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L^p-Taylor approximations characterize the Sobolev space $W^{1,p}$

Les développements de Taylor-L^p caractérisent l'espace de Sobolev $W^{1,p}$

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A R T I C L E   I N F O

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A B S T R A C T

In this note, we introduce a variant of Calderón and Zygmund's notion of L^p-differentiability – an *L^p-Taylor approximation*. Our first result is that functions in the Sobolev space $W^{1,p}(\mathbb{R}^N)$ possess a first-order L^p-Taylor approximation. This is in analogy with Calderón and Zygmund's result concerning the L^p-differentiability of Sobolev functions. In fact, the main result we announce here is that the first-order L^p-Taylor approximation characterizes the Sobolev space $W^{1,p}(\mathbb{R}^N)$, and therefore implies L^p-differentiability. Our approach establishes connections between some characterizations of Sobolev spaces due to Swanson using Calderón–Zygmund classes with others due to Bourgain, Brézis, and Mironescu using nonlocal functionals with still others of the author and Mengesha using nonlocal gradients. That any two characterizations of Sobolev spaces are related is not surprising; however, one consequence of our analysis is a simple condition for determining whether a function of bounded variation is in a Sobolev space.

R É S U M É

Dans cette note, nous introduisons une variante de la notion de Calderón et Zygmund de différentiabilité L^p – un *développement de Taylor-L^p*. Notre premier résultat est que les fonctions de l'espace de Sobolev $W^{1,p}(\mathbb{R}^N)$ possèdent un développement de Taylor-L^p au premier ordre. C'est un analogue du résultat de Calderón et Zygmund concernant la différentiabilité L^p des fonctions de Sobolev. En fait, le résultat principal que nous annonçons ici est que le développement de Taylor-L^p au premier ordre caractérise l'espace de Sobolev $W^{1,p}(\mathbb{R}^N)$, et donc implique la différentibilité L^p. Notre approche établit des liens entre les caractérisations des espaces de Sobolev dues à Swanson, qui utilisent les classes de Calderón–Zygmund, celles dues à Bourgain, Brézis et Mironescu, qui utilisent des fonctionnelles non locales, et celles dues à l'auteur et à Mengesha, qui utilisent des gradients non locaux. Que les différentes caractérisations des espaces de Sobolev soient reliées n'est pas surprenant ; cependant, notre analyse donne une condition simple pour déterminer si une fonction à variation bornée est dans un espace de Sobolev.

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1. Introduction and main results

L^p-differentiability was introduced by Calderón and Zygmund in their study of the local properties of solutions of elliptic differential equations [3,4]. It is a natural extension of classical differentiability in that it relaxes the requirement of the existence of a locally linear map in a uniform sense to its existence in an averaged sense. As the Sobolev spaces arise readily in the study of partial differential equations, it is not surprising that Sobolev functions possess an L^p-derivative. The following theorem asserting this fact was proven by Calderón and Zygmund [4, Theorem 12] (for a modern reference, see the monograph of Evans and Gariepy [5]).

Theorem 1.1 *(Calderón and Zygmund). Suppose $1 \leq p < \infty$ and $f \in W^{1,p}(\mathbb{R}^N)$. Then*

$$\lim_{\epsilon \to 0} \left(\frac{1}{\epsilon^{pq}} \fint_{B(0,\epsilon)} |f(x+h) - f(x) - \nabla f(x) \cdot h|^{pq} \, \mathrm{d}h \right)^{\frac{1}{pq}} = 0 \tag{1}$$

for $\mathcal{L}^N$ almost every $x \in \mathbb{R}^N$, where $1 \leq q \leq \frac{N}{N-p}$ if $1 \leq p < N$, $1 \leq q < \infty$ if $p = N$, and $1 \leq q \leq \infty$ if $p > N$ (when $q = \infty$ the left side of (1) is understood to be $L^\infty_h(B(0,\epsilon))$ norm applied to the integrand).

