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Composition operators on Hilbert spaces of entire functions ☆

Opérateurs de composition sur les espaces de Hilbert de fonctions entières

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ABSTRACT

In this Note, we introduce Hilbert spaces of entire functions in the complex plane $\mathbb{C}$. We study composition operators on these spaces and obtain, in particular, criteria for the boundedness and compactness of such operators. Our results contain the corresponding results of Chacón et al. (2007) [1] as particular cases.

RÉSUMÉ

Dans cette Note, nous introduisons des espaces de Hilbert de fonctions entières dans le plan complexe $\mathbb{C}$. Nous étudions les opérateurs de composition sur ces espaces et obtenons notamment des critères pour que ces opérateurs soient bornés ou compacts. Nous retrouvons les résultats correspondents de Chacón et al. (2007) [1] comme cas particuliers.

1. Introduction

Let $\beta = (\beta_n)$ be a sequence of positive real numbers, called “weights”. It is well known that the set of complex sequences

$$\ell^2_\beta = \left\{ \mathbf{a} = (a_n) : \ \|\mathbf{a}\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\} \tag{1}$$

is a Hilbert space with the inner product defined by

$$\langle \mathbf{a}, \mathbf{c} \rangle = \sum_{n=0}^{\infty} a_n \overline{c_n} \beta_n^2, \quad \mathbf{a} = (a_n), \mathbf{c} = (c_n) \in \ell^2_\beta.$$

Weighted sequence spaces have many important applications in studying operators on (weighted) function spaces. We refer the reader to [3] for more detailed information.

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Using ℓ^2_β, one can easily construct a new type of Hilbert spaces of entire functions. Note that a complex-valued power series $f(z)=\sum\limits_{n=0}^{\infty} a_n z^n$ represents an entire function if and only if

$$\lim_{n\to\infty} |a_n|^{1/n} = 0. \tag{2}$$

Let E be the set of all complex sequences (a_n) that satisfies condition (2). Since ℓ^2_β is a Hilbert space and every element in E corresponds to an entire function, it is natural to consider all coefficient sequences (a_n) in the intersection $\ell^2_\beta \cap E$. However, depending on the values of (β_n), the subspace $\ell^2_\beta \cap E$ can be incomplete in the norm (1), and so the space of all functions generated from elements of $\ell^2_\beta \cap E$ and equipped with the norm (1), is not necessarily a Hilbert space.

The following result characterizes all possible inclusion–exclusion relations between ℓ^2_β and E.

Theorem 1.1. *Let $\beta_* = \liminf\limits_{n\to\infty} \beta_n^{1/n}$ and $\beta^* = \limsup\limits_{n\to\infty} \beta_n^{1/n}$. Exactly one of the following alternative cases can happen.*

(i) $\ell^2_\beta \subsetneq E$ if and only if $\beta_* = +\infty$.
(ii) $E \subsetneq \ell^2_\beta$ if and only if $\beta^* < +\infty$.
(iii) $E \setminus \ell^2_\beta \neq \varnothing$ and $\ell^2_\beta \setminus E \neq \varnothing$ if and only if $\beta_* < \beta^* = +\infty$.

Furthermore, it is worth to remark that the set

$$\mathcal{H}(\beta, E) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \ \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

is a Hilbert space of entire functions if and only if (β_n) satisfies $\beta_* = +\infty$, i.e., if and only if condition (i) in Theorem 1.1 holds.

It should be noted that the Hilbert space of entire functions using the representation of complex functions through power series and a comparison function was initiated in [2]. More precisely, given a sequence of positive real numbers (γ_n) with $(\gamma_{n+1}/\gamma_n) \downarrow 0$, the space E^2_γ is defined as:

$$E^2_\gamma = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \ \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \gamma_n^{-2} < \infty \right\}.$$

The space E^2_γ is in fact a special case of $\mathcal{H}(\beta, E)$ by letting $\beta_n = \gamma_n^{-1}$ and $(\beta_n/\beta_{n+1}) \downarrow 0$ (see Proposition 2.1 below). Moreover, if $\frac{(n+1)\gamma_{n+1}}{\gamma_n} \downarrow \tau$ for some $\tau > 0$, then there exists an $f \in E^2_\gamma$ of order 1 and type $\sigma < \tau$, and for any $\mathfrak{b} \in \mathbb{C}$, the translation operator $T_{\mathfrak{b}} : \ f(z) \mapsto f(z+\mathfrak{b})$ acting on the space is bounded. Following these results, the authors of [1] defined composition operators C_φ acting on the spaces E^2_γ with $\frac{(n+1)\gamma_{n+1}}{\gamma_n} \downarrow \tau > 0$, and derived the criteria for boundedness and compactness of these operators.

