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Logarithmic residues along plane curves

Résidus logarithmiques des courbes planes

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ABSTRACT

Let $(D,0) \subset (\mathbb{C}^2,0)$ be a plane curve germ defined by a reduced equation f. We prove that a fractional ideal I of D satisfies a symmetry property with its dual, and then apply it to study the behavior of the module of logarithmic residues of D in equisingular deformations.

RÉSUMÉ

Soit $(D,0) \subset (\mathbb{C}^2,0)$ un germe de courbe plane défini par une équation réduite f. On démontre qu'un idéal fractionnaire I de D vérifie une propriété de symétrie avec son dual, et on applique ce résultat à l'étude du comportement du module des résidus logarithmiques de D dans le cas de déformations équisingulières.

1. Introduction

Let $(D,0) \subseteq (\mathbb{C}^2,0)$ be a plane curve germ defined by a reduced equation $f \in \mathbb{C}\{x,y\}$, with the ring of functions $\mathcal{O}_D := \mathbb{C}\{x,y\}/(f)$. Let us denote by $\mathcal{O}_{\widetilde{D}} = \bigoplus_{i=1}^{p} \mathbb{C}\{t_i\}$ its normalization, where p is the number of irreducible components of D, and $Q(\mathcal{O}_D) = \bigoplus_{i=1}^{p} \mathbb{C}\{t_i\}\left[\frac{1}{t_i}\right]$ its total ring of fractions.

The normalization gives a parameterization $\varphi_i(t_i) = \big(x_i(t_i), y_i(t_i)\big)$ of each irreducible component of D; therefore, for a non-zero divisor $g \in Q(\mathcal{O}_D)$, we can define a valuation $\mathrm{val}_i(g)$ along D_i as the order in t_i of $g \circ \varphi_i$. The element $\mathrm{val}(g) = \big(\mathrm{val}_1(g), \dots, \mathrm{val}_p(g)\big) \in \mathbb{Z}^p$ is called the value of g. Then, for a fractional ideal $I \subseteq Q(\mathcal{O}_D)$, that is to say a finite $\mathcal{O}_D$-submodule that contains a non-zero divisor, we define $\mathrm{val}(I) = \{\mathrm{val}(g);\ g \in I \text{ non-zero divisor}\} \subseteq \mathbb{Z}^p$.

For an *irreducible* plane curve, the conductor is the minimal c with $c + \mathbb{N} \subseteq \mathrm{val}(\mathcal{O}_D)$. It is well known that the semigroup $\mathrm{val}(\mathcal{O}_D)$ satisfies the following property (see [3, Exc. 5.2.25]):

$$v \in \mathrm{val}(\mathcal{O}_D) \iff c - v - 1 \notin \mathrm{val}(\mathcal{O}_D) \tag{1}$$

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For reducible plane curves, an analogous property of $\mathrm{val}(\mathcal{O}_D)$ is proved in [5]. Our main Theorem 2.4 is a generalization of this symmetry to fractional ideals $I \subseteq \mathcal{O}_D$. We then apply it to the Jacobian ideal and the module of logarithmic residues in order to study their behavior in equisingular deformations.

2. Preliminaries

As in [5], let us define the following sets.

Let $\mathcal{M} \subseteq \mathbb{Z}^p$ and $v \in \mathbb{Z}^p$. For $i \in \{1,\dots,p\}$, we define:

$$\Delta_i(v,\mathcal{M}) = \{\alpha \in \mathcal{M};\ \alpha_i = v_i \text{ and } \forall j \neq i, \alpha_j > v_j\}$$

and $\Delta(v,\mathcal{M}) = \bigcup_{i=1}^p \Delta_i(v,\mathcal{M})$. We consider the partial product order on $\mathbb{Z}^p$, so that for $\alpha,\beta \in \mathbb{Z}^p$, $\inf(\alpha,\beta) = \big(\min(\alpha_1,\beta_1),\dots,\min(\alpha_p,\beta_p)\big)$. We set $\alpha - \underline{1} = (\alpha_1 - 1,\dots,\alpha_p - 1)$.

