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Confluent Vandermonde matrices and divided differences over quaternions

Matrices de Vandermonde confluentes et différences divisées sur les quaternions

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A R T I C L E I N F O

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A B S T R A C T

We introduce the notion of a confluent Vandermonde matrix over quaternions and present the formula to compute its rank. This extends a result of T.Y. Lam (A general theory of Vandermonde matrices, Expo. Math. 4 (3) (1986) 193–215). Another contribution is the representation formula for divided differences of quaternion polynomials which extends a result of G. Gentili and D.C. Struppa (A new theory of regular functions of a quaternionic variable, Adv. Math. 216 (1) (2007) 279–301).

R É S U M É

Nous introduisons la notion de matrice de Vandermonde confluente sur les quaternions et nous calculons son rang. Ceci étend les résultats de T.Y. Lam (A general theory of Vandermonde matrices, Expo. Math. 4 (3) (1986) 193–215). Ensuite, nous montrons une formule de représentation d'ordre supérieur pour les différences divisées de polynômes à coefficients quaternions, généralisant un résultat de G. Gentili et D.C. Struppa (A new theory of regular functions of a quaternionic variable, Adv. Math. 216 (1) (2007) 279–301).

Let $\mathbb{H}$ denote the skew field of quaternions $\alpha = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the imaginary units commuting with $\mathbb{R}$ and satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. For $\alpha \in \mathbb{H}$ as above, its real and imaginary parts, the quaternion conjugate and the absolute value are defined as $\mathrm{Re}(\alpha) = x_0$, $\mathrm{Im}(\alpha) = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, $\overline{\alpha} = \mathrm{Re}(\alpha) - \mathrm{Im}(\alpha)$ and $|\alpha|^2 = \alpha\overline{\alpha} = |\mathrm{Re}(\alpha)|^2 + |\mathrm{Im}(\alpha)|^2$, respectively. Two quaternions α and β are called *equivalent* (conjugate to each other) if $\alpha = h^{-1}\beta h$ for some nonzero $h \in \mathbb{H}$; in notation, $\alpha \sim \beta$. It turns out that $\alpha \sim \beta$ if and only if $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta)$ and $|\alpha| = |\beta|$, so that the *conjugacy class* of a given $\alpha \in \mathbb{H}$ form a 2-sphere (of radius $|\mathrm{Im}(\alpha)|$ around $\mathrm{Re}(\alpha)$), which will be denoted by $[\alpha]$.

Given elements $\alpha_1, \dots, \alpha_n \in \mathbb{H}$, we denote by $V_m = \left[\alpha_i^{j-1}\right]_{i=1,\dots,n}^{j=1,\dots,m}$ the associated $n \times m$ Vandermonde matrix. Recall that the rank of a quaternion matrix is defined as the dimension of the left linear span of its rows or equivalently (by [4,

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Theorem 7]) the dimension of the right span of its columns. For a set Λ, we will write $\sharp(\Lambda)$ for its cardinality. The following result is due to T.-Y. Lam [4]:

Theorem 1. *Let $S_1, \dots, S_\ell$ be all distinct conjugacy classes having non-empty intersection with the set $\Lambda = \{\alpha_1, \dots, \alpha_n\}$, and let*

$$\kappa = \mu_1 + \ldots + \mu_\ell, \quad \text{where} \quad \mu_j = \begin{cases} 1, & \text{if } \sharp(S_j \cap \Lambda) = 1, \\ 2, & \text{if } \sharp(S_j \cap \Lambda) \geq 2. \end{cases}$$

Then $\operatorname{rank} V_m = \min(m, \kappa)$. In particular, the square Vandermonde matrix V_n is invertible if and only if all elements in Λ are distinct and none three of them belong to the same conjugacy class.

The theorem was established in [4] in a more general setting of division rings with a fixed endomorphism and the integer κ was identified with the minimally possible degree of a nonzero polynomial having left zeros at $\alpha_1, \dots, \alpha_n$. Further extensions of this result to the setting of a division ring K endowed with an endomorphism S and an S-derivation D, along with their interactions with skew polynomials from the Ore domain $K[z, S, D]$, were obtained in [5]. The objective of this paper is to present the extension of Theorem 1 in a different direction: we will introduce a meaningful notion of a *confluent Vandermonde matrix* (over $\mathbb{H}$ only, for the sake of simplicity) and establish the "confluent" version of Theorem 1.