Although the problem of composition operators on the general Hilbert spaces of entire functions (e.g., $\mathcal{H}(\beta, E)$) is involved and still remains open, this paper aims to study some classes of $\mathcal{H}(\beta, E)$. We obtain criteria for boundedness and compactness of composition operators on these classes. In particular, we generalized the corresponding results of [1] even to the cases when the condition $\frac{(n+1)\gamma_{n+1}}{\gamma_n} \downarrow \tau > 0$ does not hold.

2. Various Hilbert spaces of entire functions

Recall that for a positive sequence (β_n), we denote $\beta_* = \liminf\limits_{n\to\infty} \beta_n^{1/n}$. Also we define

$$\beta_\rho = \liminf_{n\to\infty} \frac{\log \beta_n}{n \log n},$$

and the following function $\mu : \mathbb{N} \setminus \{0\} \to \mathbb{R}^+$, defined by

$$\mu(n) = \frac{n\beta_{n-1}}{\beta_n}.$$

We list below the sets of sequences $\beta = (\beta_n)$ that will be considered in this section.

- $\mathbf{A} = \{\beta : \ \beta_* = +\infty\}$.
- $\mathbf{B} = \{\beta : \ \beta_\rho > 0\}$.

- $\mathbf{C} = \{\beta : \ \beta_\rho < +\infty\}$.
- $\mathbf{D} = \left\{\beta : \ \dfrac{\beta_n}{\beta_{n+1}} \downarrow 0\right\}$.
- $\mathbf{E} = \{\beta : \ (\mu(n)) \text{ is bounded}\}$.
- $\mathbf{G} = \{\beta : \ \exists\ \tau = \tau(\beta) > 0 \text{ such that } \mu(n) \downarrow \tau\}$.

The following result is important for characterization of some classes of $\mathcal{H}(\beta, E)$.

Proposition 2.1. *The following are true.*

(i) $\mathbf{C} \nsubseteq \mathbf{A}$ *but* $\mathbf{D} \subsetneq \mathbf{A}$
(ii) $\mathbf{D} \setminus \mathbf{E} \neq \varnothing$ *and* $\mathbf{E} \setminus \mathbf{D} \neq \varnothing$.
(iii) $\mathbf{E} \subsetneq \mathbf{B} \subsetneq \mathbf{A}$.
(iv) $\mathbf{G} \subsetneq \mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \cap \mathbf{D} \cap \mathbf{E} = \mathbf{C} \cap \mathbf{D} \cap \mathbf{E}$.

Be reminded that the condition $\beta \in \mathbf{G}$ is the main assumption in the paper [1], but $\mathbf{G}$ is rather small: it is contained even in the intersection $\mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \cap \mathbf{D} \cap \mathbf{E}$.

2.1. Spaces $\mathcal{H}(\beta, \rho)$ and $\mathcal{H}(\beta, \rho_+)$

One of the most popular characteristics of entire functions is their growth orders. Recall (see, e.g., [4]) that a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has finite order ρ if and only if

$$\rho = \limsup_{n\to\infty} \frac{-n \log n}{\log |a_n|} < +\infty. \tag{3}$$

We have the following result.

Theorem 2.2. *Every function in a space $\mathcal{H}(\beta, E)$ has a finite order if and only if $\beta \in \mathbf{B}$.*

We are also interested in a space $\mathcal{H}(\beta, E)$ in which (β_n) does not diverge too fast to infinity, so that there is some function f in the space with non-zero order (which can be $+\infty$).

Theorem 2.3. *Suppose $\beta \in \mathbf{A}$. The space $\mathcal{H}(\beta, E)$ has some function f with non-zero order if and only if $\beta \in \mathbf{C}$.*

Note that condition $\beta \in \mathbf{A}$ is needed in the statement of Theorem 2.3, because of Proposition 2.1(i).

As a consequence of Theorems 2.2 and 2.3, we have the following result.