We denote by $\mathcal{C}_D$ the conductor ideal of D, which is equal to $\mathrm{Ann}_{\mathcal{O}_D}\mathcal{O}_{\widetilde{D}}/\mathcal{O}_D$. There exists a $\gamma \in \mathbb{N}^p$, called the conductor, such that $\mathcal{C}_D = t^\gamma \mathcal{O}_{\widetilde{D}}$, where $t^\gamma = \left(t_1^{\gamma_1},\dots,t_p^{\gamma_p}\right)$. Similarly, for a fractional ideal I, we denote by $\mathcal{C}_I = \mathrm{Ann}_{\mathcal{O}_D}\mathcal{O}_{\widetilde{D}}/I$, and $\nu \in \mathbb{Z}^p$ the "conductor of I" defined by $\mathcal{C}_I = t^\nu \mathcal{O}_{\widetilde{D}}$.

The two following properties will be useful (see [5, 1.1.2 and 1.1.3]).

Proposition 2.1. *For a fractional ideal $I \subseteq Q(\mathcal{O}_D)$, if $v, v' \in \mathrm{val}(I)$, then $\inf(v,v') \in \mathrm{val}(I)$.*

Proposition 2.2. *Let $v \neq v' \in \mathrm{val}(I)$. If there exists $i \in \{1,\dots,p\}$ such that $v_i = v'_i$, then there exists $v'' \in \mathrm{val}(I)$ such that $v''_i > v_i$, and for $j \neq i$, $v''_j \geqslant \min(v_j, v'_j)$ with equality if $v_j \neq v'_j$.*

We will also need the following result, which is in fact a consequence of the previous ones:

Proposition 2.3. *Let $\alpha \in \mathbb{Z}^p$. Assume that all $v \geqslant \alpha$ are in $\mathrm{val}(I)$. Then an element $v \in \mathbb{Z}^p$ is in $\mathrm{val}(I)$ if and only if $\inf(v,\alpha) \in \mathrm{val}(I)$.*

Proof. For the implication $\Leftarrow$, we use Proposition 2.2 several times, starting with α and $\inf(v,\alpha)$ in order to obtain an element $v' \in \mathrm{val}(I)$ such that $v'_i = v_i$ if $v_i < \alpha_i$, and $v'_i \geqslant v_i$ otherwise. We then use Proposition 2.1 with v' and an element $\beta \geqslant \alpha$ satisfying $\beta_i = v_i$ if $v_i \geqslant \alpha_i$. □

Our main result is the following generalization of Theorem 2.8 of [5], where $I^\vee$ stands for the $\mathcal{O}_D$-dual of I, namely, $I^\vee = \mathrm{Hom}_{\mathcal{O}_D}(I,\mathcal{O}_D) \simeq \{m \in Q(\mathcal{O}_D);\ mI \subseteq \mathcal{O}_D\}$:

Theorem 2.4. *For a fractional ideal $I \subseteq \mathcal{O}_D$, $v \in \mathrm{val}(I^\vee)$ if and only if $\Delta(\gamma - v - \underline{1}, I) = \varnothing$.*

3. Proof of the main theorem

Let us prove the first implication $\Rightarrow$. Let $v \in \mathrm{val}(I^\vee)$ and assume that $\Delta(\gamma - v - \underline{1}, I) \neq \varnothing$. Let $w \in \Delta(\gamma - v - \underline{1}, I)$. Then, by duality, we obtain $v + w \in \mathrm{val}(\mathcal{O}_D)$. In fact, $v + w \in \Delta(\gamma - \underline{1}, \mathcal{O}_D)$, which is impossible from Corollary 1.9 of [5], whose statement is $\Delta(\gamma - \underline{1}, \mathcal{O}_D) = \varnothing$. Hence the first implication.

The implication $\Leftarrow$ is more subtle, and needs more preparation. With the first implication, we can define a set $\mathcal{V} \subseteq \mathbb{Z}^p$ by $\mathcal{V} = \{v \in \mathbb{Z}^p;\ \Delta(\gamma - v - \underline{1}, I) = \varnothing\}$. It contains $\mathrm{val}(I^\vee)$, but it could be bigger. In particular, it is not obvious that $\mathcal{V}$ is the set of values of a $\mathcal{O}_D$-module.

In [5], a way to compute the dimension of some quotients from the values is given. Let $J \subseteq Q(\mathcal{O}_D)$ be a fractional ideal, and $\alpha \in \mathbb{Z}^p$. We define $\ell(\alpha, J) = \dim_{\mathbb{C}} J/\{g \in J, \mathrm{val}(g) \geqslant \alpha\}$.

