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Algebraic geometry

On Euler–Poincaré characteristics

Sur les caractéristiques d'Euler–Poincaré

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ABSTRACT

It is well known that the Euler characteristic of the cohomology of a complex algebraic variety coincides with the Euler characteristic of its cohomology with compact support. An old result of G. Laumon asserts that a relative version of this statement is true in ℓ-adic cohomology. The purpose of this note is to extend Laumon's result to the topological setting. Some applications are also discussed.

RÉSUMÉ

Il est bien connu que la caractéristique d'Euler de la cohomologie d'une variété algébrique complexe coïncide avec celle de sa cohomologie à support compact. Un résultat déjà ancien de G. Laumon affirme une version relative de cet énoncé en cohomologie ℓ-adique. Notre propos dans cette Note est d'étendre le résultat de Laumon au cadre topologique. Nous discutons également quelques applications.

1. Introduction

It is a remarkable fact about algebraic varieties that the Euler characteristic of the cohomology of a complex algebraic variety coincides with the Euler characteristic of the cohomology with compact support. Using Whitney stratifications, one may extend this to cohomology with coefficients in any algebraically constructible sheaf. These statements cry out for a generalization to the relative setting. This, and a bit more, is what this note aims to do.

The main result is Theorem 2.1; some applications are discussed in §3–§6. Theorem 2.1 is the topological analogue of an old result of G. Laumon in ℓ-adic cohomology [5]. The argument presented here uses the same local to global technique as [5]. Thus, all the main ideas are due to Laumon. After this note was written, J. Schürmann informed me that the non-mixed case of Theorem 2.1 is also covered in [9, §6.0.6] (with a similar proof).

In Theorem 5.1, we extend Theorem 2.1 to quotient stacks (i.e. to the setting of equivariant cohomology). In ℓ-adic cohomology, for the special case of quotients by finite groups, this is due to I. Illusie and W. Zheng [4, Theorem 1.3].

2. The main result

For a complex variety X, let DX be either:

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(i) $D^b(\mathrm{MHM}(X))$ – M. Saito's bounded derived category of mixed Hodge modules on X (see [7] and [8], in particular see [8, §4]);
(ii) $D^b_c(X_{\mathrm{an}})$ – the bounded derived category of algebraically constructible sheaves of A-modules on the complex analytic site associated with X. Here A is a commutative ring (with 1).

(i) will be referred to as the mixed case, (ii) as the non-mixed case.

For $\mathcal{M} \in DX$, let $[\mathcal{M}]$ denote the class of $\mathcal{M}$ in the Grothendieck ring of DX. Let (1) denote Tate twist in the mixed case, or the identity functor in the non-mixed case. Write $\widetilde{K}X$ for the quotient of the Grothendieck ring of DX by the ideal generated by elements of the form $[\mathcal{M}(1)] - [\mathcal{M}]$.

Let $f: X \to Y$ be a morphism of varieties. Associated with f are the derived functors $f_*, f_!: DX \to DY$. These induce group morphisms $f_*, f_!: \widetilde{K}X \to \widetilde{K}Y$.

Theorem 2.1. *The maps $f_*, f_!: \widetilde{K}X \to \widetilde{K}Y$ coincide.*

Proof. Using Nagata compactification, we may factor f as $f = p \circ j$ with p proper, and j an open immersion. Thus, it suffices to only consider open immersions.

Let $j: U \hookrightarrow X$ be an open immersion, and $i: Z \hookrightarrow X$ the inclusion of the closed complement $Z := X - U$. The distinguished triangle[1] $j_! j^* \to \mathrm{id} \to i_* i^*$ applied to j_* yields the triangle $j_! \to j_* \to i_* i^* j_*$. So it suffices to show that $i^* j_* = 0$ in $\widetilde{K}Z$.

Let $\mathrm{Bl}_Z X$ be the blow up of X along Z. Then we have a commutative diagram, with Cartesian squares and proper vertical arrows,

$$\begin{tikzcd}
\mathrm{Bl}_Z X \times_X Z \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{\pi}"'] & \mathrm{Bl}_Z X \arrow[d, "\pi"'] & U \arrow[l, "\tilde{j}"'] \arrow[d, equal] \\
Z \arrow[r, "i"] & X & U \arrow[l, "j"'] .
\end{tikzcd}$$

By a proper base change, $i^* j_* = i^* \pi_* \tilde{j}_* = \tilde{\pi}_* \tilde{i}^* \tilde{j}_*$. Hence, we may assume that Z is an effective Cartier divisor.

