C. R. Acad. Sci. Paris, Ser. I 353 (2015) 511–516

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Ordinary differential equations/Calculus of variations

Existence of bound and ground states for a system of coupled nonlinear Schrödinger–KdV equations

Existence de solutions à énergie finie et énergie minimale pour des systèmes couplés d'équations de Schrödinger–KdV non linéaires

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A R T I C L E   I N F O

Article history:
Received 1 October 2014
Accepted after revision 3 March 2015
Available online 3 April 2015

Presented by Haïm Brézis

A B S T R A C T

We prove the existence of bound and ground states for a system of coupled nonlinear Schrödinger–Korteweg–de Vries equations, depending on the size of the coupling coefficient.

R É S U M É

On montre l'existence de solutions à énergie finie et énergie minimale pour des systèmes couplés d'équations de Schrödinger–Korteweg–de Vries non linéaires, en fonction de la taille du coefficient de couplage.

1. Introduction

The aim of this note is to show some existence of solutions for a system of coupled nonlinear Schrödinger–KdV equations as follows,

$$\begin{cases} \mathrm{i} f_t + f_{xx} + \alpha f g + |f|^2 f = 0 \\ g_t + g_{xxx} + g g_x + \frac{1}{2}\alpha(|f|^2)_x = 0, \end{cases} \tag{1}$$

where $f = f(x,t) \in \mathbb{C}$ while $g = g(x,t) \in \mathbb{R}$, and $\alpha < 0$ is the real coupling constant. System (1) appears in phenomena of interactions between short and long dispersive waves, arising in fluid mechanics, such as the interactions of capillary–gravity water waves. Indeed, f represents the short wave, while g stands for the long wave; see for instance [9] and references therein.

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[1] Partially supported by Ministry of Economy and Competitiveness of Spain and FEDER, project MTM2013-44123-P.

http://dx.doi.org/10.1016/j.crma.2015.03.011

If we define $f(x,t) = e^{i(\omega t + kx)} u(x - ct)$, $g(x,t) = v(x - ct)$, with $u,\ v \geq 0$ real functions, choosing $\lambda_1 = k^2 + \omega$, $\lambda_2 = c = 2k$ and $\beta = -\alpha$, we get that $u,\ v$ solve the following system

$$\begin{cases} -u'' + \lambda_1 u = u^3 + \beta uv \\ -v'' + \lambda_2 v = \frac{1}{2} v^2 + \frac{1}{2}\beta u^2. \end{cases} \tag{2}$$

We deal with the general case; λ_1 not necessarily equals λ_2. We demonstrate the existence of positive even:

- bound states when the coupling parameter $\beta > 0$ is sufficiently small; or $0 < \beta < \Lambda$ and λ_2 large enough,
- ground states provided the coupling factor $\beta > \Lambda > 0$, not proved before for none range of $\lambda_j > 0$, $j = 1, 2$; or $0 < \beta < \Lambda$ and λ_2 large enough.

Note that if $0 < \beta < \Lambda$ and if λ_2 is large enough, we show a multiplicity of positive solutions, precisely the existence of at least two positive solutions.

In the particular case when $\lambda_1 = \lambda_2$ and $\beta > \frac{1}{2}$ studied by Dias et al. in [7], the authors proved the existence of non-negative bound states. As a consequence of our existence results, we show, in that range of parameters, that there exist not only non-negative bound states, but also positive ground states. Also, we want to point out that our method is in part inspired by [1,2], that it is different from the one in [7], and that it seems to be more appropriate to study system (2); see Remarks 4, 5. Another relevant result that we show is the multiplicity result on the existence of bound and ground states for $\beta > 0$ small. This is a great novelty with respect to the previous known results and it is completely different from the more studied coupled systems of NLSE, in which there is uniqueness of positive solutions, see for instance [12]. System (2) has also been recently studied by Liu and Zheng in [11] in the dimensional case $n = 2,\ 3$, see Remark 10(ii) where we give a comparison with our results.