Let $(e_1,\dots,e_p)$ denote the canonical basis of $\mathbb{Z}^p$. For $\mathcal{M} \subseteq \mathbb{Z}^p$ and $v \in \mathbb{Z}^p$, let

$$\Lambda_i(v,\mathcal{M}) = \{\alpha \in \mathcal{M};\ \alpha_i = v_i \text{ and } \forall j \neq i, \alpha_j \geqslant v_j\}$$

We then have (see [5, Proposition 1.11]):

Proposition 3.1. *For all $\alpha \in \mathbb{Z}^p$, $\ell(\alpha + e_i, J) - \ell(\alpha, J) \in \{0,1\}$ and $\ell(\alpha + e_i, J) = \ell(\alpha, J) + 1$ if and only if $\Lambda_i(\alpha, \mathrm{val}(J)) \neq \varnothing$.*

From this proposition, we can prove the implication $\Leftarrow$ of Theorem 2.4 in three steps.

First step

Let $I \subseteq \mathcal{O}_D$ be a fractional ideal, whose conductor ideal is $\mathcal{C}_I$ and whose conductor is ν. Notice that we have the following sequences of inclusions: $\mathcal{C}_I \subseteq I \subseteq \mathcal{O}_D \subseteq \mathcal{O}_{\widetilde{D}}$ and $\mathcal{C}_D \subseteq \mathcal{O}_D \subseteq I^\vee$.

Proposition 3.2. *Assume that $\mathcal{V} \neq \mathrm{val}(I^\vee)$. Then there exists $w^{(0)} \in \mathcal{V} \backslash \mathrm{val}(I^\vee)$ such that $w^{(0)} \leqslant \gamma$. Moreover, there exists $j \in \{1, \dots, p\}$ such that $\Lambda_j(w^{(0)}, \mathrm{val}(I^\vee)) = \varnothing$ and $w_j^{(0)} < \gamma_j$.*

We need the following lemma, which is analogous to Proposition 2.1:

Lemma 3.3. *Let $w, w' \in \mathcal{V}$. Then* $\inf(w, w') \in \mathcal{V}$.

Proof. Let $v = \gamma - w - \underline{1}$, $v' = \gamma - w' - \underline{1}$, $v'' = \gamma - \inf(w, w') - \underline{1} = \sup(v, v')$. It is then easy to see that $\Delta(v'', I) \subseteq \Delta(v, I) \cup \Delta(v', I)$. The result comes from the definition of $\mathcal{V}$. $\square$

Proof of Proposition 3.2. Let $w \in \mathcal{V} \backslash \mathrm{val}(I^\vee)$. Then using Lemma 3.3, we obtain $w^{(0)} := \inf(w, \gamma) \in \mathcal{V}$. It follows from Proposition 2.3 that $\inf(w, \gamma) \notin \mathrm{val}(I^\vee)$ since all $v \geqslant \gamma$ are in $\mathrm{val}(I^\vee)$. Then, from Proposition 2.1, there exists $j \in \{1, \dots, p\}$ such that $\Lambda_j(w^{(0)}, \mathrm{val}(I^\vee)) = \varnothing$. Since $\gamma \in \mathrm{val}(I^\vee)$, necessarily, $w_j^{(0)} < \gamma_j$. $\square$

Second step

Assume from now on that $\mathcal{V} \neq \mathrm{val}(I^\vee)$, and let $w^{(0)}$ be given by Proposition 3.2. For the sake of simplicity, assume that $\Lambda_p(w^{(0)}, \mathrm{val}(I^\vee)) = \varnothing$. Let us define the following sequence $(\alpha^{(j)})_{0 \leqslant j \leqslant n_0}$ with $n_0 = \sum_{i=1}^p \nu_i$:

$$\underbrace{\gamma - \nu}_{=\alpha^{(0)}} \xrightarrow[(+e_1)^\bullet]{} \underbrace{(w_1^{(0)}, \gamma_2 - \nu_2, \dots, \gamma_p - \nu_p)}_{=\alpha^{(k_1)}} \xrightarrow[(+e_2)^\bullet]{} \cdots \xrightarrow[(+e_{p-1})^\bullet]{} \underbrace{(w_1^{(0)}, \dots, w_{p-1}^{(0)}, \gamma_p - \nu_p)}_{=\alpha^{(k_{p-1})}}$$

