a pour valeurs propres* $1, p, \ldots, p^{n-1}, p^n$ *de multiplicités* $p^n - p^{n-1}, p^{n-1} - p^{n-2}, \ldots, p-1, 1$, *respectivement.*

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1. Introduction

Fix a prime number p and consider the category $\mathcal{U}$ of unstable modules over the mod p Steenrod algebra $\mathcal{A}_p$ [13]. Let V be an elementary Abelian p-group, i.e. a group isomorphic to $(\mathbb{Z}/p)^n$ for some natural number n. By a result of Carlsson–Miller–Lannes–Zarati, the mod p cohomology of the classifying space $\mathrm{B}V$, $\mathrm{H}^*\mathrm{B}V$, is an injective object in the category $\mathcal{U}$ [9].

Let $K^{\mathrm{red}}(\mathcal{U})$ denote the Grothendieck group generated by the isomorphism classes of indecomposable injective *reduced* modules in $\mathcal{U}$. Recall that an unstable module is *reduced* if it does not contain a non-trivial suspension [13, p. 47]. By the Lannes–Schwartz classification theorem for $\mathcal{U}$-injectives, we know that indecomposable injective reduced modules in the category $\mathcal{U}$ are precisely the indecomposable $\mathcal{A}_p$-module summands of $\mathrm{H}^*\mathrm{B}V$ for some V [8].

Recall now that Lannes' T-functor $\mathrm{T}: \mathcal{U} \to \mathcal{U}$ is left adjoint to the functor given by tensoring with $\mathrm{H}^*\mathrm{B}\mathbb{Z}/p$ in the category $\mathcal{U}$ [7]. The functor T induces an operator on the Grothendieck group $K^{\mathrm{red}}(\mathcal{U})$. There is a natural filtration on $K^{\mathrm{red}}(\mathcal{U})$:

$$K_0^{\mathrm{red}}(\mathcal{U}) \subset K_1^{\mathrm{red}}(\mathcal{U}) \subset \cdots \subset K_n^{\mathrm{red}}(\mathcal{U}) \subset \cdots \subset K^{\mathrm{red}}(\mathcal{U}),$$

where the subgroup $K_n^{\mathrm{red}}(\mathcal{U})$, $n \in \mathbb{N}$, is generated by the isomorphism classes of indecomposable summands of $\mathrm{H}^*\mathrm{B}(\mathbb{Z}/p)^n$. Fundamental properties of T can be used to show that T preserves this filtration.

The purpose of this note is to describe a proof of the following conjecture of Lionel Schwartz:

Conjecture 1.1 *(L. Schwartz). The operator induced by the* T*-functor on the vector space $\mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathcal{U})$ is diagonalizable and has eigenvalues $1, p, \ldots, p^{n-1}, p^n$ with multiplicities $p^n - p^{n-1}, p^{n-1} - p^{n-2}, \ldots, p-1, 1$, respectively.*

The conjecture is supported by computations of Harris and Shank [5, Appendix] and may be checked by computer for small n by using work of Harris and Shank [5] as follows.

Let $\mathrm{M}_n(\mathbb{F}_p)$ denote the semigroup of all $n \times n$-matrices with coefficients in the finite field $\mathbb{F}_p$. Let $\mathrm{Irr}\,\mathrm{M}_n$ denote the set of isomorphism classes of irreducible modules for the semigroup ring $\mathbb{F}_p[\mathrm{M}_n(\mathbb{F}_p)]$. Note that $|\mathrm{Irr}\,\mathrm{M}_n| = p^n$ [4]. For $\lambda \in \mathrm{Irr}\,\mathrm{M}_n$, there is a corresponding indecomposable summand L_λ of $\mathrm{H}^*\mathrm{B}(\mathbb{Z}/p)^n$ given by

$$L_\lambda := \mathrm{Hom}_{\mathbb{F}_p[\mathrm{M}_n(\mathbb{F}_p)]}\big(P_\lambda, \mathrm{H}^*\mathrm{B}(\mathbb{Z}/p)^n\big), \tag{1}$$