We use the following notation: E denotes the Sobolev space $W^{1,2}(\mathbb{R})$, which can be defined as the completion of $\mathcal{C}_0^1(\mathbb{R})$ endowed with the norm $\|u\| = \sqrt{(u \mid u)}$, with the scalar product $(u \mid w) = \int_{\mathbb{R}} (u'w' + uw)\,dx$. We denote the following equivalent norms and scalar products in E,

$$\|u\|_j = \left(\int_{\mathbb{R}} (|u'|^2 + \lambda_j u^2)\,dx \right)^{\frac{1}{2}}, \quad (u|v)_j = \int_{\mathbb{R}} (u' \cdot v' + \lambda_j uv)\,dx; \quad j = 1, 2.$$

We define the product Sobolev space $\mathbb{E} = E \times E$. The elements in $\mathbb{E}$ are denoted by $\mathbf{u} = (u, v)$, and $\mathbf{0} = (0, 0)$. We take $\|\mathbf{u}\| = \sqrt{\|u\|_1^2 + \|v\|_2^2}$ as a norm in $\mathbb{E}$. For $\mathbf{u} \in \mathbb{E}$, $\mathbf{u} \geq \mathbf{0}$, $\mathbf{u} > \mathbf{0}$, means that $u, v \geq 0$, $u, v > 0$, respectively. We denote H as the space of even (radial) functions in E, and $\mathbb{H} = H \times H$. We define the functional

$$\Phi(\mathbf{u}) = I_1(u) + I_2(v) - \frac{1}{2}\beta \int_{\mathbb{R}} u^2 v\,dx, \qquad \mathbf{u} \in \mathbb{E},$$

where

$$I_1(u) = \frac{1}{2}\|u\|_1^2 - \frac{1}{4} \int_{\mathbb{R}} u^4 dx, \qquad I_2(v) = \frac{1}{2}\|v\|_2^2 - \frac{1}{6} \int_{\mathbb{R}} v^3 dx, \qquad u,\ v \in E.$$

We say that $\mathbf{u} \in \mathbb{E}$ is a non-trivial *bound state* of (2) if $\mathbf{u}$ is a non-trivial critical point of Φ. A bound state $\widetilde{\mathbf{u}}$ is called *ground state* if its energy is minimal among all the non-trivial bound states, namely

$$\Phi(\widetilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathbb{E} \setminus \{\mathbf{0}\},\ \Phi'(\mathbf{u}) = 0\}. \tag{3}$$

An expanded version of this note, with more details and further results, will appear in [6].

2. Existence of ground states

Concerning the ground state solutions of (2), our first result is the following.

Theorem 1. *There exists a real constant $\Lambda > 0$ such that for any $\beta > \Lambda$, System (2) has a positive even ground state $\widetilde{\mathbf{u}} = (\widetilde{u}, \widetilde{v})$.*

We will work in $\mathbb{H}$. Setting,

$$\Psi(\mathbf{u}) = (\nabla\Phi(\mathbf{u})|\mathbf{u}) = (I_1'(u)|u) + (I_2'(v)|v) - \frac{3}{2}\beta \int_{\mathbb{R}} u^2 v\,dx,$$

we define the corresponding Nehari manifold

$$\mathcal{N} = \{\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\} : \Psi(\mathbf{u}) = 0\}.$$

One has that

$$(\nabla\Psi(\mathbf{u}) \mid \mathbf{u}) = -\|\mathbf{u}\|^2 - \int_{\mathbb{R}} u^4 < 0, \quad \forall \mathbf{u} \in \mathcal{N}, \tag{4}$$

and thus $\mathcal{N}$ is a smooth manifold locally near any point $\mathbf{u} \neq \mathbf{0}$ with $\Psi(\mathbf{u}) = 0$. Moreover, $\Phi''(\mathbf{0}) = I_1''(0) + I_2''(0)$ is positive definite, then we infer that $\mathbf{0}$ is a strict minimum for Φ. As a consequence, $\mathbf{0}$ is an isolated point of the set $\{\Psi(\mathbf{u}) = 0\}$, proving that $\mathcal{N}$ is a smooth complete manifold of codimension 1, and there exists a constant $\rho > 0$ so that