$$\xrightarrow[(+e_p)^\bullet]{} \underbrace{(w_1^{(0)}, \dots, w_{p-1}^{(0)}, \gamma_p)}_{=\alpha^{(k_p)}} \xrightarrow[(+e_{p-1})^\bullet]{} \underbrace{(w_1^{(0)}, \dots, w_{p-2}^{(0)}, \gamma_{p-1}, \gamma_p)}_{=\alpha^{(k_{p+1})}} \xrightarrow[(+e_{p-2})^\bullet]{} \cdots \xrightarrow[(+e_1)^\bullet]{} \underbrace{\gamma}_{=\alpha^{(k_{2p-1})}}$$

where $k_{2p-1} = n_0$. More precisely, $k_1 = w_1^{(0)} - (\gamma_1 - \nu_1)$ and for $j \in \{0, \dots, k_1 - 1\}$, $\alpha^{(j+1)} = \alpha^{(j)} + e_1$, and so on.

Since $t^\nu \mathcal{O}_{\widetilde{D}} \subseteq I$, the smallest value that can appear in $\mathcal{V}$ is $\gamma - \nu$. Then from Proposition 3.1, we have:

$$\dim_{\mathbb{C}} I^\vee / \mathcal{C}_D = \mathrm{Card}\left(\left\{j \in \{0, \dots, n_0 - 1\};\ \Lambda_i(\alpha^{(j)}, \mathrm{val}(I^\vee)) \neq \varnothing,\ \text{where } \alpha^{(j+1)} = \alpha^{(j)} + e_i\right\}\right) \tag{2}$$

We define a number ℓ'_α by changing $\mathrm{val}(I^\vee)$ into $\mathcal{V}$ in (2). This number may depend on the chosen sequence α. For the sequence α defined above, since $\Lambda_p(w^{(0)}, \mathrm{val}(I^\vee)) = \varnothing$ and $\Lambda_p(w^{(0)}, \mathcal{V}) \neq \varnothing$, we have the following inequality:

$$\ell'_\alpha \geqslant 1 + \dim_{\mathbb{C}} I^\vee / \mathcal{C}_D \tag{3}$$

Third step

For the third step, we need the following property (see [3, proof of Lemma 5.2.8]):

$$\dim_{\mathbb{C}} J_1 / J_2 = \dim_{\mathbb{C}} J_2^\vee / J_1^\vee \quad J_1, J_2 \text{ fractional ideals} \tag{4}$$

With the same notations, let us consider the sequence $(\beta^{(j)})_{0 \leqslant j \leqslant n_0}$ defined by $\beta^{(j)} = \gamma - \alpha^{(n_0 - j)}$.

As for $(\alpha^{(j)})$, the sequence $(\beta^{(j)})$ can be used to compute $\dim_{\mathbb{C}} I / \mathcal{C}_I$ in the same way as (2). From the relation between the two sequences, it can be proved that for $0 \leqslant j \leqslant n_0 - 1$, $\Lambda_i(\alpha^{(j)}, \mathcal{V}) \neq \varnothing$ implies $\Lambda_i(\beta^{(n_0 - (j+1))}, \mathrm{val}(I)) = \varnothing$, which provides us with the following inequality:

$$\sum_{i=1}^p \nu_i - \dim_{\mathbb{C}} I / \mathcal{C}_I \geqslant \ell'_\alpha \tag{5}$$

However, from (4), $\dim_{\mathbb{C}} \mathcal{O}_{\widetilde{D}} / I = \dim_{\mathbb{C}} I^\vee / \mathcal{C}_D$ therefore we have:

$$\dim_{\mathbb{C}} I^\vee / \mathcal{C}_D = \dim_{\mathbb{C}} \mathcal{O}_{\widetilde{D}} / \mathcal{C}_I - \dim_{\mathbb{C}} I / \mathcal{C}_I = \sum_{i=1}^p \nu_i - \dim_{\mathbb{C}} I / \mathcal{C}_I \tag{6}$$

Therefore, $\ell'_\alpha \leqslant \dim_{\mathbb{C}} I^\vee / \mathcal{C}_D$, which is a contradiction with (3). Thus, the set $\mathcal{V}$ cannot be bigger than $\mathrm{val}(I^\vee)$, which gives us the implication $\Leftarrow$ of Theorem 2.4.