1. Divided differences

Let $\mathbb{H}[z]$ be the ring of polynomials in one formal variable z that commutes with quaternionic coefficients. The ring operations in $\mathbb{H}[z]$ are defined as in the commutative case, but as multiplication in $\mathbb{H}$ is not commutative, multiplication in $\mathbb{H}[z]$ is not commutative either. A straightforward computation verifies that for any $\alpha \in \mathbb{H}$ and $f \in \mathbb{H}[z]$,

$$f(z) = f^{\boldsymbol{e}_\ell}(\alpha) + (z - \alpha) \cdot (L_\alpha f)(z), \tag{1}$$

where $f^{\boldsymbol{e}_\ell}(\alpha)$ is the left evaluation of f at α and $L_\alpha f$ is the polynomial of degree $m-1$ given by

$$f^{\boldsymbol{e}_\ell}(\alpha) = \sum_{k=0}^{m} \alpha^k f_k, \quad (L_\alpha f)(z) = \sum_{k=0}^{m-1} z^k \left(\sum_{j=0}^{m-k-1} \alpha^j f_{k+j+1} \right) \quad \text{if} \quad f(z) = \sum_{k=0}^{m} z^k f_k. \tag{2}$$

The mapping $f \mapsto L_\alpha f$ defines a right linear operator (called in analogy to the complex case, the left backward shift) acting on $\mathbb{H}[z]$ (considered as a vector space over $\mathbb{H}$).

In what follows, we will use notation $\boldsymbol{\rho}_\alpha(z) := z - \alpha$ for a fixed $\alpha \in \mathbb{H}$. Given a polynomial $f \in \mathbb{H}[z]$, the successive application of formula (1) to elements $\alpha_1, \dots, \alpha_n \in \mathbb{H}$ and polynomials f, $L_{\alpha_1} f$, $L_{\alpha_2} L_{\alpha_1} f, \dots$ leads us to the representation

$$f = f^{\boldsymbol{e}_\ell}(\alpha_1) + \sum_{k=1}^{n-1} \boldsymbol{\rho}_{\alpha_1} \dots \boldsymbol{\rho}_{\alpha_k} \cdot (L_{\alpha_k} \cdots L_{\alpha_1} f)^{\boldsymbol{e}_\ell}(\alpha_{k+1}) + \boldsymbol{\rho}_{\alpha_1} \dots \boldsymbol{\rho}_{\alpha_n} \cdot (L_{\alpha_n} \cdots L_{\alpha_1} f), \tag{3}$$

which, being the (left) quaternionic analog of the Newton interpolation formula, suggests to define *left divided differences* by letting

$$[\alpha_1; f]_\ell = f^{\boldsymbol{e}_\ell}(\alpha_1), \qquad [\alpha_1, \dots, \alpha_k; f]_\ell = (L_{\alpha_{k-1}} \cdots L_{\alpha_1} f)^{\boldsymbol{e}_\ell}(\alpha_k) \quad \text{for} \quad k > 1. \tag{4}$$

If we denote by $f^{(k)}$ the k-th formal derivative of $f \in \mathbb{H}[z]$, then for any fixed $\alpha \in \mathbb{H}$,

$$f = \sum_{k=0}^{\deg f} \boldsymbol{\rho}_\alpha^k \cdot \frac{(f^{(k)})^{\boldsymbol{e}_\ell}(\alpha)}{k!} \quad \text{and} \quad [\underbrace{\alpha, \dots, \alpha}_{(k+1) \text{ times}}; f]_\ell = \frac{(f^{(k)})^{\boldsymbol{e}_\ell}(\alpha)}{k!} \quad \text{for} \quad k \geq 0. \tag{5}$$

The first representation is verified in a quite straightforward way, while the second equality follows by letting $\alpha_j = \alpha$ for $j = 1, \dots, n$ in (3) and comparing the obtained representation with the first formula in (5). The difference between the complex and quaternionic settings becomes transparent even in the case where $k = 2$. It is not hard to show that if $\alpha_2 \not\sim \alpha_1$, then

$$[\alpha_1, \alpha_2; f]_\ell = (\widetilde{\alpha}_2 - \alpha_1)^{-1} (f^{\boldsymbol{e}_\ell}(\widetilde{\alpha}_2) - f^{\boldsymbol{e}_\ell}(\alpha_1)), \quad \text{where} \quad \widetilde{\alpha}_2 = (\alpha_2 - \bar{\alpha}_1)^{-1} \alpha_2 (\alpha_2 - \bar{\alpha}_1). \tag{6}$$

