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Algebraic geometry

Diagonal property of the symmetric product of a smooth curve ☆

Propriété de la diagonale pour les produits symétriques d'une courbe lisse

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ABSTRACT

Let C be an irreducible smooth projective curve defined over an algebraically closed field. We prove that the symmetric product $\mathrm{Sym}^d(C)$ has the diagonal property for all $d \geq 1$. For any positive integers n and r, let $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(nr)$ be the Quot scheme parameterizing all the torsion quotients of $\mathcal{O}_C^{\oplus n}$ of degree nr. We prove that $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(nr)$ has the weak-point property.

RÉSUMÉ

Soit C une courbe irréductible, lisse, définie sur un corps algébriquement clos. Nous montrons que le produit symétrique $\mathrm{Sym}^d(C)$ a la propriété de la diagonale, pour tout $d \geq 1$. Pour tous entiers n et r, soit $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(nr)$ le schéma Quot paramétrant tous les quotients de torsion de $\mathcal{O}_C^{\oplus n}$ de degré nr. Nous montrons que $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(nr)$ a la propriété du point, faible.

1. Introduction

In [8], Pragacz, Srinivas and Pati introduced the diagonal and (weak) point properties of a variety, which we recall.

Let X be a variety of dimension d over an algebraically closed field k. It is said to have the *diagonal property* if there is a vector bundle $E \longrightarrow X \times X$ of rank d, and a section $s \in H^0(X \times X, E)$, such that the zero scheme of s is the diagonal in $X \times X$. The variety X is said to have the *weak point property* if there is a vector bundle F on X of rank d, and a section $t \in H^0(X, F)$, such that the zero scheme of t is a (reduced) point of X. The diagonal property implies the weak-point property because the restriction of the above section s to $X \times \{x_0\}$ vanishes exactly on x_0.

These properties were extensively studied in [8] and [5]. In particular, it was shown that:

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- they impose strong conditions on the variety,
- on the other hand there are many example of varieties with these properties.

Here we investigate these conditions for some varieties associated with a smooth projective curve.

Let C be an irreducible smooth projective curve over k. For any positive integer d, let $\mathrm{Sym}^d(C)$ be the quotient of C^d for the natural action of the group of permutations of $\{1, \cdots, d\}$. It is a smooth projective variety of dimension d. We prove the following (Theorem 3.1).

Theorem 1.1. *The variety* $\mathrm{Sym}^d(C)$ *has the diagonal property.*

Theorem 1 in [8, p. 1236] contains several examples of surfaces satisfying the diagonal property. We note that the surface $\mathrm{Sym}^2(C)$ is not among them.

For positive integers n and d, let $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(d)$ be the Quot scheme parameterizing the torsion quotients of $\mathcal{O}_C^{\oplus n}$ of degree d. Quot schemes were constructed in [6] (see [7] for an exposition on [6]). The variety $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(d)$ is smooth projective, and its dimension is nd. Note that $\mathcal{Q}_{\mathcal{O}_C}(d) = \mathrm{Sym}^d(C)$. These varieties $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(d)$ are extensively studied in algebraic geometry and mathematical physics (see [3,2,1,4] and references therein).

We prove the following (Theorem 2.2).

Theorem 1.2. *If d is a multiple of n, then the variety* $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(d)$ *has the weak-point property.*

2. Quot scheme and the weak-point property

We continue with the notation of the introduction.

For a locally free coherent sheaf E of rank n on C, let $\mathcal{Q}_E(d)$ be the Quot scheme parameterizing all torsion quotients of E of degree d. Equivalently, $\mathcal{Q}_E(d)$ parametrizes all coherent subsheaves of E of rank n and degree $\mathrm{degree}(E) - d$. Note that any coherent subsheaf of E is locally free because any torsion-free coherent sheaf on a smooth curve is locally free. This $\mathcal{Q}_E(d)$ is an irreducible smooth projective variety of dimension nd.

