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Integrability of the periodic Kostant–Toda flow on matrix loops of level k

Intégrabilité du flot périodique de Kostant–Toda sur des boucles de matrices de niveau k

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A B S T R A C T

In this note, we announce results on the Liouville integrability of the periodic Kostant–Toda flow on loops of matrices in $\mathfrak{sl}(n, \mathbb{C})$ of level k.

R É S U M É

Dans cette note, nous annonçons des résultats sur l'intégrabilité du flot périodique de Kostant–Toda sur des boucles de matrices de niveau k dans $\mathfrak{sl}(n, \mathbb{C})$.

Version française abrégée

Le réseau de Toda non périodique est un système de particules placées sur une droite qui a deux représentations de Lax distinctes, associées à des décompositions différentes de la même algèbre de Lie en sous-algèbres de Lie (voir [1,11]). Cela a donné lieu à des extensions naturelles sur des espaces de phase plus vastes, qui sont encore Liouville intégrables [5–7,9]. Pour la version périodique du réseau de Toda [8], s'il est connu que les deux représentations de Lax distinctes existent, une extension (dans le sens de la réalisation de Kostant) à des espaces de phase plus vastes n'a été introduite que récemment dans [4], et cela a été fait dans le contexte des algèbres de Lie simples complexes $\mathfrak{g}$. Le système obtenu, que l'auteur de [4] a appelé « réseau périodique de Kostant–Toda complet », s'ést avéré être complètement intégrable sur la variété de Poisson sous-jacente. En outre, à la fin de [4], l'auteur a introduit une famille de sous-variétés de Poisson paramétrée par $1 \le k \le m_\ell$, où m_ℓ est le plus grand exposant de $\mathfrak{g}$, et il conjecture que le système est aussi intégrable au sens de Liouville sur ces sous-variétés pour $2 \le k \le m_\ell - 1$ (le cas $k = m_\ell$ est celui dont l'intégrabilité a été etablie dans [4], tandis que le réseau de Toda périodique usuel correspond à $k = 1$.) Dans cette Note, nous annonçons des résultats qui établissent la validité de cette conjecture pour $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$.

Dans ce qui suit, nous parlerons simplement du flot périodique de Kostant–Toda, au lieu de réseau périodique de Kostant–Toda complet. Nous réservons la terminologie de réseau au cas limite $k = 1$. Le lecteur verra que le nombre k est égal à

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la largeur de bande inférieure du terme constant L_0 de l'opérateur de Lax $L(z) = E_{n1}z + L_0 + L_{-1}z^{-1}$, qui est également le nombre de superdiagonales non nulles dans le coin supérieur droit de la matrice triangulaire supérieure L_{-1}. En effet, $L(z)$ correspond à un operateur de différence périodique qui agit sur $\ell^\infty(\mathbb{Z})$ comme une matrice bande infinie de largeur de bande inférieure égale à k et de largeur de bande supérieure égale à 1 avec des 1 sur la première superdiagonale [13]. Nous appellerons k le niveau de la matrice de lacets $L(z)$. Il convient de souligner que notre approche dans ce travail est basée sur la délimitation des espaces de phase comme orbites coadjointes. À cet égard, il est intéressant de noter comment certaines conditions arithmétiques reliant k, les exposants m_i et $d_i = m_i + 1$ jouent un rôle dans notre analyse. De ces conditions arithmétiques, le plus grand diviseur commun de n et k émerge et figure dans la formule de la dimension des orbites coadjointes génériques. Il y a plusieurs questions évidentes que nous n'abordons pas dans cette Note. L'une d'elles est la construction des variables action–angle. L'autre est l'intégrabilité pour d'autres algèbres de Lie simples. Nous aborderons ces questions et d'autres dans des publications ultérieures.