The formula (6) is similar to its complex counterpart, but the element α_2 is replaced by the equivalent element $\widetilde{\alpha}_2$, which is equal to α_2 if and only if α_1 and α_2 commute. If $\alpha_1 \sim \alpha_2 \neq \bar{\alpha}_1$, then the formula for $[\alpha_1, \alpha_2; f]$ involves not only the values of f at α_1 and α_2, but also the value of f':

$$[\alpha_1, \alpha_2; f]_\ell = (\alpha_2 - \bar{\alpha}_2)^{-1} \left(f^{\boldsymbol{e}_\ell}(\alpha_2) - f^{\boldsymbol{e}_\ell}(\alpha_1) + (\alpha_2 - \bar{\alpha}_1) f'^{\boldsymbol{e}_\ell}(\alpha_1) \right).$$

Thus, the divided differences based on a spherical chain (different from that in (5)) is the object that does not appear in the commutative setting.

2. Confluent Vandermonde matrices

Vaguely speaking, the confluent Vandermonde matrix should be defined so that it will be non-singular in cases where the usual Vandermonde matrix is singular, that is (according to Theorem 1) if (1) $\alpha_{i_1}=\alpha_{i_2}$ and/or (2) $\alpha_{i_1}\sim\alpha_{i_2}\sim\alpha_{i_3}$. To achieve the latter, we introduce the notion of a *spherical chain*: a finite ordered collection $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)\subset\mathbb{H}$ such that

$$\alpha_1\sim\alpha_2\sim\dots\sim\alpha_k \quad\text{and}\quad \alpha_{j+1}\neq\bar{\alpha}_j \quad\text{for}\quad j=1,\dots,k-1. \tag{7}$$

We define the *confluent Vandermonde matrix* $V_m(\boldsymbol{\alpha})$ associated with the spherical chain (7) as follows:

$$[V_m(\boldsymbol{\alpha})]_{ij}=[\alpha_1,\alpha_2,\dots,\alpha_i;\, z^j]_{\boldsymbol{\ell}},\qquad i=1,\dots,k;\ j=1,\dots,m. \tag{8}$$

Due to formulas (5), the matrix $V_m(\boldsymbol{\alpha})$ based on the spherical chain $\boldsymbol{\alpha}=(\alpha,\dots,\alpha)$ has the same form as in the commutative complex case (see e.g., [3]). We next observe that the matrix $V_m(\boldsymbol{\alpha})$ can be defined more explicitly as the $k\times m$ matrix whose j-th column equals $\mathcal{J}_{\boldsymbol{\alpha}}^{j-1}E_k$:

$$V_m(\boldsymbol{\alpha})=\left[\,E_k\quad \mathcal{J}_{\boldsymbol{\alpha}}E_k\quad\dots\quad \mathcal{J}_{\boldsymbol{\alpha}}^{m-1}E_k\,\right],\quad\text{where}\quad \mathcal{J}_{\boldsymbol{\alpha}}=\begin{bmatrix}\alpha_1 & 0 & \dots & & 0\\ 1 & \alpha_2 & 0 & & \\ 0 & 1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & 0 & 1 & \alpha_k\end{bmatrix},\quad E_k=\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}. \tag{9}$$

Indeed, since $\deg(L_\alpha f)=\deg(f)-1$ and since the backward shift of a monic polynomial is again monic (or identical zero), it follows that the matrix $V_m(\boldsymbol{\alpha})$ is upper triangular with all diagonal entries equal to one. In particular, the leftmost column of the matrix (8) is indeed E_k. The rest follows from the recursion

$$\left[\alpha_1,\dots,\alpha_k;z^j\right]_{\boldsymbol{\ell}}=\alpha_k\left[\alpha_1,\dots,\alpha_k;z^{j-1}\right]_{\boldsymbol{\ell}}+\left[\alpha_1,\dots,\alpha_{k-1};z^{j-1}\right]_{\boldsymbol{\ell}}\quad (i,j\geq 2), \tag{10}$$

which, together with equalities $[\alpha_1;z^j]_{\boldsymbol{\ell}}=\alpha_1^j$, imply that the j-th column in the matrix (8) can be obtained by multiplying the previous column by $\mathcal{J}_{\boldsymbol{\alpha}}$ on the left, which eventually leads to formula (9). We define *the confluent Vandermonde matrix based on n spherical chains* $\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n$ by

$$V_m(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)=\begin{bmatrix}V_m(\boldsymbol{\alpha}_1)\\ \vdots\\ V_m(\boldsymbol{\alpha}_n)\end{bmatrix},\quad \boldsymbol{\alpha}_i=(\alpha_{i,1},\dots,\alpha_{i,k_i}),\quad i=1,\dots,n, \tag{11}$$

where the matrices $V_m(\boldsymbol{\alpha}_j)$ are defined via the formula (9). To formulate the "confluent" extension of Theorem 1, we need one last definition.