4. Application: logarithmic residues and equisingular deformations

Let $\mathrm{Der}(-\log D)$ and $\Omega^1(\log D)$ be respectively the $\mathbb{C}\{x,y\}$-modules of logarithmic vector fields and of logarithmic 1-forms along D at the origin. Since we consider a plane curve, these two modules are free. Let us recall some results from [7]. A meromorphic 1-form ω is logarithmic if and only if there exist a holomorphic 1-form η, and $\xi, g \in \mathbb{C}\{x,y\}$ where g does not induce a zero divisor in $\mathcal{O}_D$, such that $g\omega = \xi \frac{\mathrm{d}f}{f} + \eta$. In fact, for g one can choose every linear combination of the derivatives of f that does not induce a zero divisor in $\mathcal{O}_D$. The residue of ω is $\mathrm{res}(\omega) = \frac{\xi}{g} \in Q(\mathcal{O}_D)$, and we define $\mathcal{R}_D = \mathrm{res}\left(\Omega^1(\log D)\right)$. This module is called the module of logarithmic residues, and is a finite-type $\mathcal{O}_D$-submodule of $Q(\mathcal{O}_D)$, generated by the residues of a basis of $\Omega^1(\log D)$. We always have the inclusion $\mathcal{O}_{\widetilde{D}} \subseteq \mathcal{R}_D$.

Let $\mathcal{J}_D \subseteq \mathcal{O}_D$ be the Jacobian ideal. The following result is proved in [6]: $\mathcal{J}_D^\vee = \mathcal{R}_D$. Therefore, from Theorem 2.4, we deduce that $v \in \mathrm{val}(\mathcal{R}_D)$ if and only if $\Delta(\gamma - v - \underline{1}, \mathcal{J}_D) = \emptyset$.

Another consequence of this duality is:

$$\dim_{\mathbb{C}} \mathcal{R}_D/\mathcal{O}_{\widetilde{D}} = \tau - \delta \tag{7}$$

with τ the Tjurina number and $\delta = \dim_{\mathbb{C}} \mathcal{O}_{\widetilde{D}}/\mathcal{O}_D$. Indeed, from (4), $\dim_{\mathbb{C}} \mathcal{R}_D/\mathcal{O}_D = \dim_{\mathbb{C}} \mathcal{O}_D/\mathcal{J}_D = \tau$.

Our purpose is to study the behavior of logarithmic residues in an equisingular deformation of a plane curve germ D. Consider a deformation $F(x,s)$ of $f(x)$ with base space $(S,0) = (\mathbb{C}^k, 0)$ for a $k \in \mathbb{N}$. Denote for $s \in S$, $F_s = F(.,s)$, and $D_s = F_s^{-1}(0)$. Equisingularity means that all fibers $(D_s, 0) \subseteq (\mathbb{C}^2, 0)$ have the same Milnor number μ. From the theorem of equisingularity for plane curves (see [8, §3.7]), a parameterization $(x(t), y(t))$ of D gives rise to a deformation of the parametrization $(x_s(t), y_s(t))$.

Let us denote by $\mathcal{R}_s$ the module of logarithmic residues of D_s.

Definition 4.1. The stratification by logarithmic residues is the partition $S = \bigcup_{\mathcal{V} \subseteq \mathbb{Z}^p} S_{\mathcal{V}}$, where $s \in S_{\mathcal{V}}$ if and only if $\mathrm{val}(\mathcal{R}_s) = \mathcal{V}$.

Proposition 4.2.

(i) *If s, s' do not belong to the same stratum of the stratification by τ, they do not belong to the same stratum for the stratification by logarithmic residues. In other words, the stratification by logarithmic residues is finer than the stratification by τ.*
(ii) *The stratification by logarithmic residues is finite.*
(iii) *Each stratum $S_{\mathcal{V}}$ is locally analytic and locally closed.*

Sketch of the proof. The first point follows easily from (7). For the second point, since $\tau \leqslant \mu$, it is clear that the stratification by the Tjurina number τ is finite. Therefore, it is sufficient to consider the behavior of logarithmic residues in a τ-constant stratum. When τ is constant, it is an admissible deformation in the sense of [9], so that there exist $\delta_i(x,y,s) = a_i(x,y,s)\partial_x + b_i(x,y,s)\partial_y$, $i = 1, 2$, such that for every s, $(\delta_1(.,s), \delta_2(.,s))$ is a basis of $\mathrm{Der}(-\log D_s)$. Then, for a convenient choice of $\alpha(s), \beta(s)$, the residues of D_s are generated over $\mathcal{O}_{D_s}$ by:

$$\rho_i = \frac{-\beta(s)a_i(s) + \alpha(s)b_i(s)}{\alpha(s)F'_x(s) + \beta(s)F'_y(s)}, \quad i = 1, 2$$