1. Introduction

The nonperiodic Toda lattice is a system of particles on the line, which has two distinct Lax representations, associated with different splittings of the same Lie algebra into Lie subalgebras [1,11], and has given rise to natural extensions on bigger phase spaces that are still Liouville integrable [5–7,9]. For the periodic version of the Toda lattice [8], while it is also well known that two distinct Lax representations exist, an extension (in Kostant's realization) to bigger phase spaces was however only introduced recently in [4] and this was done in the context of complex simple Lie algebras $\mathfrak{g}$. The resulting system, which the author in [4] called the periodic full-Kostant–Toda lattice, was shown to be completely integrable on the underlying Poisson manifold. Moreover, at the end of [4], the author introduced a family of Poisson submanifolds parameterized by $1 \le k \le m_\ell$, where m_ℓ is the largest exponent of $\mathfrak{g}$, and made the conjecture that the system is also Liouville integrable on these submanifolds for $2 \le k \le m_\ell - 1$. (The $k = m_\ell$ case is the full case whose integrability was established in [4], while the usual periodic Toda lattice corresponds to $k = 1$.) In this note, we announce results that will establish the validity of this conjecture for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$.

In what follows, instead of calling the system the periodic full Kostant–Toda lattice, we will refer to it as the periodic Kostant–Toda flow, reserving the word lattice to the limiting case where $k = 1$. As the reader will see, the number k is equal to the lower bandwidth of the constant term L_0 of the Lax operator $L(z) = E_{n1}z + L_0 + L_{-1}z^{-1}$, which is also equal to the number of nonzero superdiagonals in the upper right-hand corner of the upper triangular matrix L_{-1}. Indeed, $L(z)$ corresponds to a periodic difference operator that acts on $\ell^\infty(\mathbb{Z})$ as an infinite band matrix with lower bandwidth equal to k and upper bandwidth equal to 1 with 1's on the first superdiagonal [13]. We will call k the level of the matrix loop $L(z)$. We should point out that our approach in this work is based on delineating the phase spaces as coadjoint orbits. In this connection, it is interesting to note how certain arithmetic conditions involving k, the exponents m_i, and $d_i = m_i + 1$ play a role in our analysis. From these arithmetic conditions, the greatest common divisor of n and k emerges and this figures in the formula for the dimension of the generic coadjoint orbits. There are several obvious questions that we do not address in this Note. One of these is the construction of action–angle variables. The other one is the integrability for other simple Lie algebras. We will address these and other issues in subsequent publications.

2. The periodic Kostant–Toda flow on $\mathfrak{sl}(n, \mathbb{C})$

Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ with pairing $(\cdot, \cdot)$ given by $(X, Y) = \mathrm{tr}(XY)$ and let $\mathfrak{h}$ be its Cartan subalgebra consisting of diagonal matrices with zero trace. For $1 \le i, j \le n$, let E_{ij} denote the $n \times n$ matrix whose (i, j) entry is 1 and zero elsewhere. Then we have the following Lie subalgebras of $\mathfrak{g}$:

$$\mathfrak{n}_- = \bigoplus_{i>j} \mathbb{C}E_{ij}, \qquad \mathfrak{n}_+ = \bigoplus_{i<j} \mathbb{C}E_{ij}, \qquad \mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm. \tag{2.1}$$

Moreover, we have the (vector space) direct sum decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}_+. \tag{2.2}$$

Put $\epsilon = \sum_{i=1}^{n-1} E_{i,i+1}$ and let $\Pi_{\mathfrak{n}_-}$ be the projection onto $\mathfrak{n}_-$ relative to the decomposition in (2.2), then the periodic Kostant–Toda flow is given by the Lax equation:

$$\dot{L}(z) = \big[L(z), B(z)\big], \tag{2.3}$$

where

$$L(z) = E_{n1}z + L_0 + L_{-1}z^{-1}, \quad L_0 \in \epsilon + \mathfrak{b}_-, \; L_{-1} \in \mathfrak{n}_+ \tag{2.4}$$

and

$$B(z) = \Pi_{\mathfrak{n}_-} L_0 + L_{-1}z^{-1}. \tag{2.5}$$

As a special case of the general result in [4] valid for all simple Lie algebras, the Lax equation above is Hamiltonian and Liouville integrable for generic matrices $L_0 \in \epsilon + \mathfrak{b}_-$, $L_{-1} \in \mathfrak{n}_+$. Now let

$$\mathfrak{g}^{(j)} = \bigoplus_{i=1}^{n-j} \mathbb{C}E_{i,i+j}, \qquad \mathfrak{g}^{(-j)} = \bigoplus_{i=1}^{n-j} \mathbb{C}E_{i+j,i}, \quad j = 1, \cdots, n-1, \tag{2.6}$$