Let us assume that a conjugacy class S contains $d\geq 2$ (not necessarily distinct) spherical chains (11). Let $\boldsymbol{\alpha}_i=(\alpha_{i,1}\dots,\alpha_{i,k_i})$ be the longest chain. For any other chain $\boldsymbol{\alpha}_j=(\alpha_{j,1}\dots,\alpha_{j,k_j})\subset S$, we define the number

$$\nu_j=\begin{cases}0, & \text{if } \alpha_{j,1}\neq\alpha_{i,1},\\ \max\{r:\alpha_{j,\mu}=\alpha_{i,\mu}\ (1\leq\mu\leq r)\}, & \text{if } \alpha_{j,1}=\alpha_{i,1},\end{cases}\quad\text{and let}\quad \mu(S)=k_i+\max_{j\neq i}\{k_j-\nu_j\}. \tag{12}$$

If the leftmost elements $\alpha_{1,1},\dots,\alpha_{d,1}$ in the chains are all distinct, then $\mu(S)$ is equal to the total number of elements in the two longest chains. We also observe that if there are several chains of the maximal length, then the values of ν_j in (12) depend on which one of the longest chains has been chosen for comparison. It is not hard to show, however, that the integer $\max_{j\neq i}\{k_j-\nu_j\}$ is independent of this choice. In case S contains only one chain, we let $\mu(S)$ be equal to its length.

Theorem 2. *Let $V_m(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ be the confluent Vandermonde matrix based on spherical chains (11). To each conjugacy class S_j containing at least one of these chains, assign the integer $\mu(S_j)$ as has been explained above. Then* $\operatorname{rank}(V_m(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n))=\min(m,\kappa)$, *where $\kappa=\sum_{j=1}^{\ell}\mu(S_j)$.*

In particular, the square matrix $V_m(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ is invertible if and only if all leftmost chain elements $\alpha_{1,1},\alpha_{2,1},\dots,\alpha_{n,1}$ are distinct and none three of them belong to the same conjugacy class.

The integer κ in Theorem 2 admits a nice interpretation in terms of certain polynomials associated with the matrix $V_m(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$. Recall that any (left or right) ideal in $\mathbb{H}[z]$ is principal. Let $\langle h\rangle_{\mathbf{r}}:=\{hq:\,q\in\mathbb{H}[z]\}$ denote the right ideal generated by h. Given two polynomials $f,g\in\mathbb{H}[z]$, their *least right common multiple* $h=\mathbf{lrcm}(f,g)$ is defined as a (unique) monic polynomial such that $\langle h\rangle_{\mathbf{r}}=\langle f\rangle_{\mathbf{r}}\cap\langle g\rangle_{\mathbf{r}}$. Here is a more compact reformulation of Theorem 2 (which appears in [4] for Vandermonde matrices over general division rings).

Theorem 3. *Let $V_m(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n)$ be the confluent Vandermonde matrix based on spherical chains (11). With each chain $\boldsymbol{\alpha}_j=(\alpha_{j,1}\dots,\alpha_{j,k_j})$, we associate the polynomial $P_j=\boldsymbol{\rho}_{\alpha_{j,1}}\boldsymbol{\rho}_{\alpha_{j,2}}\cdots\boldsymbol{\rho}_{\alpha_{j,k_j}}$ for $j=1,\dots,n$. Then $\operatorname{rank}(V_m(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n))=\min(m,\kappa)$, where $\kappa=\deg(\mathbf{lrcm}(P_1,\dots,P_n))$.*