Let $\{U_1, \dots, U_n\}$ be a Zariski open cover of X such that, for each i, $Z \cap U_i$ is a principal divisor in U_i. For each subset $I \subseteq \{1, \dots, n\}$, set $V_I := \bigcap_{i \in I} V_i$, $U_I := U \cap V_I$, and $Z_I := Z \cap V_I$. Let $u_I: U_I \hookrightarrow U$, $j_I: U_I \hookrightarrow X$, $z_I: Z_I \hookrightarrow Z$, and $i_I: Z_I \hookrightarrow X$ be the evident inclusions. Then the usual argument, à la Čech, yields

$$i^* j_* = \sum_{\substack{I \subseteq \{1,\dots,n\}, \\ I \neq \varnothing}} (-1)^{|I|-1} z_{I*} i_I^* j_{I*} u_I^*$$

in the Grothendieck group. Hence, we may further assume that Z is principal.

Let $f: X \to \mathbf{A}^1$ be a defining equation for Z. Then we have a distinguished triangle (see Appendix A):

$$\psi_{f,1} j_* \to \psi_{f,1} j_*(-1) \to i_* i^* j_*,$$

where $\psi_{f,1}: DX \to DX$ denotes the (unipotent part of the) nearby cycles functor associated with f (see [8, 2.3.4]),[2] and (-1) is the inverse of (1). Applying i^* to this triangle yields $i^* j_* = 0$ in $\widetilde{K}Z$. □

Clearly, the proof of Theorem 2.1 holds in any 'sheaf setting' admitting a partial yoga of the six functors and an appropriate formalism of nearby cycles. Possible candidates (other than those under consideration): holonomic (not necessarily regular) D-modules, twistor D-modules, p-adic cohomology, motivic sheaves, etc.

3. Hodge polynomials and Euler characteristics

Mixed Hodge modules yield functorial mixed Hodge structures on the (rational) cohomology $H^*(X)$ as well as the cohomology with compact support $H^*_c(X)$. Set

$$h^{i,j;k} := \dim \mathrm{Gr}^i_F \mathrm{Gr}^W_{i+j} H^k(X) \quad \text{and} \quad h^{i,j;k}_c := \dim \mathrm{Gr}^i_F \mathrm{Gr}^W_{i+j} H^k_c(X),$$

where Gr_F (resp. Gr^W) denotes the associated graded of the Hodge (resp. weight filtration). Define polynomials

[1] A convenient summary of the functors, distinguished triangles, and properties we are using may be found in [1, §1.2]. For the mixed case the distinguished triangle we are using is [8, (4.4.1)].

[2] In the non-mixed case our $\psi_{f,1}$ is often denoted by ${}^p\psi_{f,1}$. In other words our nearby cycles functor is normalized so as to be t-exact for the perverse t-structure.

$$P(X;u,v) := \sum_{i,j,k}(-1)^k h^{i,j;k} u^i v^j \quad \text{and} \quad E(X;u,v) := \sum_{i,j,k}(-1)^k h_c^{i,j;k} u^i v^j.$$

Corollary 3.1. $P(X;t,t^{-1}) = E(X;t,t^{-1})$.

Let

$$\chi(X) := P(X;1,1) \quad \text{and} \quad \chi_c(X) := E(X;1,1).$$

So $\chi(X)$ is the ordinary Euler characteristic of X, and $\chi_c(X)$ is the Euler characteristic of X with compact support. Thus, we recover the identity alluded to in the introduction:

Corollary 3.2. $\chi(X) = \chi_c(X)$.

4. Exceptional pullback

From now on we assume that when in the non-mixed case our underlying ring of coefficients A is of finite global dimension (so that the exceptional pullback $f^!$ below is well defined). Then with a morphism of varieties $f: X \to Y$ we may also associate the triangulated functors $f^*, f^!: DY \to DX$. Denote the induced group morphisms $\widetilde{K}Y \to \widetilde{K}X$ by f^* and $f^!$ also.

Proposition 4.1. *The maps $f^*, f^!: \widetilde{K}Y \to \widetilde{K}X$ coincide.*

Proof. First, we handle the case of a closed immersion. Let $i: D \hookrightarrow Z$ be a closed immersion and $j: U \hookrightarrow Z$ the complementary open immersion. Apply i^* to the distinguished triangle $i_*i^! \to \mathrm{id} \to j_*j^*$ to get the triangle $i^! \to i^* \to i^*j_*j^*$. By Theorem 2.1, $i^*j_* = i^*j_! = 0$. So $i^! = i^*$.