and let $\mathfrak{g}^{(0)} := \mathfrak{h}$. Towards the end of [4], the author introduced the Poisson submanifolds

$$\mathcal{T}^{(k)} = \left\{ E_{n1}z + L_0 + L_{-1}z^{-1} \mid L_0 \in \epsilon + \oplus_{j=0}^{k} \mathfrak{g}^{(-j)}, L_{-1} \in \oplus_{j=1}^{k} \mathfrak{g}^{(n-j)} \right\}, \tag{2.7}$$

of $\mathcal{T} = \mathcal{T}^{(n-1)}$ and conjectured that the periodic Kostant–Toda flow restricted to $\mathcal{T}^{(k)}$ is also Liouville integrable for each $1 < k \le n-2$. We will see that this is the case in subsequent sections.

3. Lie groups and coadjoint orbits

Let G, N_-, and B_+ be the connected and simply-connected Lie groups that integrate the Lie algebras $\mathfrak{g}, \mathfrak{n}_-$ and $\mathfrak{b}_+$ respectively. In contrast to the framework in [4], we will use the loop algebra $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ that has an accompanying loop group LG. In this way, we find that there is no need to use the notion of integrability in the context of Poisson manifolds [2], as the phase spaces of the periodic Kostant–Toda flows are coadjoint orbits. To define the Lie subalgebras of $L\mathfrak{g}$ that we need, let $L^+\mathfrak{g}$ (resp. $L^-\mathfrak{g}$) denote the Lie subalgebra of $L\mathfrak{g}$ consisting of loops of the form $X(z) = \sum_{j>0} X_j z^j$ (resp. $X(z) = \sum_{j<0} X_j z^j$), $z \in S^1$ and denote by L^+G (resp. L^-G) the corresponding Lie group. Then we have the decomposition

$$L\mathfrak{g} = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{b}}_+, \tag{3.1}$$

where $\widetilde{\mathfrak{n}}_-$ and $\widetilde{\mathfrak{b}}_+$ are the Lie subalgebras of $L\mathfrak{g}$, given respectively by

$$\begin{aligned} \widetilde{\mathfrak{n}}_- &= \left\{ X = X_- + X_0 \in L\mathfrak{g} \mid X_- \in L^-\mathfrak{g}, X_0 \in \mathfrak{n}_- \right\}, \\ \widetilde{\mathfrak{b}}_+ &= \left\{ X = X_+ + X_0 \in L\mathfrak{g} \mid X_+ \in L^+\mathfrak{g}, X_0 \in \mathfrak{b}_+ \right\}. \end{aligned} \tag{3.2}$$

Denote by $\Pi_{\widetilde{\mathfrak{b}}_+}$ and $\Pi_{\widetilde{\mathfrak{n}}_-}$ the projection maps relative to the splitting in (3.1), then $R = \Pi_{\widetilde{\mathfrak{b}}_+} - \Pi_{\widetilde{\mathfrak{n}}_-}$ is a solution to the modified Yang–Baxter equation (mYBE) so that we can equip $L\mathfrak{g}$ with the R-bracket $[\cdot,\cdot]_R$ [12]. In what follows, we will use the following nondegenerate ad-invariant pairing on $L\mathfrak{g}$:

$$(X, Y)_{L\mathfrak{g}} = \oint_{|z|=1} \left(X(z), Y(z) \right) \frac{\mathrm{d}z}{2\pi i z}, \tag{3.3}$$

and the Lie algebra $(L\mathfrak{g}, [\cdot,\cdot]_R)$ will be denoted by $\widetilde{\mathfrak{g}}$. Now let $\widetilde{\mathfrak{g}}^*$ be the algebraic dual of $\widetilde{\mathfrak{g}}$, we will make the identification $\widetilde{\mathfrak{g}}^* \simeq \widetilde{\mathfrak{g}}$ through the invariant pairing in (3.3). Then $\mathcal{T}$ is a Poisson submanifold of $\widetilde{\mathfrak{g}} \simeq \widetilde{\mathfrak{g}}^*$ equipped with the Lie–Poisson structure $\{F_1, F_2\}(X) = (X, [\mathrm{d}F_1(X), \mathrm{d}F_2(X)]_R)_{L\mathfrak{g}}$ and the Lax equation in (2.3) is generated by the Hamiltonian $H(L) = \frac{1}{2}(L, L)_{L\mathfrak{g}}$ in the induced structure on $\mathcal{T}$.