Corollary 2.4. *Suppose $\beta \in \mathbf{A}$. Exactly one of the following alternative cases can happen to the space $\mathcal{H}(\beta, E)$.*

(i) *Every function $f \in \mathcal{H}(\beta, E)$ has order 0 if and only if $\beta_\rho = +\infty$, i.e., $\beta \in \mathbf{B} \setminus \mathbf{C}$.*
(ii) *There exists a function $f \in \mathcal{H}(\beta, E)$ that does not have finite order if and only if $\beta_\rho = 0$, i.e., $\beta \in \mathbf{C} \setminus \mathbf{B}$.*
(iii) *Every function in $\mathcal{H}(\beta, E)$ has finite order and there exists a function f having positive order if and only if $\beta_\rho \in (0, +\infty)$, i.e. $\beta \in \mathbf{B} \cap \mathbf{C}$.*

Now we denote by $\mathcal{H}(\beta, \rho)$ a space $\mathcal{H}(\beta, E)$ with $\beta \in \mathbf{B}$, i.e., a space $\mathcal{H}(\beta, E)$ of entire functions with finite orders. We also denote by $\mathcal{H}(\beta, \rho_+)$ a space $\mathcal{H}(\beta, E)$ with $\beta \in \mathbf{B} \cap \mathbf{C}$, i.e., a space $\mathcal{H}(\beta, E)$ of entire functions with finite orders that contains at least one function with positive order.

Obviously, $\mathcal{H}(\beta, \rho_+)$ is a special case of $\mathcal{H}(\beta, \rho)$, in the sense that any space $\mathcal{H}(\beta, \rho_+)$ is a space $\mathcal{H}(\beta, \rho)$, but the reverse is not true. Similarly, $\mathcal{H}(\beta, \rho)$ is a special case of $\mathcal{H}(\beta, E)$.

2.2. A comparison between spaces E^2_γ and $\mathcal{H}(\beta, \rho_+)$; the space $\mathcal{H}(\beta, \rho_+, T)$

From definitions, it follows easily that E^2_γ is a space $\mathcal{H}(\beta, E)$ with $\beta \in \mathbf{D}$. This class of spaces was introduced in [2] when the authors investigated the cyclic behavior of the translation operators. In particular, they proved that for any $\mathfrak{b} \in \mathbb{C} \setminus \{0\}$, the translation operator $T_\mathfrak{b} : f(z) \mapsto f(z + \mathfrak{b})$ is bounded on E^2_γ if and only if $\beta \in \mathbf{E}$.

Let $E^2_\gamma(T)$ denote a space $\mathcal{H}(\beta, E)$ with $\beta \in \mathbf{D} \cap \mathbf{E}$. With an appropriate modification, the condition $\beta \in \mathbf{D}$ can be omitted and the following results (which generalizes the corresponding results in [2]) still hold.

Proposition 2.5. *Let D be the derivative operator acting on a space $\mathcal{H}(\beta, E)$, i.e. $D : f \mapsto f'$, and let $\mathfrak{b} \neq 0$ be a complex number. The following are equivalent*

(i) *D is bounded on $\mathcal{H}(\beta, E)$.*
(ii) *The translation operator $T_{\mathfrak{b}}$ is bounded on $\mathcal{H}(\beta, E)$.*
(iii) *The sequence $(\mu(n))$ is bounded.*

Proposition 2.6. *The derivative operator D acting on $\mathcal{H}(\beta, E)$ is compact if and only if $\lim_{n\to\infty} \mu(n) = 0$.*

Denote by $\mathcal{H}(\beta, \rho_+, T)$ a space $\mathcal{H}(\beta, E)$ with $\beta \in \mathbf{C} \cap \mathbf{E}$, thus $\mathcal{H}(\beta, \rho_+, T)$ is a special case of $\mathcal{H}(\beta, \rho_+)$ (since $\mathbf{B} \subsetneq \mathbf{E}$ from Proposition 2.1). According to Proposition 2.5, any translation operator acts boundedly from $\mathcal{H}(\beta, \rho_+, T)$ into itself.

We also denote by $E^2_\gamma(T, \tau)$ a Hilbert space $\mathcal{H}(\beta, E)$ with $\beta \in \mathbf{G}$.

By Proposition 2.1, $E^2_\gamma(T, \tau)$ is a particular case of $E^2_\gamma(T)$, and in turn, $E^2_\gamma(T)$ is a particular case of E^2_γ. Moreover, $E^2_\gamma(T, \tau)$ is a special case of $\mathcal{H}(\beta, \rho_+, T)$. The following inclusion–exclusion chains, as a corollary of Proposition 2.1, summarize what we have discussed in this section.

