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Algebraic geometry

On the homeomorphism type of some spaces of valuations

Sur le type d'homéomorphisme des espaces de valuations

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A R T I C L E I N F O

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A B S T R A C T

Let X be an algebraic variety defined over an algebraically closed field. We study the fiber of the Riemann–Zariski space above a closed point $x \in X$. If x is regular, we prove that its homeomorphism type only depends on the dimension of X. If x is a singular point of a normal surface, we show that it only depends on the dual graph of a good resolution of (X, x) up to some precise equivalence. Both results also hold for the normalized non-Archimedean link of x in X.

R É S U M É

Soit X une variété algébrique définie sur un corps algébriquement clos. On étudie la fibre de l'espace de Riemann–Zariski au-dessus d'un point fermé $x \in X$. Si x est régulier, on démontre que son type d'homéomorphisme ne dépend que de la dimension de X. Si x est un point singulier d'une surface normale, on démontre qu'il ne dépend que de la classe du graphe d'une bonne résolution de (X, x) modulo une relation d'équivalence précise. Ces deux résultats sont aussi vrais pour l'entrelac non archimédien normalisé de x dans X.

1. Introduction

Valuations are a fundamental tool in algebraic geometry. Historically they played an important role in Zariski's approach to the problem of resolution of singularities of an algebraic variety. In [14], Zariski endowed the set of all valuation rings of the function field of the variety containing the base field with a topology and established its quasi-compactness. This was a key point in his program for resolution. It turns out to be also a key result in some recent attempts to solve this problem in positive characteristic following new strategies also using local uniformization (see [2,12]).

In this note we consider an algebraic variety X defined over an algebraically closed field k (i.e., an integral separated scheme of finite type over k) with function field K and we fix a closed point x in X. We initiate the study of the homeomorphism type of the space $\mathrm{RZ}(X, x)$ consisting of all valuations rings of K dominating the local ring $\mathcal{O}_{X,x}$, endowed with the topology induced by the Zariski topology. We call $\mathrm{RZ}(X, x)$ the *Riemann–Zariski space* of X at x. Our goal is to clarify the relation between the topological properties of this space and the local geometry of X at x. Note that the one-dimensional

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case is well understood. If X is an algebraic curve then $\mathrm{RZ}(X, x)$ is in bijection with the local analytic branches of X at x. However, the situation is richer in higher dimension.

Similar preoccupations have appeared in the context of the theory of analytic spaces as developed by Berkovich and others after [1]. Adopting this point of view, one associates with X is analytification X^{an}. A point of X^{an} is an absolute value (giving rise to a rank-one valuation by taking minus the logarithm) on the residue field of a point of X, extending the trivial absolute value of k. We may consider the subspace $\mathrm{L}(X, x)$ of all points in X^{an} that specialize to x excepting the trivial one. One nice feature of this space, established by Thuillier in [13], is that it has the homotopy type of the dual complex associated with the exceptional divisor of a resolution of singularities of (X, x) whose exceptional divisor has simple normal crossings.

In fact, the space $\mathrm{RZ}(X, x)$ is closely related to the *normalized non-Archimedean link* $\mathrm{NL}(X, x)$ of x in X, which is obtained from $\mathrm{L}(X, x)$ by identifying points defining equivalent valuations (see [3]). There is a canonical continuous surjective map from $\mathrm{RZ}(X, x)$ to $\mathrm{NL}(X, x)$, and the latter appears to be the largest Hausdorff quotient of the former space in the case of normal surfaces. A detailed proof of these facts will be given in a forthcoming paper of the author.

Our first main result is the following:

Theorem A. *Let $x \in X$, $y \in Y$ be regular closed points of two algebraic varieties defined over k. The following statements are equivalent:*

(i) The spaces $\mathrm{RZ}(X, x)$ and $\mathrm{RZ}(Y, y)$ are homeomorphic.
(ii) The spaces $\mathrm{NL}(X, x)$ and $\mathrm{NL}(Y, y)$ are homeomorphic.
(iii) The varieties X and Y have the same dimension.

In particular, the homeomorphism type of $\mathrm{RZ}(X, x)$ and $\mathrm{NL}(X, x)$ depends only on the dimension of the variety X and the base field k. In dimension two, one can be more specific. A topological model for $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}}, 0)$ has already been proposed in [5, Section 3.2.3]. The homeomorphism type of an arbitrary Berkovich curve is also treated in [7] under a countability assumption on the base field. Since $\mathrm{NL}(\mathbb{A}^2_k, 0)$ is homeomorphic to the closed unit ball over the discrete valued field $k((t))$, their result shows that $\mathrm{NL}(\mathbb{A}^2_k, 0)$ is a Ważewski universal dendrite when k is countable.

Next, we consider the normal surface singularity situation. We shall say that two finite connected graphs are equivalent if either they are both trees or neither is a tree, and the topological realizations of their cores, in the sense of [11], are homeomorphic.