$$\|\mathbf{u}\|^2 > \rho, \qquad \forall \mathbf{u} \in \mathcal{N}. \tag{5}$$

Furthermore, (4) and (5) plainly imply that $\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\}$ is a critical point of Φ if and only if $\mathbf{u} \in \mathcal{N}$ is a critical point of Φ constrained on $\mathcal{N}$.

Note that by the previous arguments, the Nehari manifold $\mathcal{N}$ is a natural constraint of Φ. Also it is remarkable that working on the Nehari manifold, the functional Φ takes the form:

$$\Phi|_{\mathcal{N}}(\mathbf{u}) = \frac{1}{6}\|\mathbf{u}\|^2 + \frac{1}{12}\int_{\mathbb{R}^n} u^4 \mathrm{d}x =: F(\mathbf{u}), \tag{6}$$

and by (5) we have

$$\Phi|_{\mathcal{N}}(\mathbf{u}) \ge \frac{1}{6}\|\mathbf{u}\|^2 > \frac{1}{6}\rho. \tag{7}$$

Then (7) shows that the functional Φ is bounded from below on $\mathcal{N}$, so one can try to minimize it on the Nehari manifold $\mathcal{N}$. With respect to he Palais–Smale (PS for short) condition, we remember that in the one-dimensional case, one cannot expect a compact embedding of E into $L^q(\mathbb{R})$ for any q verifying $2 < q < \infty$. Indeed, working on H (the even case) it is not true too. However, we will show that for a PS sequence, we can find a subsequence for which the weak limit is a solution. This fact, jointly with some properties of the Schwarz symmetrization, will permit us to prove Theorem 1. By the previous lack of compactness, we enunciate a measure result given in [10] that we will use in the proof of Theorem 1.

Lemma 2. *If $2 < q < \infty$, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} |u|^q \,\mathrm{d}x \le C\left(\sup_{z\in\mathbb{R}} \int_{|x-z|<1} |u(x)|^2 \,\mathrm{d}x\right)^{\frac{q-2}{2}} \|u\|_E^2, \quad \forall\, u \in E. \tag{8}$$

Let V denote the unique positive even solution of $-v'' + v = v^2$, $v \in H$. Setting

$$V_2(x) = 2\lambda_2\, V(\sqrt{\lambda_2}\, x) = 3\lambda_2 \,\mathrm{sech}^2\left(\frac{\sqrt{\lambda_2}}{2}x\right), \tag{9}$$

one has that V_2 is the unique positive solution of $-v'' + \lambda_2 v = \frac{1}{2}v^2$ in H. Hence $\mathbf{v}_2 := (0, V_2)$ is a particular solution of (2) for any $\beta \in \mathbb{R}$. We also put

$$\mathcal{N}_2 = \left\{v \in H : (I_2'(v)|v) = 0\right\} = \left\{v \in H : \|v\|_2^2 - \frac{1}{2}\int_{\mathbb{R}} v^3 \mathrm{d}x = 0\right\}.$$

Let us denote $T_{\mathbf{v}_2}\mathcal{N}$ the tangent space of $\mathbf{v}_2$ on $\mathcal{N}$. Since

$$\mathbf{h} = (h_1, h_2) \in T_{\mathbf{v}_2}\mathcal{N} \Longleftrightarrow (V_2|h_2)_2 = \frac{3}{4}\int_{\mathbb{R}} V_2^2 h_2 \,\mathrm{d}x,$$

it follows that

$$(h_1, h_2) \in T_{\mathbf{v}_2}\mathcal{N} \Longleftrightarrow h_2 \in T_{V_2}\mathcal{N}_2. \tag{10}$$

Lemma 3. *There exists $\Lambda > 0$ such that for $\beta > \Lambda$, then $\mathbf{v}_2$ is a saddle point of Φ constrained on $\mathcal{N}$.*

The proof follows the ideas in [2, Proposition 4.1(ii)]. We omit it for short.