where P_λ is a projective cover of the irreducible $\mathbb{F}_p[\mathrm{M}_n(\mathbb{F}_p)]$-module λ. The summands L_λ, $\lambda \in \mathrm{Irr}\,\mathrm{M}_n$, form a basis for $K_n^{\mathrm{red}}(\mathcal{U})$ [8]. Harris and Shank [5] proved that

$$\mathrm{T}(L_\lambda) \cong \bigoplus_{\mu \in \mathrm{Irr}\,\mathrm{M}_n} L_\mu^{\oplus a_{\lambda\mu}}, \quad \lambda \in \mathrm{Irr}\,\mathrm{M}_n, \tag{2}$$

where $a_{\lambda\mu}$ is the multiplicity of λ in the tensor product of μ with the p-truncated symmetric algebra $\bar{\mathrm{S}}^*(\mathbb{F}_p^n) := \mathbb{F}_p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$, considered as a representation of $\mathrm{M}_n(\mathbb{F}_p)$ as usual. The coefficients $a_{\lambda\mu}$ are not known in general. However it follows from (2) that the matrix of T (in the basis $\{L_\lambda\}$) is the transpose of the matrix of the endomorphism (in the basis $\{\lambda\}$)

$$\mathbf{t}: \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{F}_p}\big(\mathrm{M}_n(\mathbb{F}_p)\big) \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{F}_p}\big(\mathrm{M}_n(\mathbb{F}_p)\big), \qquad \mu \mapsto \mu \otimes \bar{\mathrm{S}}^*\big(\mathbb{F}_p^n\big).$$

Here $\mathrm{R}_{\mathbb{F}_p}(\mathrm{M}_n(\mathbb{F}_p))$ is the representation ring of the semigroup $\mathrm{M}_n(\mathbb{F}_p)$ over $\mathbb{F}_p$. This fact can be used to build the matrix for the operator $\mathbf{t}$ by computer. For example, for $p = 2$, $\mathrm{R}_{\mathbb{F}_2}(\mathrm{M}_n(\mathbb{F}_2))$ is generated by the exterior powers $\Lambda^k := \Lambda^k(\mathbb{F}_2^n)$, $0 \le k \le n$, subjected to the relations $\Lambda^k \otimes \Lambda^k = \Lambda^k - 2\sum_{i=1}^k (-1)^i \Lambda^{k+i} \otimes \Lambda^{k-i}$, and so has a $\mathbb{Z}$-basis $\{1\} \cup \{\Lambda^{k_1} \otimes \cdots \otimes \Lambda^{k_t} \mid 1 \le k_1 < \cdots < k_t \le n,\ 1 \le t \le n\}$. The operator $\mathbf{t}$ is now the endomorphism of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{F}_2}(\mathrm{M}_n(\mathbb{F}_2))$ given by the multiplication by $\sum_{k=0}^n \Lambda^k$. In this way K. Delamotte has checked the conjecture to be true up to $n = 9$.

2. A strategy for the proof of Conjecture 1.1

In this section we prove Conjecture 1.1 by assuming two propositions that will be treated in Sects. 3 and 4. Recall that the tensor product of two reduced $\mathcal{U}$-injectives is a reduced $\mathcal{U}$-injective [9]. Taking tensor product with $\mathrm{H} := \mathrm{H}^*\mathrm{B}(\mathbb{Z}/p)$ in $\mathcal{U}$ induces then a homomorphism that we denote also by H:

$$\mathrm{H}: K^{\mathrm{red}}(\mathcal{U}) \to K^{\mathrm{red}}(\mathcal{U}).$$

We observe that H increases the filtration on $K^{\mathrm{red}}(\mathcal{U})$.