In the language of Calderón and Zygmund, Theorem 1.1 states that $f \in W^{1,p}(\mathbb{R}^N)$ implies $f \in t^{1,pq}(x)$ (which is essentially defined by the condition (1)) for almost every $x \in \mathbb{R}^N$. The converse is false, readily seen through the "improved" L^{pq}-differentiability of functions in the Sobolev space $W^{1,p}(\mathbb{R}^N)$. A natural question is then whether one can characterize the Sobolev spaces in the spirit of condition (1). Several results have been given in this direction using the class $T^{1,p}(x)$ (for precise definitions of $t^{1,p}(x)$ and $T^{1,p}(x)$, we refer to Ziemer [10, Chapter 3, p. 132]), also introduced by Calderón and Zygmund. We mention, for instance, a sort of converse to Theorem 1.1 due to Bagby and Ziemer [1], as well as some characterizations due to Swanson [8,9]. As the classes of Calderón and Zygmund alone do not provide one with necessary and sufficient conditions for inclusion in a Sobolev space, the latter papers explore additional assumptions that enable one to capture the essential property that characterizes Sobolev functions (and therefore are deeply connected to our viewpoint). Here we will commence by examining the condition (1) from a different perspective.

Our approach begins with the observation that the key ingredients to Theorem 1.1 are the Sobolev embedding theorem and the Lebesgue differentiation theorem for L^p functions. As in the statement of Theorem 1.1, the latter is typically stated as a pointwise almost everywhere convergence result. However, a variant of the Lebesgue differentiation theorem is the following $L^p(\mathbb{R}^N)$ convergence result. If $u \in L^p(\mathbb{R}^N)$ and we define

$$h_\epsilon(x) := \left(\fint_{B(x,\epsilon)} |u(x) - u(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}},$$

then one has the convergence $h_\epsilon \to 0$ in $L^p(\mathbb{R}^N)$ as $\epsilon \to 0$.

From this perspective, it would be natural to expect an analogous L^p-type convergence to be true for functions in the Sobolev space $W^{1,p}(\mathbb{R}^N)$. We therefore introduce the following definition.

Definition 1.1. A function $f : \mathbb{R}^N \to \mathbb{R}$ is said to have a first order L^p-Taylor approximation if $f \in L^p(\mathbb{R}^N)$ and there exists a function $v \in L^p(\mathbb{R}^N; \mathbb{R}^N)$ such that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x) - v(x) \cdot h|^p}{|h|^p} \, \mathrm{d}h \, \mathrm{d}x = 0. \tag{2}$$

Our first result is the following theorem, which asserts that functions in the Sobolev space $W^{1,p}(\mathbb{R}^N)$ possess a first-order L^p-Taylor approximation.

Theorem 1.2. *Let $1 \leq p < \infty$ and $f \in W^{1,p}(\mathbb{R}^N)$. Then f has a first order L^p-Taylor approximation, and moreover, one has the stronger estimate*

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \left(\fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x) - \nabla f(x) \cdot h|^{pq}}{|h|^{pq}} \, \mathrm{d}h \right)^{\frac{1}{q}} \mathrm{d}x = 0, \tag{3}$$

where $1 \leq q \leq \frac{N}{N-p}$ if $1 \leq p < N$, $1 \leq q < \infty$ if $p = N$, and $1 \leq q \leq \infty$ if $p > N$.

What is quite surprising, and one of the main results we announce here, is that this property—the existence of a Taylor approximation in this L^p sense—in fact characterizes Sobolev functions. Precisely, we have the following theorem characterizing the Sobolev space $W^{1,p}(\mathbb{R}^N)$ in terms of the $L^p(\mathbb{R}^N)$ convergence (2).

Theorem 1.3. *Let $1 \le p < +\infty$ and suppose $f \in L^p(\mathbb{R}^N)$. Then $f \in W^{1,p}(\mathbb{R}^N)$ if and only if f has a first order L^p-Taylor approximation.*

Actually, for $1 < p < +\infty$, we will see in the proof that it is a consequence of the work of Bourgain, Brézis, and Mironescu [2], stating that the assumption of the finiteness of the limit (2) is sufficient to deduce that $f \in W^{1,p}(\mathbb{R}^N)$. Alternatively, one can follow the approach of Swanson in [8] or the author and Mengesha [7] in deducing the result. The unifying principle of these several works is the use of an appropriate form of integration by parts, which is done along an approximating sequence, and to utilize a condition of the type (2) as a uniform bound in passing the limit. Here, the approximations could be through standard mollification, as is the case with Swanson [8], or nonlocal functionals as was the approach of Bourgain, Brézis, and Mironescu [2], or nonlocal gradients, as shown by the author and Mengesha in [7].