Remark 4. If one consider $\lambda_1 = \lambda_2$ as in [7], taking $\mathbf{h}_0 = (V_2, 0) \in T_{\mathbf{v}_2}\mathcal{N}$ in the proof of Lemma 3, one finds that

$$\Phi''(\mathbf{v}_2)[\mathbf{h}_0]^2 = \|V_2\|_2^2 - \beta \int_{\mathbb{R}} V_2^3 \, dx = (1-2\beta)\|V_2\|_2^2 < 0 \quad \text{provided} \quad \beta > \frac{1}{2}.$$

See also Remark 5.

Proof of Theorem 1. We start proving that $\inf_{\mathcal{N}} \Phi$ is achieved at some positive function $\widetilde{\mathbf{u}} \in \mathbb{H}$. To do so, by Ekeland's variational principle [8], there exists a PS sequence $\{\mathbf{u}_k\}_{k\in\mathbb{N}} \subset \mathcal{N}$, i.e.,

$$\Phi(\mathbf{u}_k) \to c = \inf_{\mathcal{N}} \Phi, \qquad \nabla_{\mathcal{N}} \Phi(\mathbf{u}_k) \to 0. \tag{11}$$

By (6), one finds that $\{\mathbf{v}_k\}$ is a bounded sequence on $\mathbb{H}$, and without relabeling, we can assume that $\mathbf{u}_k \rightharpoonup \mathbf{u}$ weakly in $\mathbb{H}$, $\mathbf{u}_k \to \mathbf{u}$ strongly in $\mathbb{L}^q_{\text{loc}}(\mathbb{R}) = L^q_{\text{loc}}(\mathbb{R}) \times L^q_{\text{loc}}(\mathbb{R})$ for every $1 \le q < \infty$ and $\mathbf{u}_k \to \mathbf{u}$ a.e. in $\mathbb{R}^2$. Moreover, the constrained gradient $\nabla_{\mathcal{N}} \Phi(\mathbf{u}_k) = \Phi'(\mathbf{u}_k) - \eta_k \Psi'(\mathbf{u}_k) \to 0$, where η_k is the corresponding Lagrange multiplier. Taking the scalar product with $\mathbf{u}_k$ and recalling that $(\Phi'(\mathbf{u}_k) \mid \mathbf{u}_k) = \Psi(\mathbf{u}_k) = 0$, we find that $\eta_k(\Psi'(\mathbf{u}_k) \mid \mathbf{u}_k) \to 0$ and this, jointly with (4)–(5), implies that $\eta_k \to 0$. Since, in addition, $\|\Psi'(\mathbf{u}_k)\| \le C < +\infty$, we deduce that $\Phi'(\mathbf{u}_k) \to 0$.

Let us define $\mu_k = u_k^2 + v_k^2$, where $\mathbf{u}_k = (u_k, v_k)$. By Lemma 2, applied in a similar way as in [4], we can prove that there exist $R, C > 0$ so that

$$\sup_{z\in\mathbb{R}} \int_{|z|<R} \mu_k \ge C > 0, \quad \forall k \in \mathbb{N}. \tag{12}$$