We next introduce the Lie subgroups of LG that integrate $\widetilde{\mathfrak{n}}_-$ and $\widetilde{\mathfrak{b}}_+$:

$$\begin{aligned} \widetilde{N}_- &= \left\{ g \in LG \mid g(z) = n\widetilde{g}(z), n \in N_-, \widetilde{g} \in L^-G \right\}, \\ \widetilde{B}_+ &= \left\{ g \in LG \mid g(z) = b\widetilde{g}(z), b \in B_+, \widetilde{g} \in L^+G \right\}. \end{aligned} \tag{3.4}$$

Put

$$\widetilde{G} = \left\{ g \in LG \mid g = g_+ g_-^{-1}, \text{ where } g_+ \in \widetilde{B}_+, g_- \in \widetilde{N}_- \right\}. \tag{3.5}$$

Then following the procedure in [6], we can endow $\widetilde{G}$ with a Lie group structure by defining the multiplication $g * h = g_+ h g_-^{-1}$ and $(\widetilde{G}, *)$ is a Lie group that corresponds to the Lie algebra $\widetilde{\mathfrak{g}}$. Moreover, the coadjoint action of $\widetilde{G}$ on $\widetilde{\mathfrak{g}}^* \simeq \widetilde{\mathfrak{g}}$ is given by

$$Ad^*_{\widetilde{G}}\left(g^{-1}\right)X = \Pi_{\widetilde{\mathfrak{b}}_+^\perp} Ad_{g_-} X + \Pi_{\widetilde{\mathfrak{n}}_-^\perp} Ad_{g_+} X, \tag{3.6}$$

where $\Pi_{\widetilde{\mathfrak{b}}_+^\perp}, \Pi_{\widetilde{\mathfrak{n}}_-^\perp}$ are the projection maps relative to the splitting $L\mathfrak{g} = \widetilde{\mathfrak{b}}_+^\perp \oplus \widetilde{\mathfrak{n}}_-^\perp$. The coadjoint orbits of $\widetilde{G}$ which are relevant for us are finite-dimensional, and can be identified with coadjoint orbits of a finite-dimensional Lie group. Indeed, the subgroup B_+ acts on $\mathfrak{g}$ via $\rho(b)X = Ad_b X = bXb^{-1}$. Let S be the semidirect product group $B_+ \ltimes_\rho \mathfrak{g}$. If $\mathfrak{s}$ is the Lie algebra of S and we identify its dual $\mathfrak{s}^*$ with $(\epsilon + \mathfrak{b}_-) \times \mathfrak{g}$, then we have the following result.

Proposition 3.1. *(a) For each $1 \le k \le n-1$, $\mathcal{T}^{(k)}$ is invariant under $Ad^*_{\widetilde{G}}(g^{-1})$ for any $g \in \widetilde{G}$. Moreover, if $L \in \mathcal{T} = \mathcal{T}^{(n-1)}$, the coadjoint orbit $\mathcal{O}_L = \{Ad^*_{\widetilde{G}}(g^{-1})L \mid g \in \widetilde{G}\}$ can be identified with the coadjoint orbit of S through $(L_0, L_{-1}) \in \mathfrak{s}^*$.*
(b) Let $\gcd(n,k)$ be the greatest common divisor of n and k. Then for a generic element $L \in \mathcal{T}^{(k)}$, there exists $g \in \widetilde{G}$ such that $Ad^*_{\widetilde{G}}(g^{-1})L$ is a loop of the form

$$E_{n1}z + \epsilon + \sum_{i=1}^{k} C_i E_{i+1,1} + \sum_{j=2}^{\gcd(n,k)} C_{k+j-1} E_{k+j,j} + \sum_{j=\gcd(n,k)+1}^{n-k} E_{k+j,j} + \sum_{j=2}^{\gcd(n,k)} E_{k+j-1,j} + \left(\sum_{i=1}^{k} E_{i,n-k+i}\right) z^{-1}, \tag{3.7}$$

where, by convention, the two terms involving $\sum_{j=2}^{\gcd(n,k)}$ are zero if $\gcd(n,k)=1$. Thus the dimension of the generic coadjoint orbits in $\mathcal{T}^{(k)}$ is given by

$$\dim \mathcal{T}^{(k)} - \big(k + \gcd(n,k) - 1\big) = nk + n - k - \gcd(n,k). \tag{3.8}$$

Remark 1. With the exception of Proposition 3.1(b), what we do in this section can also be done in the general context of general simple Lie algebras.