3. Representation formulas of higher order

By a result from [2], evaluations of $f\in\mathbb{H}[z]$ at any two elements in the same conjugacy class S uniquely determine the value of f at every element in S. More precisely, for $f\in\mathbb{H}[z]$ and any three distinct equivalent elements $\alpha_1,\alpha_2,\alpha_3\in\mathbb{H}$,

$$f^{\boldsymbol{e}_\ell}(\alpha_3)=(\alpha_3-\alpha_2)(\alpha_1-\alpha_2)^{-1}f^{\boldsymbol{e}_\ell}(\alpha_1)+(\alpha_1-\alpha_3)(\alpha_1-\alpha_2)^{-1}f^{\boldsymbol{e}_\ell}(\alpha_2). \tag{13}$$

Formula (13) (termed in [2] the "representation formula") admits a natural confluent extension. It is clear that the same formula holds for formal derivatives of f, that is, for divided differences of f based on the spherical chains from the same conjugacy class of the special form (5). Our next objective is to present the analog of formula (13) for divided differences based on general spherical chains from the same conjugacy class. With a spherical chain $\boldsymbol{\alpha}$, we associate the square matrices

$$V_{\boldsymbol{\alpha}}:=V_k(\boldsymbol{\alpha})\quad\text{and}\quad T_{\boldsymbol{\alpha}}=V_{\boldsymbol{\alpha}}^{-1}\mathcal{J}_{\boldsymbol{\alpha}}^kV_{\boldsymbol{\alpha}},\qquad \boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k), \tag{14}$$

where $\mathcal{J}_{\boldsymbol{\alpha}}$ is given in (9). The matrix $V_{\boldsymbol{\alpha}}$ is invertible as the square upper triangular matrix with all diagonal entries equal one. Also, we will use notation

$$\Delta_\ell(\boldsymbol{\alpha};f)=\begin{bmatrix}[\alpha_1;f]_\ell\\ \vdots\\ [\alpha_1,\alpha_2,\dots,\alpha_k;f]_\ell\end{bmatrix},\quad \boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k),\quad f\in\mathbb{H}[z] \tag{15}$$

for the column of divided differences of a given polynomial f based on the chain $\boldsymbol{\alpha}$. The next theorem shows that the columns $\Delta(\boldsymbol{\alpha};f)$ and $\Delta(\boldsymbol{\beta};f)$ associated with the chains of the same length k from the same conjugacy class $S\subset\mathbb{H}$ and with distinct leftmost entries define all divided differences of f of order up to k associated with *any* chain $\boldsymbol{\gamma}\subset S$.

Theorem 4. *Let $\boldsymbol{\alpha}_i=(\alpha_{i,1},\dots,\alpha_{i,k})$ $(i=1,2,3)$ be three spherical chains of the same length and in the same conjugacy class $S\subset\mathbb{H}$ and let us assume that $\alpha_{1,1}\neq\alpha_{2,1}$. Then for any $f\in\mathbb{H}[z]$,*

$$\begin{aligned}\Delta_\ell(\boldsymbol{\alpha}_3;f)&=V_{\boldsymbol{\alpha}_3}(T_{\boldsymbol{\alpha}_3}-T_{\boldsymbol{\alpha}_2})(T_{\boldsymbol{\alpha}_1}-T_{\boldsymbol{\alpha}_2})^{-1}V_{\boldsymbol{\alpha}_1}^{-1}\Delta_\ell(\boldsymbol{\alpha}_1;f)\\&\quad+V_{\boldsymbol{\alpha}_3}(T_{\boldsymbol{\alpha}_3}-T_{\boldsymbol{\alpha}_1})(T_{\boldsymbol{\alpha}_2}-T_{\boldsymbol{\alpha}_1})^{-1}V_{\boldsymbol{\alpha}_2}^{-1}\Delta_\ell(\boldsymbol{\alpha}_2;f)\end{aligned} \tag{16}$$

where $V_{\boldsymbol{\alpha}_i}$ and $T_{\boldsymbol{\alpha}_i}$ are defined as in (14) and do not depend on f.