We observe that we can find a sequence of points $\{z_k\} \subset \mathbb{R}^2$ so that by (12), the translated sequence $\overline{\mu}_k(x) = \mu_k(x + z_k)$ satisfies

$$\liminf_{k\to\infty} \int_{B_R(0)} \overline{\mu}_k \ge C > 0.$$

Taking into account that $\overline{\mu}_k \to \overline{\mu}$ strongly in $L^1_{\text{loc}}(\mathbb{R})$, we obtain that $\overline{\mu} \not\equiv 0$. Therefore, defining $\bar{\mathbf{u}}_k(x) = \mathbf{u}_k(x + z_k)$, we have that $\bar{\mathbf{u}}_k$ is also a PS sequence for Φ on $\mathcal{N}$, in particular the weak limit of $\bar{\mathbf{u}}_k$, denoted by $\bar{\mathbf{u}}$, is a non-trivial critical point of Φ constrained on $\mathcal{N}$, so $\bar{\mathbf{u}} \in \mathcal{N}$. Thus, using (6) again, we find

$$\Phi(\bar{\mathbf{u}}) = F(\bar{\mathbf{u}}) \le \liminf_{k\to\infty} F(\bar{\mathbf{u}}_k) = \liminf_{k\to\infty} \Phi(\bar{\mathbf{u}}_k) = c.$$

Furthermore, by Lemma 3 we know that necessarily $\Phi(\bar{\mathbf{u}}) < \Phi(\mathbf{v}_2)$. Clearly $\widetilde{\mathbf{u}} = |\bar{\mathbf{u}}| = (|\bar{u}|, |\bar{v}|) \in \mathcal{N}$ with

$$\Phi(\widetilde{\mathbf{u}}) = \Phi(\bar{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{N}\}, \tag{13}$$

and $\widetilde{\mathbf{u}} \ge \mathbf{0}$. Finally, by the maximum principle applied to each single equation and the fact that $\Phi(\widetilde{\mathbf{u}}) < \Phi(\mathbf{v}_2)$, we get $\widetilde{\mathbf{u}} > \mathbf{0}$.

To finish, taking into account that $\mathcal{N}$ is defined on $\mathbb{H}$, we need to show that indeed

$$\Phi(\widetilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{N}\} = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathbb{E}, \ \Phi'(\mathbf{u}) = 0\}, \tag{14}$$

i.e., $\widetilde{\mathbf{u}}$ is in fact a ground state of (2). To do so, we assume for a contradiction that there exists $\mathbf{w}_0 \in \mathbb{E}$ a non-trivial critical point of Φ such that $\Phi(\mathbf{w}_0) < \Phi(\widetilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{N}\}$. Setting $\mathbf{w} = |\mathbf{w}_0|$, for $\mathbf{w} = (w_1, w_2)$, we set $\mathbf{w}^\star = (w_1^\star, w_2^\star)$, where $w_j^\star$ is the Schwarz symmetric function associated to $w_j \ge 0$; $j = 1, 2$. Then, by the classical properties of the Schwarz symmetrization, there hold

$$\|\mathbf{w}^\star\|^2 \le \|\mathbf{w}\|^2, \qquad G_\beta(\mathbf{w}^\star) \ge G_\beta(\mathbf{w}),$$

where

$$G_\beta(\mathbf{w}) = \tfrac{1}{4} \int_{\mathbb{R}} w_1^4 \, dx + \tfrac{1}{6} \int_{\mathbb{R}} w_2^3 \, dx + \tfrac{1}{2}\beta \int_{\mathbb{R}} w_1^2 w_2 \, dx.$$

Thus, in particular, $\Psi(\mathbf{w}^\star) \le \Psi(\mathbf{w})$ and by the fact that $\mathbf{w}_0$ is a critical point of Φ, we get $\Psi(\mathbf{w}) = \Psi(\mathbf{w}_0) = 0$. Furthermore, after some computations,[2] we have that there exists a unique $0 < t_0 \le 1$, so that $t_0 \mathbf{w}^\star \in \mathcal{N}$. Therefore,

[2] The complete details can be seen in the expanded version [6].