There is a natural morphism

$$\varphi' : \mathcal{Q}_E(d) \longrightarrow \mathcal{Q}_{\bigwedge^n E}(d)$$

that sends any subsheaf $S \subset E$ of rank n and degree $\mathrm{degree}(E) - d$ to the subsheaf $\bigwedge^n S \subset \bigwedge^n E$. Next note that $\mathcal{Q}_{\bigwedge^n E}(d)$ is identified with the symmetric product $\mathrm{Sym}^d(C)$ by sending any subsheaf $S' \subset \bigwedge^n E$ to the scheme theoretic support of the quotient sheaf $(\bigwedge^n E)/S'$. Let

$$\varphi : \mathcal{Q}_E(d) \longrightarrow \mathrm{Sym}^d(C) \tag{2.1}$$

be the composition of φ' with this identification of $\mathcal{Q}_{\bigwedge^n E}(d)$ with $\mathrm{Sym}^d(C)$. It should be mentioned that for a subsheaf $S \subset E$ of rank n and $\mathrm{degree}(E) - d$, the image $\varphi(S) \in \mathrm{Sym}^d(C)$ does not, in general, coincide with the scheme theoretic support of the quotient sheaf E/S.

The symmetric product $\mathrm{Sym}^d(C)$ is the moduli space of effective divisors of degree d on C. Let

$$D \subset Y := C \times \mathrm{Sym}^d(C) \tag{2.2}$$

be the universal divisor. So the fiber of D over a point $a \in \mathrm{Sym}^d(C)$ is the zero dimensional subscheme of C of length d defined by a. Let

$$\mathcal{D} = (\mathrm{Id}_C \times \varphi)^{-1}(D) \subset C \times \mathcal{Q}_E(d) \tag{2.3}$$

be the inverse image of D, where φ is constructed in (2.1).

Remark 2.1. Let L be a line bundle on C. For E as above, if $S \subset E$ is a subsheaf of rank n and degree $\mathrm{degree}(E) - d$, then

$$S \otimes L \subset E \otimes L$$

is a subsheaf of rank n and degree $\mathrm{degree}(E \otimes L) - d$. Therefore, we get an isomorphism

$$\mathcal{Q}_E(d) \xrightarrow{\sim} \mathcal{Q}_{E \otimes L}(d)$$

by sending any subsheaf $S \subset E$ to the subsheaf $S \otimes L \subset E \otimes L$.

Theorem 2.2. *For positive integers d, n such that d is a multiple of n, the Quot scheme* $\mathcal{Q}_{\mathcal{O}_C^n}(d)$ *satisfies the weak-point property.*

Proof. Let $r \in \mathbb{N}$ be such that $d = rn$. Fix a closed point x_0 in C. The line bundle $\mathcal{O}_C(rx_0)$ on C will be denoted by L. By Remark 2.1 it is enough to prove the weak-point property for $\mathcal{Q}_{L^{\oplus n}}(d)$.

Let $\mathcal{D} \hookrightarrow C \times \mathcal{Q}_{L^{\oplus n}}(d)$ be the divisor constructed in (2.3). Let

$$p : \mathcal{D} \longrightarrow C \quad \text{and} \quad q : \mathcal{D} \longrightarrow \mathcal{Q}_{L^{\oplus n}}(d) \tag{2.4}$$

be the natural projections. Taking the direct sum of copies of the natural inclusion

$$\iota : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(rx_0),$$

we get a short exact sequence of sheaves on C

$$0 \longrightarrow \mathcal{O}_C^{\oplus n} \xrightarrow{\iota^{\oplus n}} \mathcal{O}_C(rx_0)^{\oplus n} \longrightarrow T \longrightarrow 0, \tag{2.5}$$

where T is a torsion sheaf on C of degree $nr = d$. Therefore, this quotient T is represented by a point of $\mathcal{Q}_{L^{\oplus n}}(d)$. Let

$$t_0 \in \mathcal{Q}_{L^{\oplus n}}(d) \tag{2.6}$$

be the point representing T.

The direct image

$$F := q_* p^* L^{\oplus n} \longrightarrow \mathcal{Q}_{L^{\oplus n}}(d)$$

is a vector bundle of rank nd, where p and q are the projections in (2.4). We will construct a section of F. The section of $\mathcal{O}_C$ given by the constant function 1 will be denoted by s_0. Consider the section

$$s := \iota^{\oplus n}(s_0^{\oplus n}) \in H^0(C, L^{\oplus n}),$$

where $\iota^{\oplus n}$ is the homomorphism in (2.5). We have

$$\widetilde{s} := q_* p^* s \in H^0(\mathcal{Q}_{L^{\oplus n}}(d), F). \tag{2.7}$$