Although it is not obvious, the coefficient matrices

$$V_{\boldsymbol{\alpha}_3}(T_{\boldsymbol{\alpha}_3}-T_{\boldsymbol{\alpha}_2})(T_{\boldsymbol{\alpha}_1}-T_{\boldsymbol{\alpha}_2})^{-1}V_{\boldsymbol{\alpha}_1}^{-1}\quad\text{and}\quad V_{\boldsymbol{\alpha}_3}(T_{\boldsymbol{\alpha}_3}-T_{\boldsymbol{\alpha}_1})(T_{\boldsymbol{\alpha}_2}-T_{\boldsymbol{\alpha}_1})^{-1}V_{\boldsymbol{\alpha}_2}^{-1}$$

in (16) are lower triangular. On can see from (14) and (9) that if $k=1$ and $\boldsymbol{\alpha}_i=(\alpha_i)$, we have $V_{\boldsymbol{\alpha}_i}=1$ and $T_{\boldsymbol{\alpha}_i}=\mathcal{J}_{\boldsymbol{\alpha}_i}=\alpha_i$, while formulas (15) and (4) show that $\Delta_\ell(\boldsymbol{\alpha}_i;f)=[\alpha_i;f]_\ell=f^{\boldsymbol{e}_\ell}(\alpha_i)$. Now it is readily seen that in case $k=1$, formula (16) amounts to (13).

4. Left Vandermonde matrices versus right

Vandermonde matrices considered above should be termed "left" as they arise in the context of left polynomial interpolation [4,1]. For example, to find a polynomial $f\in\mathbb{H}[z]$ satisfying interpolation conditions $f^{\boldsymbol{e}_\ell}(\alpha_i)=c_i$ $(i=1,\dots,n)$, we may use (2) (with $m-1$ instead of m) to write these conditions as the system of linear equations

$$f_0+\alpha_if_1+\ldots+\alpha_i^{m-1}f_{m-1}=c_i\quad\text{for}\quad i=1,\dots,n$$

and the matrix of this system is the Vandermonde matrix $V_m=\left[\alpha_i^{j-1}\right]_{i=1,\dots,n}^{j=1,\dots,m}$. The confluent matrix (11) appears similarly in the course of the Lagrange–Hermite problem with interpolation conditions

$$[\alpha_{i,1},\dots,\alpha_{i,j};f]_\ell=c_{i,j}\quad\text{for}\quad i=1,\dots,n;\ j=1,\dots,k_i. \tag{17}$$

There exists a parallel "right" theory: similarly to (1) and (2), for any $\alpha\in\mathbb{H}$ and $f\in\mathbb{H}[z]$,

$$f(z)=f^{\boldsymbol{e}_r}(\alpha)+(R_\alpha f)(z)\cdot(z-\alpha),$$

where $f^{\boldsymbol{e}_r}(\alpha)$ is the right evaluation of f at α and $R_\alpha f$ is the polynomial of degree $m-1$ given by

$$f^{\boldsymbol{e}_r}(\alpha)=\sum_{k=0}^m f_k\alpha^k,\quad (R_\alpha f)(z)=\sum_{k=0}^{m-1}z^k\left(\sum_{j=0}^{m-k-1}\alpha^jf_{k+j+1}\right)\quad\text{if}\quad f(z)=\sum_{k=0}^m z^kf_k.$$

The mapping $f \mapsto R_\alpha f$ defines a left linear operator on $\mathbb{H}[z]$ (the right backward shift) and allows us to introduce *right divided differences* by letting

$$[\alpha_1; f]_{\mathbf{r}} = f^{\boldsymbol{e}_{\mathbf{r}}}(\alpha_1), \qquad [\alpha_1, \dots, \alpha_k; f]_{\mathbf{r}} = (R_{\alpha_{k-1}} \cdots R_{\alpha_1} f)^{\boldsymbol{e}_{\mathbf{r}}}(\alpha_k) \quad \text{for} \quad k > 1.$$

A polynomial $f \in \mathbb{H}[z]$ satisfying right interpolation conditions $f^{\boldsymbol{e}_{\mathbf{r}}}(\alpha_i) = d_i$ $(i = 1, \dots, n)$ can be found from the linear system

$$f_0 + f_1\alpha_i + \ldots + f_{m-1}\alpha_i^{m-1} = d_i \quad \text{for} \quad i = 1, \dots, n,$$

and the matrix of this linear system (the right Vandermonde matrix) $V_m^{\mathbf{r}} = \left[\alpha_j^{i-1}\right]_{i=1,\dots,m}^{j=1,\dots,n} \in \mathbb{H}^{m\times n}$ is the transpose of the left Vandermonde matrix $V_m = V_m^{\boldsymbol{\ell}}$ considered above. More generally, a polynomial f satisfying "right" interpolation conditions

$$[\alpha_{i,1}, \dots, \alpha_{i,j}; f]_{\mathbf{r}} = c_{i,j} \quad \text{for} \quad i = 1, \dots, n; \; j = 1, \dots, k_i$$