Proposition 2.1. *The homomorphism $\mathrm{H}: K^{\mathrm{red}}(\mathcal{U}) \to K^{\mathrm{red}}(\mathcal{U})$ is injective.*

Proposition 2.2. *The homomorphism* $\mathrm{T}: \mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathscr{U}) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathscr{U})$ *satisfies:*

$$\dim_{\mathbb{Q}} \ker(\mathrm{T} - 1) \geq p^n - p^{n-1}.$$

Proof of Conjecture 1.1 assuming Propositions 2.1 and 2.2. By induction on n, we prove that, for $0 \leq i \leq n$, the dimension of the eigenspace

$$E_n^i := \{X \in \mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathscr{U}) \mid \mathrm{T}(X) = p^i X\}$$

is $p^{n-i} - p^{n-i-1}$. (Here, by convention, $p^{-1} = 0$ so that $\dim_{\mathbb{Q}} E_n^n = p^0 - p^{-1} = 1$.) This is trivial if $n = 0$ and we suppose this holds for $n - 1$. By Proposition 2.2,

$$\dim_{\mathbb{Q}} E_n^0 \geq p^n - p^{n-1}.$$

For $1 \leq i \leq n$, we have $\mathrm{H}(E_{n-1}^{i-1})$ is a subspace of E_n^i since T commutes with the tensor product and $\mathrm{T}(\mathrm{H}^*\mathrm{B}\mathbb{Z}/p) \cong [\mathrm{H}^*\mathrm{B}\mathbb{Z}/p]^{\oplus p}$ [7]. By inductive hypothesis on the dimension of E_{n-1}^{i-1} and by injectivity of the homomorphism H (Proposition 2.1), we get

$$\dim_{\mathbb{Q}} E_n^i \geq \dim_{\mathbb{Q}} \mathrm{H}(E_{n-1}^{i-1}) = p^{n-i} - p^{n-i-1}.$$

We conclude that the dimension of the direct sum $\bigoplus_{i=0}^n E_n^i$ is $\geq p^n$, and since $\mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathscr{U})$ is of dimension p^n, we obtain what we need to prove. □

3. Injectivity of the homomorphism $\mathrm{H}: K^{\mathrm{red}}(\mathscr{U}) \to K^{\mathrm{red}}(\mathscr{U})$

In this section, we prove Proposition 2.1. Given M and N two finite direct sums of indecomposable reduced $\mathscr{U}$-injectives [8], we need to prove that, if $H \otimes M$ and $H \otimes N$ are isomorphic as unstable modules, then M and N are isomorphic as unstable modules.

For M an unstable module, let $|M|$ denote the connectivity of M, i.e. the lowest degree d for which M^d is non-trivial. In order to prove Proposition 2.1, we need to introduce the following definition.

Definition 3.1. An **additive** functor $F: \mathscr{U} \to \mathscr{U}$ is called **connectively increasing** if there exists a functor $\bar{F}: \mathscr{U} \to \mathscr{U}$ such that, for each $M \in \mathscr{U}$,

(i) there is a natural short exact sequence of unstable modules $0 \to M \to F(M) \to \bar{F}(M) \to 0$, and
(ii) $|\bar{F}(M)| > |M|$.

It is clear that the functor $M \mapsto \mathrm{H}^*\mathrm{B}\mathbb{Z}/p \otimes M \cong M \oplus (\bar{\mathrm{H}}^*\mathrm{B}\mathbb{Z}/p \otimes M)$ is connectively increasing, so in order to prove Proposition 2.1, it suffices to prove the following:

Lemma 3.2. *Let F be a connectively increasing functor, N an unstable module and M a finite direct sum of indecomposable $\mathscr{U}$-injectives. Suppose that $F(M) \cong F(N)$ as unstable modules of finite type. Then $M \cong N$ as unstable modules.*

Proof. It is clear that if $F(L)$ is trivial then L is trivial. Thus it suffices to prove that M is a direct factor of N. The isomorphism $F(M) \cong F(N)$ *as graded vector spaces* together with Condition (ii) of Definition 3.1 imply that $|M| = |N|$.

Take an indecomposable injective summand I of M such that I is non-trivial in degree $|M|$. Since M can be considered as a submodule of $F(M)$, it follows that I is a direct summand of $F(M)$, and thus a direct summand of $F(N)$. But since $|I| = |M| = |N| < |\bar{F}(N)|$, the indecomposable injective I must be a direct summand of N.