In fact, thanks to the equisingularity condition, it is possible to choose $\alpha, \beta \in \mathbb{C}^2$ such that the value of $\alpha F'_x(s) + \beta F'_y(s)$ is independent of s. To prove this, one can use the theorem of equisingularity (see [8, §3.7]), Teissier's lemma (see [2, 2.3]) and Theorem 2.7 of [4]. All values of $\mathcal{R}_s$ are then greater than $\mathrm{val}(\alpha F'_x(0) + \beta F'_y(0))$, and the finiteness follows from this and from Proposition 2.3, since $\mathcal{O}_{\widetilde{D_s}} \subseteq \mathcal{R}_s$. For the third point, recall from the appendix by Teissier in [10] that the strata of the stratification by the Tjurina number are locally analytic and locally closed. Then the result about the stratification by logarithmic residues is also a consequence of the existence of this denominator. □

Let us look at some examples.

Example 1. Consider $f(x,y) = x^5 - y^6$ and the equisingular deformation of f given by $F(x,y,s_1,s_2,s_3) = x^5 - y^6 + s_1x^2y^4 + s_2x^3y^3 + s_3x^3y^4$. The stratification by τ is composed of three strata, $S_{\tau=20} = \{0\}$, $S_{\tau=19} = \{(0,0,s_3), s_3 \neq 0\}$ and $S_{\tau=18} = \{(s_1,s_2,s_3), (s_1,s_2) \neq (0,0)\}$. The computation of the values of $\mathcal{J}_{D_s}$ is quite easy in this case, and it can be seen that the stratum $S_{\tau=18}$ divides into two strata for the values of $\mathcal{J}_{D_s}$: $S_1 = \{(0,s_2,s_3), s_2 \neq 0\}$ and $S_2 = \{(s_1,s_2,s_3), s_1 \neq 0\}$, and the same goes for the stratification by logarithmic residues thanks to Theorem 2.4. Therefore, the stratification by logarithmic residues is not the same as the stratification by τ.

Example 2. The following proposition can be obtained by an explicit computation of $\mathrm{val}(\mathcal{O}_D)$:

Proposition 4.3. *Let* $f(x,y)=\prod_{j=1}^{p}(x^a-\lambda_\ell y^b+\sum_{ib+ja>ab}a_{ij}^{(\ell)}x^iy^j)$ *be a reduced equation, with the* $\lambda_\ell\in\mathbb{C}$ *pairwise distinct,* $\gcd(a,b)=1$ *and* $a_{ij}^{(\ell)}\in\mathbb{C}$*. Let* γ *be the conductor of* D*.*

Then $\gamma+(\mathrm{val}(\mathcal{O}_D)\backslash\{0\})-\underline{1}\subseteq\mathrm{val}(\mathcal{J}_D)$*.*

Let us consider the deformation $F(x,y,s_1,s_2)=x^{10}+y^8+s_1x^5y^4+s_2x^3y^6$. It is given in [1], as an example of the stratification by the b-function not satisfying the frontier condition. A stratification $S=\bigcup_\alpha S_\alpha$ satisfies the frontier condition if for $\alpha\neq\beta$, $S_\alpha\cap\overline{S_\beta}\neq\varnothing$ implies $S_\alpha\subseteq\overline{S_\beta}$, with $\overline{S_\beta}$ the closure of S_β.

A computation shows that there are three strata for the stratification by τ in a neighborhood of the origin of $\mathbb{C}^2$: $S_{\tau=63}=\{(s_1,0)\}$, $S_{\tau=54}=\{(0,s_2),s_2\neq 0\}$ and $S_{\tau=53}=\{(s_1,s_2),s_1s_2\neq 0\}$. From Proposition 4.3, the semigroup of values of $\mathcal{J}_{D_s}$ does not change in the stratum $S_{\tau=63}$, so that the latter is exactly a stratum of the stratification by logarithmic residues. However, there exists a stratum $S'\subseteq S_{\tau=54}$ whose closure contains the origin, but not the whole stratum $S_{\tau=63}$. Therefore, the stratification by logarithmic residues does not satisfy the frontier condition.

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