Now suppose that $f: X \to Y$ is a morphism with X smooth. Via its graph, f factors as a closed immersion $X \hookrightarrow X \times Y$ followed by the projection $p: X \times Y \to Y$. As X is smooth, $p^! = p^*$ in $\widetilde{K}(X \times Y)$. Consequently, $f^! = f^*$.

We now deal with the general case. Stratify X by closed subvarieties

$$\varnothing = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$$

such that each $Z_i := X_i - X_{i-1}$ is smooth. Let $j_i: Z_i \hookrightarrow X$ be the inclusion, and set $f_i := f \circ j_i$. Then

$$f^! = \sum_{i=1}^{n} j_{i*} f_i^! = \sum_{i=1}^{n} j_{i*} f_i^* = \sum_{i=1}^{n} j_{i!} f_i^* = f^*. \qquad \square$$

Corollary 4.2. *Let $\mathbb{1}_X \in DX$ be the monoidal unit (i.e., the 'constant sheaf') and let $\omega_X \in DX$ be the dualizing object. Then $[\mathbb{1}_X] = [\omega_X]$ in $\widetilde{K}X$.*

5. Quotient stacks

Suppose a linear algebraic group G acts on X. Let D_GX denote the analogue of DX for the stack $[X/G]$ (for instance, see [2]). Associated with a G-equivariant morphism $f: X \to Y$, one has functors $f^*, f_*, f_!, f^!$ between D_GX and D_GY satisfying the usual formalism. Write $\widetilde{K}_GX$ for the Grothendieck ring of D_GX modulo the same ideal as before.

Theorem 5.1. *The induced maps $f_*, f_!: \widetilde{K}_GX \to \widetilde{K}_GY$ coincide. Similarly, the induced maps $f^!, f^*: \widetilde{K}_GY \to \widetilde{K}_GX$ coincide.*

Proof. We will only demonstrate the assertion for $f_*, f_!$. The proof for $f^*, f^!$ is similar and left to the reader.

Using Steifel varieties, as in [2, §3.1], one sees that property $(*)$ of [2, §2.2.4] holds for Y (alternatively, see [6, Lemme 18.7.5]). In particular, we obtain a G-torsor $Z \to \overline{Z}$ along with a smooth equivariant morphism $p: Z \to Y$ of some relative dimension d with connected fibers.[3] Let $\mathcal{M}_GY$, $\mathcal{M}_GZ$ denote the hearts of the perverse t-structure on D_GY and D_GZ, respectively. Then the functor $p^*[d]: \mathcal{M}_GY \to \mathcal{M}_GZ$ is full, faithful, and its essential image is closed under taking subquotients. Consequently, $p^*: \widetilde{K}_GY \to \widetilde{K}_GZ$ is injective. Thus, using a smooth base change, we may assume that there exist

[3] There is a subtlety here. In general, $\overline{Z}$ (which is constructed as an associated bundle) is only an algebraic space and not necessarily a variety. In the non-mixed setting this is a non-issue. In the mixed setting there are two ways to deal with this. One can either restrict to only quasi-projective varieties, in which case $\overline{Z}$ is a variety. Or, one may observe that as mixed Hodge modules are étale local, the formalism of DX and D_GX extends to algebraic spaces. However, note that the details of such an extension are not yet available in the literature.

G-torsors $X \to \overline{X}$ and $Y \to \overline{Y}$. But then we have canonical equivalences $D_G X \simeq D\overline{X}$ and $D_G Y \simeq D\overline{Y}$ that commute with the functors f_* and $f_!$. Hence, the assertion reduces to the non-equivariant statement of Theorem 2.1. □

The above proof applies more generally to 'Bernstein–Lunts stacks' (see [6, §18.7.4]). However, I do not know of any examples of such stacks other than quotient stacks.

6. Traces

Let G^0 denote the identity component of G and let $\Gamma := G/G^0$ be the finite group of components. Let $\mathcal{M} \in D_G X$. Then the G^0-equivariant cohomology $H^*_{G^0}(X; \mathcal{M})$ and the G^0-equivariant cohomology with compact support $H^*_{G^0,c}(X; \mathcal{M})$ are Γ-modules. The Γ-action respects the Hodge structures in the mixed case.