$$\Phi(t_0\mathbf{w}^\star)=\frac{1}{6}t_0^2\|\mathbf{w}^\star\|^2+\frac{1}{12}t_0^4\int_{\mathbb{R}}(w_1^\star)^4\,\mathrm{d}x\le\frac{1}{6}\|\mathbf{w}\|^2+\frac{1}{12}\int_{\mathbb{R}}w_1^4\,\mathrm{d}x=\Phi(\mathbf{w}),$$

proving that $\Phi(t_0\mathbf{w}^\star)\le\Phi(\mathbf{w})<\Phi(\widetilde{\mathbf{u}})=\min\{\Phi(\mathbf{u}):\mathbf{u}\in\mathcal{N}\}$, which is a contradiction with $t_0\mathbf{w}^\star\in\mathcal{N}$. $\square$

Remark 5. As we anticipated in the introduction (see also Remark 4), in the range of parameters studied by [7], $\lambda_1=\lambda_2$ and $\beta>\frac{1}{2}$, we have found a positive ground state in contrast with the non-negative bound state founded by [7].

Theorem 6. *There exists $M>0$ such that if $\lambda_2>M$, System (2) has an even ground state $\widetilde{\mathbf{u}}>\mathbf{0}$ for every $\beta>0$.*

Proof. Arguing in the same way as in the proof of Theorem 1, we initially have that there exists an even ground state $\widetilde{\mathbf{u}}\ge\mathbf{0}$. Moreover, in Theorem 1, for $\beta>\Lambda$, we proved that $\widetilde{\mathbf{u}}>\mathbf{0}$. In order to show that for $\beta\le\Lambda$, by the maximum principle, $\widetilde{\mathbf{u}}>\mathbf{0}$ provided $\widetilde{\mathbf{u}}\ne\mathbf{v}_2$. Arguing in a similar way as in [2, Proposition 4.1(i)], $\mathbf{v}_2$ is a strict local minimum of Φ on $\mathcal{N}$, but this does not allow us to prove that $\widetilde{\mathbf{u}}\ne\mathbf{v}_2$. The new idea here consists in proving the existence of a function $\mathbf{u}_1=(u_1,v_1)\in\mathcal{N}$ with $\Phi(\mathbf{u}_1)<\Phi(\mathbf{v}_2)$. To do so, since $\mathbf{v}_2=(0,V_2)$ is a strict local minimum of Φ on $\mathcal{N}$ provided $0<\beta<\Lambda$, we cannot find $\mathbf{u}_1$ in a neighborhood of $\mathbf{v}_2$ on $\mathcal{N}$. Thus, we define $\mathbf{u}_1=t(V_2,V_2)$ where $t>0$ is the unique value, so that $\mathbf{u}_1\in\mathcal{N}$. To finish, after some computations[3] comparing the energies of $\mathbf{u}_1$, $\mathbf{v}_2$, we find that $\Phi(\mathbf{u}_1)<\Phi(\mathbf{v}_2)$ provided $\lambda_2>M$ for some constant $M>0$, which concludes the result. $\square$

Note that we have found a positive even ground state for every $\beta>0$.

3. Existence of bound sates

Finally, we establish the existence of positive bound states to (2) provided the coupling parameter is small by a perturbation argument. Let us set $\mathbf{u}_0=(U_1,V_2)$, where V_2 is given by (9) and $U_1(x)=\sqrt{2\lambda_1}\,\mathrm{sech}\left(\sqrt{\lambda_1}x\right)$ is the unique positive solution of $-u''+\lambda_1u=u^3$ in H. Then we have the following.

Theorem 7. *There exists $\varepsilon_0>0$ such that, for any $0<\varepsilon<\varepsilon_0$ and $\beta=\varepsilon\widetilde{\beta}>0$, System (2) has an even bound state $\mathbf{u}_\varepsilon>\mathbf{0}$ with $\mathbf{u}_\varepsilon\to\mathbf{u}_0$ as $\varepsilon\to0$.*

In order to prove this result, we can follow some ideas of the proof of [5, Theorem 4.2] with appropriate modifications. To be short, the idea is that by the non-degeneracy of U_1 and V_2 as critical points of their corresponding energy functionals on the radial space H, plainly $\mathbf{u}_0$ is a non-degenerate critical point of Φ on $\mathbb{H}$, hence, an application of the local inversion theorem and some energy computations permit us to prove the existence of $\varepsilon_0>0$ and a convergent sequence of critical points $\mathbf{u}_\varepsilon\to\mathbf{u}_0$ as $\varepsilon\to0$ for $0<\varepsilon<\varepsilon_0$. It remains to show the positivity of $\mathbf{u}_\varepsilon$, which relies on variational techniques in a similar way as in [5], with appropriate changes.