We write $M = I \oplus M'$ and $N = I \oplus N'$. The additivity of F gives rise to an isomorphism *of unstable modules*:

$$F(I) \oplus F(M') \cong F(I) \oplus F(N'). \tag{3}$$

As $F(I) \oplus F(M')$ is of finite type, it follows that $F(M') \cong F(N')$ as *graded vector spaces* and again by connectivity reason, we have $|M'| = |N'|$.

Take an indecomposable injective summand I' of M' such that I' is non-trivial in degree $|M'|$. As we can cancel the $\mathscr{U}$**-injectives** isomorphic to I' appearing in the factor $F(I)$ on both sides of the isomorphism (3), we can suppose that $F(I)$ does not contain I' as a direct factor. Thus I' is a direct summand of $F(M')$ and the isomorphism (3) implies that $F(N')$ must contain a summand isomorphic to I'. Since $|I'| = |M'| = |N'| < |\bar{F}(N')|$, the indecomposable injective I' must be a summand of N'.

We write $M = I \oplus I' \oplus M''$ and $N = I \oplus I' \oplus N''$. Since M is a finite direct sum of indecomposable injectives, the argument above can be repeated to show that the injective module M is a direct factor of N. The lemma follows. □

4. Lower bound for the dimension of the eigenspace of T associated with 1

In this section, we describe a proof of Proposition 2.2. We need to establish the inequality $\dim_{\mathbb{Q}} \ker(\mathrm{T}-1) \geq p^n - p^{n-1}$, where T is considered as an operator on $\mathbb{Q} \otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathscr{U})$. For this, we need to introduce some operators on $K^{\mathrm{red}}(\mathscr{U})$, which are induced by taking the functional dual of stable summands.

Given G a finite group, let B_+G denote the union of the classifying space $\mathrm{B}G$ with an added disjoint basepoint. For simplicity, B_+G will also denote the suspension spectrum $\Sigma^\infty \mathrm{B}_+G$. Given spectra X, Y, let $\mathrm{F}(X, Y)$ denote the functional spectrum and $\mathrm{D}(X) = \mathrm{F}(X, S^0)$ the functional dual. In the sequel, all spaces and spectra are implicitly completed at the prime p, and cohomology is taken with $\mathbb{Z}/p$-coefficients.

4.1. Dual of stable summands derived from irreducible $\mathrm{GL}_n(\mathbb{F}_p)$-modules

By the Segal Conjecture [1,2,11] for the elementary Abelian group V, there is a $\mathrm{GL}(V)$-equivariant equivalence of spectra:

$$\mathrm{DB}_+V \simeq \bigvee_{W \subseteq V} \mathrm{B}_+(V/W), \tag{4}$$

where the wedge sum is taken over all subgroups of V. We refer the reader to Olga Stroilova's thesis [14] for a thorough discussion on the action of $\mathrm{GL}(V)$ upon the functional dual DB_+V. For our purposes, it suffices to know that each $g \in \mathrm{GL}(V)$ sends the summand indexed by the subspace W in the wedge sum to the summand indexed by the subspace $g^{-1}W$.

Denote by $\mathrm{Irr}\,\mathrm{GL}_n$ the set of isomorphism classes of irreducible modules for the group ring $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$. Given $\lambda \in \mathrm{Irr}\,\mathrm{GL}_n$, let e_λ denote a primitive idempotent of $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$ such that $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]e_\lambda$ is a projective cover of the irreducible $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$-module λ. The corresponding stable summand, $e_\lambda \mathrm{B}_+(\mathbb{Z}/p)^n$, of $\mathrm{B}_+(\mathbb{Z}/p)^n$ is given by a telescope construction. Its cohomology is denoted by M_λ, and this is a direct factor of $\mathrm{H}^*\mathrm{B}(\mathbb{Z}/p)^n$. The $\mathscr{A}_p$-modules M_λ, $\lambda \in \coprod_{0 \leq m \leq n} \mathrm{Irr}\,\mathrm{GL}_m$, then form a basis for $K_n^{\mathrm{red}}(\mathscr{U})$ [4].