When $p = 1$, we observe that Theorem 1.3 characterizes $W^{1,1}(\mathbb{R}^N)$, analogously to the paper of Swanson [9] and in contrast to the paper of Bourgain, Brézis, and Mironescu [2], which characterizes $BV(\mathbb{R}^N)$. This difference is a consequence of the stricter requirement of possessing a first-order L^1-Taylor approximation, which, unlike L^1-differentiability, generically does not hold for functions of bounded variation. However, one still has the following theorem concerning the finiteness of the integrated infinitesimal, which provides one with a characterization of the space of functions of bounded variation in a similar spirit to that of [2].

Theorem 1.4. *Suppose $f \in L^1(\mathbb{R}^N)$. Then $f \in BV(\mathbb{R}^N)$ if and only if there exists a function $v \in L^1(\mathbb{R}^N;\mathbb{R}^N)$ such that*

$$\limsup_{\epsilon \to 0} \int_{\mathbb{R}^N} \left(\fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x) - v(x)\cdot h|^q}{|h|^q} \,\mathrm{d}h \right)^{\frac{1}{q}} \mathrm{d}x < +\infty \tag{4}$$

for any $1 \le q \le \frac{N}{N-1}$.

We will see in the proof of Theorem 1.3 that the assumption (2) is a simple condition that allows us to obtain the equi-integrability (actually, strong convergence) of the approximating sequence, allowing us to conclude that there is no singular portion of the measure limit that arises in the non-reflexive space $L^1(\mathbb{R}^N;\mathbb{R}^N)$. As a result, we have the following condition for determining whether a function of bounded variation is in a Sobolev space.

Corollary 1.5. *Suppose $f \in BV(\mathbb{R}^N)$. Then $f \in W^{1,1}(\mathbb{R}^N)$ if and only if (2) holds with $p = 1$.*

Remark 1.1. For a general $f \in BV(\mathbb{R}^N)$, the limit in Eq. (4) can be bounded above by the total mass of the singular part of the measure Df.

We will shortly give a proof of Theorems 1.2 and 1.3 in the case $1 \le p < N$, while the proofs of the regime $p \ge N$, Theorem 1.4, and Corollary 1.5 will be deferred to a later work. First, let us state without proof the following Lemma, which is a variation of a calculation implicit in Evans and Gariepy [5, Chapter 6, p. 231].

Lemma 1.6. *Suppose $f \in W^{1,p}_{loc}(\mathbb{R}^N)$ for some $1 \le p < \infty$, and that $1 \le q \le \frac{N}{N-p}$ if $1 \le p < N$. Then there exists a $C = C(p,q,N) > 0$ such that for all $0 < t < 1$*

$$\frac{1}{t^{N+pq}} \int_{B(0,t)} |f(x+h) - f(x) - \nabla f(x)\cdot h|^{pq} \,\mathrm{d}h \le C \left(\fint_{B(0,t)} |\nabla f(x+h) - \nabla f(x)|^p \,\mathrm{d}h \right)^q + C \left(\int_0^1 \fint_{B(0,st)} |\nabla f(x+sz) - \nabla f(x)|^p \,\mathrm{d}z\,\mathrm{d}s \right)^q.$$

We now give a proof of Theorem 1.2.

Proof. For any $0 < \epsilon < 1$, we expand the integrand on concentric rings

$$\fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x) - \nabla f(x) \cdot h|^{pq}}{|h|^{pq}} \, dh$$

$$= \sum_{i=0}^{\infty} \frac{1}{\epsilon^N |B(0,1)|} \int_{B(0,\frac{\epsilon}{2^i}) \setminus B(0,\frac{\epsilon}{2^{i+1}})} \frac{|f(x+h) - f(x) - \nabla f(x) \cdot h|^{pq}}{|h|^{pq}} \, dh.$$