The last result dealing with bound states is the following.

Theorem 8. *In the hypotheses of Theorem 6 and $0<\beta<\Lambda$, there exists an even bound state $\mathbf{u}^*>\mathbf{0}$ with $\Phi(\mathbf{u}^*)>\Phi(\mathbf{v}_2)$.*

Proof. Following the ideas of [2, Proposition 4.1(i)] it is not difficult to show that $\mathbf{v}_2$ is a strict local minimum of Φ on $\mathcal{N}$ for $0<\beta<\Lambda$. The positive ground state $\widetilde{\mathbf{u}}$ found in Theorem 6 satisfies $\Phi(\widetilde{\mathbf{u}})<\Phi(\mathbf{v}_2)$. As a consequence, we have the Mountain Pass (MP in short) geometry of Φ between $\widetilde{\mathbf{u}}$ and $\mathbf{v}_2$ on $\mathcal{N}$. We define the set of all continuous paths joining $\widetilde{\mathbf{u}}$ and $\mathbf{v}_2$ on the Nehari manifold by

$$\Gamma=\{\gamma:[0,1]\to\mathcal{N}\text{ continuous}\mid\gamma(0)=\widetilde{\mathbf{u}},\ \gamma(1)=\mathbf{v}_2\}.$$

Thanks to the MP Theorem by Ambrosetti and Rabinowitz; [3], there exists a PS sequence $\{\mathbf{u}_k\}\subset\mathcal{N}$, i.e.,

$$\Phi(\mathbf{u}_k)\to c=\inf_{\mathcal{N}}\Phi,\qquad\nabla_{\mathcal{N}}\Phi(\mathbf{u}_k)\to0,$$

where

$$c=\inf_{\gamma\in\Gamma}\max_{0\le t\le1}\Phi(\gamma(t)).\tag{15}$$

Plainly, by (6) the sequence $\{\mathbf{u}_k\}$ is bounded on $\mathbb{H}$, and we obtain a weakly convergent subsequence $\mathbf{u}_k\rightharpoonup\mathbf{u}^*\in\mathcal{N}$.

[3] The complete details can be seen in the expanded version [6].

The difficulty of the lack of compactness, due to work in the one-dimensional case, can be circumvented in a similar way as in the proof of Theorem 6. Thus, we find that the weak limit $\mathbf{u}^* = (u^*, v^*) \not\equiv \mathbf{0}$ is an even bound state of (2), and clearly, $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$. It remains to prove that $\mathbf{u}^* > \mathbf{0}$; to do so, we consider the new problem:

$$\begin{cases} -u'' + \lambda_1 u = (u^+)^3 + \beta u^+ v \\ -v'' + \lambda_2 v = \frac{1}{2}v^2 + \frac{1}{2}\beta(u^+)^2. \end{cases} \tag{16}$$

By the maximum principle applied to the second equation, we have $v > 0$, and of course the first component $u \geq 0$ by definition of problem (16). Repeating the previous arguments we show the existence of a MP critical point $\mathbf{u}^*$ of (16). Also we can show that $\mathbf{v}_2$ is a strict local minimum of the corresponding functional Φ^+ on the associated Nehari manifold $\mathcal{N}^+$, besides the new difficulty that Φ^+ is not $\mathcal{C}^2$, without using the second derivative of Φ on $\mathcal{N}$. In order to finish, the positivity of u^* and hence of $\mathbf{u}^*$ follows by the maximum principle applied to the first equation and the fact that $\Phi(\mathbf{u}^*) > \Phi(\mathbf{v}_2)$. □

As a consequence of Theorems 6, 8, we have the following novelty and surprising result about non-uniqueness of positive solutions for (2).