For each stable summand X of $\mathrm{B}_+(\mathbb{Z}/p)^n$ derived from an irreducible $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$-module, it follows from (4) that the dual spectrum $\mathrm{D}(X)$ is a wedge sum of stable summands of $\mathrm{B}_+(\mathbb{Z}/p)^m$ with $m \leq n$. By identifying a stable summand with its cohomology [6], we obtain a filtration-preserving operator:

$$\mathrm{D} : K^{\mathrm{red}}(\mathscr{U}) \to K^{\mathrm{red}}(\mathscr{U}).$$

Theorem 4.1. *Let e'_λ denote the image of e_λ under the antipode $\sum_g a_g g \mapsto \sum_g a_g g^{-1}$ of the Hopf algebra $\mathbb{F}_p[\mathrm{GL}_n(\mathbb{F}_p)]$. Then*

$$\mathrm{D}\big(e_\lambda \mathrm{B}_+(\mathbb{Z}/p)^n\big) \simeq e'_\lambda \mathrm{B}_+(\mathbb{Z}/p)^n \vee \textit{ stable summands of } \mathrm{B}_+(\mathbb{Z}/p)^m \quad \textit{with } m < n.$$

This follows from the discussion in [14] about the action of $\mathrm{GL}(V)$ on the functional dual DB_+V.

In order to prove Proposition 2.2, we need to define a variant of the dual operator D. By the Segal conjecture for $\mathbb{Z}/p \oplus V$, we have an equivalence:

$$\mathrm{DB}_+(\mathbb{Z}/p \oplus V) \simeq \bigvee_{W \subseteq \mathbb{Z}/p \oplus V} \mathrm{B}_+\big((\mathbb{Z}/p \oplus V)/W\big).$$

Considering $\mathrm{GL}(V)$ as a subgroup of $\mathrm{GL}(\mathbb{Z}/p \oplus V)$, we have a $\mathrm{GL}(V)$-equivariant decomposition:

$$\mathrm{DB}_+(\mathbb{Z}/p \oplus V) \simeq \Big[\bigvee_{W \supseteq \mathbb{Z}/p} \mathrm{B}_+\big((\mathbb{Z}/p \oplus V)/W\big)\Big] \vee \Big[\bigvee_{W \subseteq V} \mathrm{B}_+\big((\mathbb{Z}/p \oplus V)/W\big)\Big] \vee \Big[\bigvee_{\mathbb{Z}/p \not\subseteq W \not\subseteq V} \mathrm{B}_+\big((\mathbb{Z}/p \oplus V)/W\big)\Big].$$

By the Segal conjecture for V, we can identify the first wedge sum on the right with DB_+V and the second with $\mathrm{B}_+\mathbb{Z}/p \wedge \mathrm{DB}_+V$. For the third one, let W be a subgroup of $\mathbb{Z}/p \oplus V$ such that $\mathbb{Z}/p \not\subseteq W \not\subseteq V$. Denote by $\widehat{W}$ the image of the composite $W \hookrightarrow \mathbb{Z}/p \oplus V \twoheadrightarrow V$. We observe that $\widehat{W} \cong W$ since W does not contain $\mathbb{Z}/p$ and that $\dim \widehat{W} \geq 1$ since W is not contained in V. We have

$$\mathrm{B}_+\big((\mathbb{Z}/p \oplus V)/W\big) \simeq \mathrm{B}_+\big((\mathbb{Z}/p \oplus V)/\widehat{W}\big) \simeq \mathrm{B}_+\mathbb{Z}/p \wedge \mathrm{B}_+(V/\widehat{W}).$$

Thus we obtain a $\mathrm{GL}(V)$-equivariant equivalence:

$$\mathrm{DB}_+(\mathbb{Z}/p \oplus V) \simeq (\mathrm{DB}_+V) \vee (\mathrm{B}_+\mathbb{Z}/p \wedge \mathrm{DB}_+V) \vee (\mathrm{B}_+\mathbb{Z}/p \wedge \mathrm{D}_1\mathrm{B}_+V), \tag{5}$$

where

$$\mathrm{D}_1\mathrm{B}_+V \simeq \bigvee_{\mathbb{Z}/p\not\subseteq W\not\subseteq V} \mathrm{B}_+(V/\widehat{W}). \tag{6}$$

In particular, we can define a spectrum D_1X for X a stable summand of B_+V, and thus obtain an endomorphism by identifying a stable summand with its cohomology:

$$\mathrm{D}_1 : K^{\mathrm{red}}(\mathscr{U}) \to K^{\mathrm{red}}(\mathscr{U}).$$

We note that D_1 decreases the filtration.