We now make estimates for $i \in \mathbb{N}$ fixed. We have

$$\frac{1}{\epsilon^N} \int_{B(0,\frac{\epsilon}{2^i}) \setminus B(0,\frac{\epsilon}{2^{i+1}})} \frac{|f(x+h) - f(x) - \nabla f(x) \cdot h|^{pq}}{|h|^{pq}} \, dh$$

$$\leq \frac{1}{\epsilon^N} \left(\frac{\epsilon}{2^{i+1}}\right)^{-pq} \int_{B(0,\frac{\epsilon}{2^i}) \setminus B(0,\frac{\epsilon}{2^{i+1}})} |f(x+h) - f(x) - \nabla f(x) \cdot h|^{pq} \, dh$$

$$\leq \frac{2^{pq}}{2^{iN}} \left(\frac{\epsilon}{2^i}\right)^{-N-pq} \int_{B(0,\frac{\epsilon}{2^i})} |f(x+h) - f(x) - \nabla f(x) \cdot h|^{pq} \, dh.$$

Lemma 1.6 further implies that

$$\left(\frac{\epsilon}{2^i}\right)^{-N-pq} \int_{B(0,\frac{\epsilon}{2^i})} |f(x+h) - f(x) - \nabla f(x) \cdot h|^{pq} \, dh \leq C \left(\fint_{B(0,\frac{\epsilon}{2^i})} |\nabla f(x+h) - \nabla f(x)|^p \, dh \right)^q$$

$$+ C \left(\int_0^1 \fint_{B(0,s\frac{\epsilon}{2^i})} |\nabla f(x+sz) - \nabla f(x)|^p \, dz \, ds \right)^q.$$

Therefore, summing in i and applying the basic inequality $\left(\sum_i |a_i|\right)^{\frac{1}{q}} \leq \sum_i |a_i|^{\frac{1}{q}}$ (which follows from the subadditivity of the function $t \mapsto t^{\frac{1}{q}}$), we have

$$\left(\fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x) - \nabla f(x) h|^{pq}}{|h|^{pq}} \, dh \right)^{\frac{1}{q}} \leq C \sum_{i=0}^{\infty} \left(\frac{1}{2^i}\right)^{N/q} \fint_{B(0,\frac{\epsilon}{2^i})} |\nabla f(x+h) - \nabla f(x)|^p \, dh$$

$$+ C \sum_{i=0}^{\infty} \left(\frac{1}{2^i}\right)^{N/q} \int_0^1 \fint_{B(0,s\frac{\epsilon}{2^i})} |\nabla f(x+sz) - \nabla f(x)|^p \, dz \, ds.$$

Integrating the preceding inequality over $x \in \mathbb{R}^N$ and making use of Tonelli's theorem, we obtain

$$\int_{\mathbb{R}^N} \left(\fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x) - \nabla f(x) h|^{pq}}{|h|^{pq}} \, dh \right)^{\frac{1}{q}} dx$$

$$\leq C \sum_{i=0}^{\infty} \left(\frac{1}{2^i}\right)^{N/q} \fint_{B(0,\frac{\epsilon}{2^i})} \int_{\mathbb{R}^N} |\nabla f(x+h) - \nabla f(x)|^p \, dx \, dh$$

$$+ C \sum_{i=0}^{\infty} \left(\frac{1}{2^i}\right)^{N/q} \int_0^1 \fint_{B(0,s\frac{\epsilon}{2^i})} \int_{\mathbb{R}^N} |\nabla f(x+sz) - \nabla f(x)|^p \, dx \, dz \, ds.$$

However, if $h, z \in B(0,\epsilon)$ and $s \in (0,1)$ we have

$$\max\left\{\int_{\mathbb{R}^N} |\nabla f(x+h) - \nabla f(x)|^p \, dx, \int_{\mathbb{R}^N} |\nabla f(x+sz) - \nabla f(x)|^p \, dx\right\} \leq \sup_{w\in B(0,\epsilon)} \int_{\mathbb{R}^N} |\nabla f(x+w) - \nabla f(x)|^p \, dx,$$

and observe that this bound is independent of $i \in \mathbb{N}$. Thus,

$$\int_{\mathbb{R}^N} \left(\fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x) - \nabla f(x)h|^{pq}}{|h|^{pq}} \, dh \right)^{\frac{1}{q}} dx \leq \sup_{w\in B(0,\epsilon)} \int_{\mathbb{R}^N} |\nabla f(x+w) - \nabla f(x)|^p \, dx \left(C \sum_{i=0}^{\infty} \left(\frac{1}{2^i}\right)^{N/q} \right).$$

As the infinite series is summable, the result follows from sending $\epsilon \to 0$ and using continuity of translation in $L^p(\mathbb{R}^N)$. □

We conclude with a proof of Theorem 1.3.