For the point t_0 in (2.6), the scheme theoretic inverse image

$$q^{-1}(t_0) \subset \mathcal{D} \subset C \times \mathcal{Q}_{L^{\oplus n}}(d)$$

is $(rx_0) \times t_0$, where q is the projection in (2.4). Since the section $\iota(s_0)$ of L vanishes exactly on rx_0, this implies that the section $\widetilde{s}$ in (2.7) vanishes exactly on the reduced point t_0. Therefore, $\mathcal{Q}_{L^{\oplus n}}(d)$ has the weak-point property. □

3. Diagonal property for symmetric product of curves

Theorem 3.1. *For any $d \geq 1$, the symmetric product $\mathrm{Sym}^d(C)$ of a smooth projective curve C has the diagonal property.*

Proof. Consider the divisor D in (2.2). Let

$$L = \mathcal{O}_Y(D) \longrightarrow Y \tag{3.1}$$

be the line bundle. Now consider $Z := Y \times \mathrm{Sym}^d(C) = C \times \mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C)$. Let

$$\alpha : Z \longrightarrow C, \ \beta : Z \longrightarrow \mathrm{Sym}^d(C) \quad \text{and} \quad \gamma : Z \longrightarrow \mathrm{Sym}^d(C) \tag{3.2}$$

be the projections defined by $(x, y, z) \longmapsto x$, $(x, y, z) \longmapsto y$ and $(x, y, z) \longmapsto z$ respectively. Let

$$\widetilde{D} := (\alpha \times \gamma)^{-1}(D) \subset C \times \mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C) = Z \tag{3.3}$$

be the inverse image, where D is defined in (2.2), and $\alpha \times \gamma : Z \longrightarrow C \times \mathrm{Sym}^d(C)$ sends any (x, y, z) to (x, z).

Let

$$p : \widetilde{D} \longrightarrow \mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C)$$

be the projection defined by $b \longmapsto (\beta(b), \gamma(b))$, where β and γ are defined in (3.2), and $\widetilde{D}$ is constructed in (3.3). Consider the direct image

$$V := p_*(((\alpha \times \beta)^* L)|_{\widetilde{D}}) \longrightarrow \mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C), \tag{3.4}$$

where L is the line bundle in (3.1). The natural projection

$$D \longrightarrow \mathrm{Sym}^d(C), \ (x, y) \longmapsto y,$$

where D is defined in (2.2), is a finite morphism of degree d. This implies that p is a finite morphism of degree d. Consequently, the direct image V is a vector bundle on $\mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C)$ of rank d.

Consider the natural inclusion $\mathcal{O}_Y \hookrightarrow \mathcal{O}_Y(D) = L$ (see (3.1)). Let

$$\sigma_0 \in H^0(Y,\ L) \tag{3.5}$$

be the section given by the constant function 1 using this inclusion. Let

$$\sigma := p_*(((\alpha \times \beta)^* \sigma_0)|_{\widetilde{D}}) \in H^0(\mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C),\ V)$$

be the section of V (constructed in (3.4)) given by σ_0.

We will show that the scheme theoretic inverse image

$$\sigma^{-1}(0) \subset \mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C)$$

is the diagonal.

Take any point $(a\,,b) \in \mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C)$ such that $a \neq b$. Then there is a point $z \in C$ such that the multiplicity of z in a is strictly smaller than the multiplicity of z in b. We note that the scheme theoretic inverse image

$$p^{-1}((a\,,b)) \subset \widetilde{D} \subset Z = C \times \mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C)$$

is $\{(a\,,b)\} \times \widehat{b}$, where $\widehat{b}$ is the zero dimensional subscheme of C of length d defined by b. On the other hand, for the section σ_0 in (3.5), the intersection $\sigma_0^{-1}(0) \bigcap (C \times \{a\})$ is the zero dimensional subscheme $\widehat{a}$ of C of length d defined by a. Since the multiplicity of z in a is strictly smaller than the multiplicity of z in b, we have:

$$\sigma_0((z_0\,,b)) \neq 0.$$

Consequently, $\sigma((a\,,b)) \neq 0$.

Now take a point $(a\,,a)$ on the diagonal of $\mathrm{Sym}^d(C) \times \mathrm{Sym}^d(C)$. We have observed above that the inverse image

$$p^{-1}((a\,,a)) \subset C$$

coincides with the intersection $\sigma_0^{-1}(0) \bigcap (C \times a)$. This implies that

- $\sigma((a\,,a)) = 0$, and
- $\sigma^{-1}(0)$ is the reduced diagonal.

Therefore, $\mathrm{Sym}^d(C)$ has the diagonal property. □

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