4.2. The fundamental equation

We know now that the operators T, H, D and D_1 act on the Grothendieck group $K^{\mathrm{red}}(\mathscr{U})$. It is natural to ask how these operators are related. Here is a relation which is sufficient for our purposes.

Theorem 4.2 *(The fundamental equation). For $X \in K^{\mathrm{red}}(\mathscr{U})$, we have:*

$$\mathrm{T}\big(\mathrm{D}(X)\big) = \mathrm{D}(X) + \mathrm{H}\big(\mathrm{D}_1(X)\big). \tag{7}$$

We defer the proof of this theorem for a while, and use it now to verify Proposition 2.2 (which is an essential ingredient in the proof of the conjecture).

Proof of Proposition 2.2. Consider T, D and D_1 as operators on $\mathbb{Q}\otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathscr{U})$. We need to prove that

$$\dim_{\mathbb{Q}} \ker(\mathrm{T}-1) \geq p^n - p^{n-1}.$$

We have

$$\mathrm{T}(\mathrm{D}X) = \mathrm{D}X \iff \mathrm{H}(\mathrm{D}_1X) = 0 \iff \mathrm{D}_1X = 0,$$

where the first equivalence comes from the fundamental equation and the second is a consequence of the injectivity of the homomorphism H. By Theorem 4.1, D is an isomorphism. It follows that

$$\dim_{\mathbb{Q}} \ker(\mathrm{T}-1) = \dim_{\mathbb{Q}} \ker \mathrm{D}_1.$$

On the other hand, by definition, D_1 maps $\mathbb{Q}\otimes_{\mathbb{Z}} K_n^{\mathrm{red}}(\mathscr{U})$ into $\mathbb{Q}\otimes_{\mathbb{Z}} K_{n-1}^{\mathrm{red}}(\mathscr{U})$, so

$$\dim_{\mathbb{Q}} \ker \mathrm{D}_1 \geq p^n - p^{n-1}.$$

The proposition follows. □

In order to prove Theorem 4.2, we use the adjunction equivalence of spectra:

$$\mathrm{D}(\mathrm{B}_+\mathbb{Z}/p \wedge X) \simeq \mathrm{F}(\mathrm{B}_+\mathbb{Z}/p, \mathrm{D}X) \tag{8}$$

for each stable summand X of B_+V. Theorem 4.2 is now an immediate consequence of the following two propositions.

Proposition 4.1. *For X a stable summand of B_+V, there is an isomorphism of unstable modules:*

$$\bar{\mathrm{H}}^*\mathrm{D}(\mathrm{B}_+\mathbb{Z}/p \wedge X) \cong \big[\bar{\mathrm{H}}^*\mathrm{D}X\big] \oplus \big[\mathrm{H}\otimes\bar{\mathrm{H}}^*\mathrm{D}X\big] \oplus \big[\mathrm{H}\otimes\bar{\mathrm{H}}^*\mathrm{D}_1X\big]. \tag{9}$$

This follows directly from the definition of the operator D_1.

Proposition 4.2. *For X a stable summand of B_+V, there is an isomorphism of unstable modules:*

$$\bar{\mathrm{H}}^*\mathrm{F}(\mathrm{B}_+\mathbb{Z}/p, \mathrm{D}X) \cong \big[\mathrm{T}\bar{\mathrm{H}}^*\mathrm{D}X\big] \oplus \big[\mathrm{H}\otimes\bar{\mathrm{H}}^*\mathrm{D}X\big]. \tag{10}$$

The proof of this result makes use of the fact that $\mathrm{D}X$ is a suspension spectra (which is a consequence of the Segal conjecture), the theory of Lannes [7], the relation between Lannes' T-functor and Singer's R_s-functors [10,15,12], and calculations of the mod p cohomology of infinite loop spaces [3].

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