Now assume that we are in the non-mixed case and that our coefficient ring is a field (i.e., we are dealing with cohomology with field coefficients). For $g \in \Gamma$ and $k \in \mathbf{Z}$, consider the traces

$$\mathrm{Tr}(g, H^k_{G^0}(X; \mathcal{M})) \quad \text{and} \quad \mathrm{Tr}(g, H^k_{G^0,c}(X; \mathcal{M}))$$

of the g-action. Set

$$\chi_g(\mathcal{M}) := \sum_k (-1)^k \mathrm{Tr}(g, H^k_{G^0}(X; \mathcal{M})),$$
$$\chi_{g,c}(\mathcal{M}) := \sum_k (-1)^k \mathrm{Tr}(g, H^k_{G^0,c}(X; \mathcal{M})).$$

Note that we are implicitly assuming that the right hand side above makes sense. I.e., $H^k_{G^0}(X; \mathcal{M})$ and $H^k_{G^0,c}(X; \mathcal{M})$ vanish for large k.

Corollary 6.1. *We have $\chi_g(\mathcal{M}) = \chi_{g,c}(\mathcal{M})$. In particular, if G is finite, then for each $g \in G$:*

$$\sum_k (-1)^k \mathrm{Tr}(g, H^k(X; \mathcal{M})) = \sum_k (-1)^k \mathrm{Tr}(g, H^k_c(X; \mathcal{M})).$$

In the case of rational coefficients, this may be refined as follows. Assume we are in the mixed case. To prevent the notation from getting too painful, we will only consider the case of $\mathcal{M}$ being the 'constant sheaf' (i.e., the trivial weight 0 mixed Hodge module whose underlying rational structure is the constant sheaf). As in the non-equivariant situation, we obtain functorial mixed Hodge structures on the (rational) equivariant cohomology $H^*_{G^0}(X)$ as well as the equivariant cohomology with compact support $H^*_{G^0,c}(X)$. For $g \in \Gamma$, set

$$h_g^{i,j;k} := \mathrm{Tr}(g, \mathrm{Gr}^i_F \mathrm{Gr}^W_{i+j} H^k_{G^0}(X)) \quad \text{and} \quad h_{g,c}^{i,j;k} := \mathrm{Tr}(g, \mathrm{Gr}^i_F \mathrm{Gr}^W_{i+j} H^k_{G^0,c}(X)),$$

where Gr_F (resp. Gr^W) denotes the associated graded of the Hodge (resp. weight filtration). Define formal power series

$$P_g(X; u, v) := \sum_{i,j,k} (-1)^k h_g^{i,j;k} u^i v^j \quad \text{and} \quad E_g(X; u, v) := \sum_{i,j,k} (-1)^k h_{g,c}^{i,j;k} u^i v^j.$$

Corollary 6.2. $P_g(X; t, t^{-1}) = E_g(X; t, t^{-1})$.

If G is a finite group and g the identity element, then this recovers Corollary 3.1.

Appendix A

Although well known, the distinguished triangle $\psi_{f,1} j_* \to \psi_{f,1} j_*(-1) \to i_* i^* j_*$ does not seem to be explicitly stated in any of the usual references. Regardless, it is easy to derive it from more familiar triangles. To avoid any notational confusion, I will only deal with the mixed case here. The proof for the non-mixed case is the same (modulo sorting out notational differences) and the distinguished triangles used can be found (amongst other places) in [3].

Let $f: X \to \mathbf{A}^1$ be a morphism of varieties. Set $Z := f^{-1}(0)$, and $U := X - Z$. Write $i: Z \hookrightarrow X$, and $j: U \hookrightarrow X$ for the evident inclusions. Then we have the usual nearby cycles setup

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] & U \arrow[l, hook', "j"'] \arrow[d] \\ \{0\} \arrow[r] & \mathbf{A}^1 & \mathbf{A}^1 - \{0\} \arrow[l] \end{tikzcd}$$

with all squares Cartesian. Let

$$\psi_{f,1} \colon DX \to DX \quad \text{and} \quad \phi_{f,1} \colon DX \to DX$$

denote the (unipotent part of the) nearby cycles functor and vanishing cycles functor associated with f. Then we have distinguished triangles (see [8, 2.24.2]):

$$\psi_{f,1} \xrightarrow{\text{can}} \phi_{f,1} \to i_* i^* \quad \text{and} \quad i_* i^! \to \phi_{f,1} \xrightarrow{\text{var}} \psi_{f,1}(-1).$$

Applying these triangles to j_* we get the distinguished triangle

$$\psi_{f,1} j_* \xrightarrow{\text{can}} \phi_{f,1} j_* \to i_* i^* j_*$$

and the isomorphism

$$\text{var} \colon \phi_f j_* \xrightarrow{\sim} \psi_f j_*(-1).$$

These yield the desired distinguished triangle. $\psi_{f,1} j_* \to \psi_{f,1} j_*(-1) \to i_* i^* j_*$.

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