4. Liouville integrability

Since $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, its exponents are $m_i = i$ for $i = 1, \cdots, n-1$. Let $d_i = m_i + 1$, $i = 1, \cdots, n-1$, then the Coxeter number is given by $h = d_{n-1} = n$. In order to discuss the integrability of the periodic Kostant–Toda flow on generic coadjoint orbits in $\mathcal{T}^{(k)}$, we introduce the polynomial

$$\det\big(L(z) - wI\big) = (-1)^n w^n + \sum_{r=1}^{n-1} \left(\sum_{j=0}^{r} I_{rj}(L) z^{-j} + cz\delta_{r,n-1} \right) w^{n-1-r} \tag{4.1}$$

where we have used Proposition 10 in [4], and where c is a constant. For us, since $L \in \mathcal{T}^{(k)}$, $2 \le k \le n-2$, a more refined analysis of the above expansion yields the following result.

Lemma 4.1. *(a) $c = (-1)^{n+1}$.*
(b) For $L \in \mathcal{T}^{(k)}$, we have $I_{rj}(L) \equiv 0$ for $j > \lfloor \frac{kd_r}{h} \rfloor$. Moreover, if $I_{rj^*}(L) \neq 0$, then for all $j < j^*$, I_{rj} is not a Casimir on $\mathcal{T}^{(k)}$.
(c) For $L \in \mathcal{T}^{(k)}$, we have $I_{r,\lfloor \frac{kd_r}{h} \rfloor}(L) \neq 0$.
(d) Let $E = \{r \mid \lfloor \frac{kd_r}{h} \rfloor = 1 + \lfloor \frac{km_{r-1}}{h} \rfloor\}$. Then $I_{r,\lfloor \frac{kd_r}{h} \rfloor}$ is a Casimir on $\mathcal{T}^{(k)}$ if and only if $r \in E$.
(e) $\mathrm{Card}\,(E) = k + \gcd(n,k) - 1$. Hence $\mathrm{Card}\,(E)$ is the same as the number of C's in Proposition 3.1 (b).

Remark 2. If we replace $\mathfrak{sl}(n, \mathbb{C})$ by a general complex simple Lie algebra, and the elementary symmetric functions with a generating set of primitive invariants of the algebra of ad-invariant polynomial functions on the simple Lie algebra, then the analog of statement (b) above is also true.

Theorem 4.2. *(a) Generic coadjoint orbits of $\widetilde{G}$ in $\mathcal{T}^{(k)}$ are level sets of the Casimirs $I_{r,\lfloor \frac{kd_r}{h} \rfloor}$, $r \in E$.*
(b) The functions I_{rj}, $0 \le j \le \lfloor \frac{kd_r}{h} \rfloor - 1$, $1 \le r \le n-1$ and $I_{r,\lfloor \frac{kd_r}{h} \rfloor}$, $r \in \{1, \cdots, n-1\} \setminus E$, provide a collection of $\frac{1}{2}(nk + n - k - \gcd(n,k))$ integrals in involution on the generic coadjoint orbits in $\mathcal{T}^{(k)}$. Furthermore, the integrals are functionally independent on an open dense set in each generic coadjoint orbit.

Remark 3. Note that the projective curve C whose affine part is defined by the equation $\det(zL(z) - wI) = 0$ is irreducible but is never smooth because the point $[\zeta : z : w] = [1 : 0 : 0]$ is a multiple point of order $n-k$. By Baker's theorem [3,10] or otherwise [13], the genus of the nonsingular model $\widetilde{C}$ of C is given by $g(\widetilde{C}) = \frac{1}{2}(nk + n - k - \gcd(n,k))$. In this case, we establish the functional independence of the integrals by computing the Poisson brackets between the integrals and the (putative) angles that are constructed from a basis of holomorphic 1-forms on $\widetilde{C}$ and an appropriate divisor (cf. [6]).

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