Proof. As we have shown that $f \in W^{1,p}(\mathbb{R}^N)$ implies the L^p-convergence (2), it remains to show the converse. We first treat the case $1 < p < +\infty$. Let us therefore suppose that there exists a function $v \in L^p(\mathbb{R}^N; \mathbb{R}^N)$ such that (2) holds. We then estimate

$$\begin{aligned}
&\int_{\mathbb{R}^N} \fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x)|^p}{|h|^p} \, dh \, dx \\
&\quad \leq 2^{p-1} \int_{\mathbb{R}^N} \fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x) - v(x)\cdot h|^p}{|h|^p} \, dh \, dx \\
&\quad + 2^{p-1} \int_{\mathbb{R}^N} \fint_{B(0,\epsilon)} \left| v(x) \cdot \frac{h}{|h|} \right|^p dh \, dx.
\end{aligned}$$

Now our assumption is that the first term on the right-hand side tends to zero as $\epsilon \to 0$, while the second is bounded by a constant times the L^p norm of v. We then have that for a sequence $\epsilon_n \to 0$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \fint_{B(0,\epsilon_n)} \frac{|f(x+h) - f(x)|^p}{|h|^p} \, dh \, dx < +\infty,$$

and so by a version of the result of Bourgain, Brézis, and Mironescu [2] for $\mathbb{R}^N$, we conclude that $f \in W^{1,p}(\mathbb{R}^N)$.

For the case $p = 1$, a similar argument allows us to deduce that $f \in BV(\mathbb{R}^N)$ (and can be used to demonstrate one direction of Theorem 1.4). It therefore remains to show that $v = Df$ in the sense of distributions and we can conclude $f \in W^{1,1}(\mathbb{R}^N)$. However, we observe that $f \in BV(\mathbb{R}^N)$ implies that the nonlocal gradient

$$\mathcal{G}_\epsilon f(x) := N \fint_{B(0,\epsilon)} \frac{f(x+h) - f(x)}{|h|} \frac{h}{|h|} \, dh$$

is well defined as a Lebesgue integral and that $\mathcal{G}_\epsilon f \in L^1(\mathbb{R}^N; \mathbb{R}^N)$ (see the paper of the author and Mengesha [7]). If we could show that $\mathcal{G}_\epsilon f \to v$ in $L^1(\mathbb{R}^N; \mathbb{R}^N)$, we would be finished, since the convergence $\mathcal{G}_\epsilon f \overset{*}{\rightharpoonup} Df$ in $\left(C_0(\mathbb{R}^N; \mathbb{R}^N)\right)'$ would then imply that $v = Df$ in the sense of distributions, which is the desired result. However, we observe that

$$v_i(x) N \fint_{B(0,\epsilon)} \frac{h_i h_j}{|h|^2} = v_i(x)\delta_{ij}$$

and therefore we have

$$(\mathcal{G}_\epsilon f)_i(x) - v_i(x) = N \fint_{B(0,\epsilon)} \frac{f(x+h) - f(x) - v(x)\cdot h}{|h|} \frac{h_i}{|h|} \, dh,$$

and thus we can estimate

$$\int_{\mathbb{R}^N} |(\mathcal{G}_\epsilon f)_i(x) - v_i(x)| \leq N \int_{\mathbb{R}^N} \fint_{B(0,\epsilon)} \frac{|f(x+h) - f(x) - v(x)\cdot h|}{|h|} \, dh,$$

which tends to zero by our assumption and the result is demonstrated. □

In forthcoming work we will complete the proofs of our claims, as well as discuss several applications and variations of the result in the context of the assumptions of Bourgain, Brézis, and Mironescu [2]. In particular, we will specifically address the use of different approximations of the identity, local convergence results, and the case of characterizations for domains $\Omega \subset \mathbb{R}^N$ open. We also will give a proof of a result related to a claim in the paper [6].

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