Corollary 2.7. *Denote by $\{\mathcal{H}(\beta, E)\}$ the set of all $\mathcal{H}(\beta, E)$ spaces, and similarly to the notations $\{\mathcal{H}(\beta, \rho)\}$, $\{\mathcal{H}(\beta, \rho_+)\}$, $\{\mathcal{H}(\beta, \rho_+, T)\}$, $\{E^2_\gamma\}$, $\{E^2_\gamma(T)\}$ and $\{E^2_\gamma(T, \tau)\}$. The following exclusion–inclusion diagram is true.*

$$\begin{array}{ccccccc}
\{E^2_\gamma(T,\tau)\} & & & \subsetneq & \{E^2_\gamma(T)\} & \subsetneq & \{E^2_\gamma\} \\
\rotatebox{90}{$\subsetneq$} & & & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} \\
\{\mathcal{H}(\beta,\rho_+,T)\} & \subsetneq & \{\mathcal{H}(\beta,\rho_+)\} & \subsetneq & \{\mathcal{H}(\beta,\rho)\} & \subsetneq & \{\mathcal{H}(\beta,E)\}.
\end{array}$$

The paper [1] only dealt with problems of composition operators on the smallest class of spaces $E^2_\gamma(T, \tau)$. In the next section, we study these problems for larger classes of Hilbert spaces of entire functions, namely $\{\mathcal{H}(\beta, \rho_+)\}$ and $\{\mathcal{H}(\beta, \rho_+, T)\}$.

3. Composition operators on $\mathcal{H}(\beta, \rho_+)$ and $\mathcal{H}(\beta, \rho_+, T)$

Let $\mathcal{H}(\beta, E)$ be a Hilbert space induced by a sequence $\beta = (\beta_n) \in \mathbf{A}$ and φ be an entire function. Define the composition operator C_φ, induced by φ, acting on $\mathcal{H}(\beta, E)$ as

$$C_\varphi(f) = f \circ \varphi, \quad f \in \mathcal{H}(\beta, E).$$

Properties such as boundedness, compactness, closed range, essential norms, topological structures, etc. are usual topics in the study of composition operators (see, e.g., [3]). For any space $E^2_\gamma(T, \tau)$, i.e. a space $\mathcal{H}(\beta, E)$ with $\beta \in \mathbf{G}$, the criteria for the boundedness and compactness of C_φ have been discovered in [1]. It should be noted that with the techniques of [1], one cannot remove the assumption $\mu(n) \downarrow \tau > 0$.

A question to ask is whether in a general case (when β is in some superset of $\mathbf{G}$), these results are still valid.

This section generalize these results to the spaces $\mathcal{H}(\beta, \rho_+)$ (i.e. $\beta \in \mathbf{B} \cap \mathbf{C}$) and $\mathcal{H}(\beta, \rho_+, T)$ (i.e., $\beta \in \mathbf{C} \cap \mathbf{E}$), whose proofs undoubtedly do not regard the existence of the number τ.

3.1. Boundedness

One of the most useful results about composition of entire functions is the following lemma by G. Pólya [5].

Lemma 3.1. *Let g and h be entire functions such that $f = g \circ h$ is an entire function of finite order. Then either*

(1) *h is a polynomial and g is of finite order, or*
(2) *h is not a polynomial, but a function of finite order, and g is of order 0.*

For each $n \in \mathbb{N}$, define $h_n(z) = z^n/\beta_n$. Clearly, $h_n \in \mathcal{H}(\beta, \rho_+)$ and has norm $\|h_n\| = 1$ and order 0. The sequence of functions (h_n) in fact forms an orthonormal basis for $\mathcal{H}(\beta, \rho_+)$. Using Lemma 3.1, we can achieve the following result.

Theorem 3.2. *If a composition operator C_φ, induced by some entire function φ, is bounded on $\mathcal{H}(\beta, \rho_+)$, then $\varphi(z) = \mathfrak{a}z + \mathfrak{b}$ ($\mathfrak{a}, \mathfrak{b} \in \mathbb{C}$) and $|\mathfrak{a}| \leq 1$.*

Remark 3.3. In the proof of the necessity for boundedness of composition operators on $E^2_\gamma(T, \tau)$ obtained in [1], the authors closely followed the hypothesis $\mu(n) \downarrow \tau > 0$, i.e. $\beta \in \mathbf{G}$, to prove that $\varphi(z) = \mathfrak{a}z + \mathfrak{b}$. This result is similar to our Theorem 3.2, but the method of proof cannot be applied for the space $\mathcal{H}(\beta, \rho_+)$ with $\mu(n)$ not decreasing or with $\mu(n)$ converging to 0.