can be found via solving a linear system whose matrix (the right confluent Vandermonde matrix based on spherical chains (11)) is given by

$$V_m^{\mathbf{r}}(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n) = [\, V_m^{\mathbf{r}}(\boldsymbol{\alpha}_1) \quad \dots \quad V_m^{\mathbf{r}}(\boldsymbol{\alpha}_n)\,], \quad \boldsymbol{\alpha}_i = (\alpha_{i,1}, \dots, \alpha_{i,k_i}), \tag{18}$$

where (compare with (8))

$$V_m^{\mathbf{r}}(\boldsymbol{\alpha}_i) = \left[[\alpha_{i,1}, \dots, \alpha_{i,j}; z^j]_{\mathbf{r}}\right]_{j=1,\dots,m}^{i=1,\dots,k_i}, \quad i = 1, \dots, n. \tag{19}$$

In contrast to the basic case, the confluent left and right Vandermonde matrices based on the same spherical chains are not transposed to each other. For example, if $\boldsymbol{\alpha} = (\alpha, \beta)$ $(\alpha \sim \beta \neq \bar{\alpha})$, then it follows from (8) and (19) that

$$V_4^{\boldsymbol{\ell}}(\boldsymbol{\alpha}) = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 0 & 1 & \beta + \alpha & \beta^2 + \beta\alpha + \alpha^2 \end{bmatrix}, \quad (V_4^{\mathbf{r}}(\boldsymbol{\alpha}))^\top = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 0 & 1 & \beta + \alpha & \beta^2 + \alpha\beta + \alpha^2 \end{bmatrix}.$$

Comparing the $(2, 4)$ entries in the latter matrices, we conclude that $V_m^{\boldsymbol{\ell}}(\boldsymbol{\alpha}) \neq (V_m^{\mathbf{r}}(\boldsymbol{\alpha}))^\top$ unless $\alpha\beta = \beta\alpha$, which occurs (since $\alpha \sim \beta \neq \bar{\alpha}$) only if $\beta = \alpha$. However, the results concerning left confluent Vandermonde matrices can be translated to their right counter-parts due to the following observation.

Lemma 5. *Let $V_m^{\boldsymbol{\ell}}(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ and $V_m^{\mathbf{r}}(\bar{\boldsymbol{\alpha}}_1, \dots, \bar{\boldsymbol{\alpha}}_n)$ be the left and the right confluent Vandermonde matrices based on spherical chains $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \dots, \alpha_{i,k_i})$ and $\bar{\boldsymbol{\alpha}}_i := (\bar{\alpha}_{i,1}, \dots, \bar{\alpha}_{i,k_i})$, respectively. Then $V_m^{\boldsymbol{\ell}}(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n) = (V_m^{\mathbf{r}}(\bar{\boldsymbol{\alpha}}_1, \dots, \bar{\boldsymbol{\alpha}}_n))^*$.*

We recall that as in the complex case, the adjoint to the matrix $A = (a_{ij})$ is defined by $A^* = (\bar{a}_{ji})$. Finally, it was observed in [4] that, although in general the quaternion matrices A and $A^\top$ may have different ranks, the equality $\operatorname{rank} V_m = \operatorname{rank} V_m^\top$ holds for Vandermonde matrices. We do not know if this result extends to confluent Vandermonde matrices. However, since $V_m^\top = V_m^{\mathbf{r}}$, the above result can be interpreted as follows: *the left and the right Vandermonde matrices have the same rank*. This version does admit the "confluent" extension.

Remark 6. The left and right confluent Vandermonde matrices $V_m^{\boldsymbol{\ell}}(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ and $V_m^{\mathbf{r}}(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ based on the same spherical chains have the same rank.

Indeed, replacing the spherical chains $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n$ by $\bar{\boldsymbol{\alpha}}_1, \dots, \bar{\boldsymbol{\alpha}}_n$, we come up with the same integer κ as for the original chains; therefore, $\operatorname{rank} V_m^{\boldsymbol{\ell}}(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n) = \operatorname{rank} V_m^{\boldsymbol{\ell}}(\bar{\boldsymbol{\alpha}}_1, \dots, \bar{\boldsymbol{\alpha}}_n)$. Since $\operatorname{rank} A = \operatorname{rank} A^*$ for any matrix A over $\mathbb{H}$, the statement follows by Lemma 5.

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