Corollary 9. *Assume $0 < \beta < \Lambda$ and $\lambda_2 > M$. Then there exist at least two positive solutions of (2), given by the ground state $\widetilde{\mathbf{u}}$ (in Theorem 6) and the bound state $\mathbf{u}^*$ (in Theorem 8).*

Remark 10. (i) This result makes a great difference with the more studied systems of coupled NLS equations

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1 \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, \end{cases}$$

for which it is known that there is uniqueness of positive solutions, under appropriate conditions on the parameters including the case $\beta > 0$ small; see for instance [12]. Indeed, for $\beta > 0$ small, the ground state is not positive, and it is given by one of the two semi-trivial solutions $(U^{(1)}, 0)$ or $(0, U^{(2)})$ depending on whether $\Phi(U^{(1)}, 0)$ is lower or greater than $\Phi(0, U^{(2)})$. Here $U^{(j)}$ is the unique positive radial solution of $-\Delta u_j + \lambda_j u_j = \mu_j u_j^3$ in $W^{1,2}(\mathbb{R}^n)$, for $n = 1, 2, 3$ and $j = 1, 2$.

(ii) Following some ideas by Ambrosetti and Colorado in [2], Liu and Zheng proved in [11] a partial result on the existence of solutions to the corresponding system (2) in the dimensional case $n = 2, 3$. More precisely, in [11] the authors showed that the infimum of the energy functional on the corresponding Nehari manifold (defined on the radial Sobolev space) is achieved, but they do not proved that it is positive, and it was not shown that the infimum on the Nehari Manifold is a ground state, i.e., the least energy solution as we have proved here for $n = 1$; see the expanded version [6] for details and the more dimensional case $n = 1, 2, 3$. Also, in [11], the existence of other bound states was not investigated, as we have done in this manuscript in the one-dimensional case, $n = 1$, and in the expanded version [6] for the non-critical dimensions $n = 2, 3$, too.

Acknowledgement

The author likes to thank an anonymous referee for the improvement of the presentation of the manuscript.

References

[1] A. Ambrosetti, E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, C. R. Acad. Sci. Paris, Ser. I 342 (7) (2006) 453–458.
[2] A. Ambrosetti, E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, J. Lond. Math. Soc. (2) 75 (1) (2007) 67–82.
[3] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[4] E. Colorado, Existence results for some systems of coupled fractional nonlinear Schrödinger equations, in: Recent Trends in Nonlinear Partial Differential Equations. II. Stationary Problems, in: Contemp. Math., vol. 595, Amer. Math. Soc., Providence, RI, USA, 2013, pp. 135–150.
[5] E. Colorado, Positive solutions to some systems of coupled nonlinear Schrödinger equations, Nonlinear Anal. 110 (2014) 104–112.
[6] E. Colorado, On the existence of bound and ground states for some coupled nonlinear Schrödinger–Korteweg–de Vries equations, Preprint, arXiv:1411.7283v3, 2014.
[7] J.-P. Dias, M. Figueira, F. Oliveira, Existence of bound states for the coupled Schrödinger–KdV system with cubic nonlinearity, C. R. Acad. Sci. Paris, Ser. I 348 (19–20) (2010) 1079–1082.
[8] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974) 324–353.
[9] M. Funakoshi, M. Oikawa, The resonant interaction between a long internal gravity wave and a surface gravity wave packet, J. Phys. Soc. Jpn. 52 (1) (1983) 1982–1995.
[10] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 1 (1984) 223–283.
[11] C. Liu, Y. Zheng, On soliton solutions to a class of Schrödinger–KdV systems, Proc. Amer. Math. Soc. 141 (10) (2013) 3477–3484.
[12] J. Wei, W. Yao, Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations, Commun. Pure Appl. Anal. 11 (3) (2012) 1003–1011.