Example 3.4. Let $\beta_0 = 1$ and $\beta_n = n^{2n}$ for all $n \geq 1$, then $\beta_\rho = 2$ and $\lim_{n\to\infty} \frac{n\beta_{n-1}}{\beta_n} = 0$. Thus this (β_n) satisfies $\beta \in \mathbf{C} \cap \mathbf{E}$ but $\beta \notin \mathbf{G}$, so it is an $\mathcal{H}(\beta, \rho_+)$ space (in fact, an $\mathcal{H}(\beta, \rho_+, T)$ space), but not an $E^2_\gamma(T, \tau)$ space. Hence, there is no conclusion drawn from the result of [1] about boundedness of composition operators for this example. Nevertheless, Theorem 3.2 still holds, which states that any composition operator C_φ bounded on this $\mathcal{H}(\beta, \rho_+)$ space must be induced by some φ of the form $\mathfrak{a}z + \mathfrak{b}$ with $|\mathfrak{a}| \leq 1$.

We have obtained the necessity for boundedness of C_φ on any space $\mathcal{H}(\beta, \rho_+)$. Particularly for the space $\mathcal{H}(\beta, \rho_+, T)$, the reverse is also true. The scheme of the proof is standard, that is it makes use of the boundedness of the translation operator $T_\mathfrak{b}$.

Theorem 3.5. *Let C_φ be a composition operator acting on $\mathcal{H}(\beta, \rho_+, T)$ induced by some entire function φ. If $\varphi(z) = \mathfrak{a}z + \mathfrak{b}$ $(\mathfrak{a}, \mathfrak{b} \in \mathbb{C})$ where $|\mathfrak{a}| \leq 1$, then C_φ is bounded on $\mathcal{H}(\beta, \rho_+, T)$.*

3.2. Compactness

For the compactness of an operator C_φ on $\mathcal{H}(\beta, \rho_+)$, we have the following results.

Theorem 3.6. *If a composition operator C_φ, induced by some entire function φ, is compact on $\mathcal{H}(\beta, \rho_+)$, then $\varphi(z) = \mathfrak{a}z + \mathfrak{b}$ $(\mathfrak{a}, \mathfrak{b} \in \mathbb{C})$ and $|\mathfrak{a}| < 1$.*

Similarly to the result of boundedness, particularly for the space $\mathcal{H}(\beta, \rho_+, T)$, the necessity for the compactness of an operator C_φ is also its sufficiency.

Theorem 3.7. *Let C_φ be a composition operator acting on $\mathcal{H}(\beta, \rho_+, T)$ induced by some entire function φ. If $\varphi(z) = \mathfrak{a}z + \mathfrak{b}$ $(\mathfrak{a}, \mathfrak{b} \in \mathbb{C})$ where $|\mathfrak{a}| < 1$, then C_φ is compact on $\mathcal{H}(\beta, \rho_+, T)$.*

Remark 3.8. A comparable result in [1] states that if $\varphi(z) = \mathfrak{a}z + \mathfrak{b}$ with $|\mathfrak{a}| < 1$, then the composition operator C_φ acting on $E^2_\gamma(T, \tau)$ is compact. However, in order to prove this statement, the authors again adhered to the hypothesis $\mu(n) \downarrow \tau$. More explicitly, the proof introduced the Hilbert space $\mathfrak{H}_{W_\tau}$ of all entire functions f satisfying

$$\|f\|^2_W = \int_{\mathbb{C}} |f(z)|^2 W_\tau(|z|) \mathrm{d}A(z) < \infty,$$

where $W_\tau(z) = \mathrm{e}^{-2\tau z}$. The proof is then complicated and is only applicable to the case $\beta \in \mathbf{G}$. Proposition 3.7, however, only utilizes fundamental results in functional analysis, without taking τ into consideration.

Example 3.9. Consider the space $\mathcal{H}(\beta, E)$ in Example 3.4, with $\beta_0 = 1$ and $\beta_n = n^{2n}$ for all $n \geq 1$. We have seen that $\beta \in (\mathbf{C} \cap \mathbf{E}) \setminus \mathbf{G}$ so it is an $\mathcal{H}(\beta, \rho_+, T)$ space but not an $E^2_\gamma(T, \tau)$ space. While [1] does not provide us with information about the compactness of a composition operator C_φ acting on this space, we can see that C_φ is compact if and only if it is induced by some $\varphi(z) = \mathfrak{a}z + \mathfrak{b}$, with $|\mathfrak{a}| < 1$, by Theorem 3.7.

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