Theorem B. *Let $x \in X$ and $y \in Y$ be singular points of normal algebraic surfaces defined over k and $\Gamma_{X'}$, $\Gamma_{Y'}$ the dual graphs associated with two good resolutions of (X, x) and (Y, y), respectively. The following statements are equivalent:*

(i) the spaces $\mathrm{RZ}(X, x)$ and $\mathrm{RZ}(Y, y)$ are homeomorphic.
(ii) the spaces $\mathrm{NL}(X, x)$ and $\mathrm{NL}(Y, y)$ are homeomorphic.
(iii) the graphs $\Gamma_{X'}$ and $\Gamma_{Y'}$ are equivalent.

Observe that this statement implies that the spaces of valuations $\mathrm{RZ}(X, x)$ and $\mathrm{NL}(X, x)$ associated with *any* rational surface singularity (X, x) are homeomorphic to $\mathrm{RZ}(\mathbb{A}^2_k, 0)$ and $\mathrm{NL}(\mathbb{A}^2_k, 0)$ respectively. In order to obtain more precise information on the singularity (X, x), it will be necessary to explore finer structures of $\mathrm{RZ}(X, x)$. In fact, the spaces of valuations $\mathrm{RZ}(X, x)$ and $\mathrm{NL}(X, x)$ have more structure than just topology. Actually they are both locally ringed spaces. The second carries a natural analytic structure locally modeled on affinoid spaces over $k((t))$. Note that these local $k((t))$-analytic structures are not canonical and cannot in general be glued to get a global one. This structure was studied in [3] and shown (proof of Lemma 9.3) to determine the completion of the local ring $\mathcal{O}_{X,x}$.

2. Homeomorphism type in the regular case

Throughout this section, $x \in X$ and $y \in Y$ are regular closed points of two algebraic varieties X, Y defined over the same algebraically closed field k. If X and Y are reduced to x and y respectively, then all spaces of valuations are singletons. Therefore we may assume that X and Y have dimension at least one. We indicate how Theorem A can be proved.

(i) ⇒ (iii) Recall that the Krull dimension of a topological space Z is the supremum of the lengths of all chains of irreducible closed subspaces of Z. A chain $\emptyset \subsetneq Z_0 \subsetneq \ldots \subsetneq Z_l \subseteq Z$ is of length l. Then we show that $\mathrm{RZ}(X, x)$ has Krull dimension $\dim X - 1$, which proves that *(i)* implies *(iii)*.
(ii) ⇒ (iii) First observe that the space $\mathrm{NL}(X, x)$ has Krull dimension zero since it is Hausdorff. We look instead at its covering dimension as defined in [10, Ch. 3, Definition 1.1], and we show that $\mathrm{NL}(X, x)$ has covering dimension $\dim X - 1$. This proves that *(ii)* implies *(iii)*.
(iii) ⇒ (ii) Under our assumptions, if X and Y have the same dimension then the formal completions of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as k-algebras.

Observe also that a point in $\mathrm{NL}(X, x)$ defines in a canonical way a multiplicative seminorm on the completion of $\mathcal{O}_{X,x}$ whose restriction to k is trivial and suitably normalized. These two observations show that *(iii)* implies *(ii)*.

(iii) ⇒ *(i)* In the Riemann–Zariski setting, the proof is more involved since a valuation on $\mathcal{O}_{X,x}$ does not extend in general to a valuation on the completion of that ring in a unique way. To prove that *(iii)* implies *(i)*, we rely on [6, Theorem 7.1] that allows to extend valuations to the henselization of $\mathcal{O}_{X,x}$ in a canonical way. We conclude by using the fact that the henselizations of the regular local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic as k-algebras since $\dim X = \dim Y$.

3. The core of a graph

We now introduce the notions necessary to state Theorem B.

By a *graph*, we mean a finite connected graph with at least one vertex, without loops and without multiple edges. Recall that a graph Γ is a purely combinatorial object that can be seen as a finite one-dimensional CW-complex. To be precise, we endow the set of vertices V of Γ and its set of edges E with the discrete topology and the unit interval $[0,1]$ with the induced topology from the standard topology of the real line. The topological space $|\Gamma|$, which we call the *topological realization* of Γ, is the quotient space of the disjoint union $\mathrm{V} \sqcup (\mathrm{E} \times [0,1])$ under the natural identifications $v \sim (e,0)$ and $v' \sim (e,1)$ given by incidence of vertices and edges.

We say that a graph is a *tree* if its topological realization is simply connected. Following [11, Section 7] we associate with any graph its core (see also the definition of the skeleton of a quasipolyhedron given in [1, p. 76]). By the degree of a vertex we mean the number of edges connected to it.

Definition 3.1. The core of a graph Γ that is not a tree is the subgraph of Γ obtained by repeatedly deleting a vertex of degree one and the edge incident to it, until no vertex of degree one remains. We denote the core of Γ by $\mathrm{Core}(\Gamma)$.

By convention we define the core of a tree to be empty. Let Γ be a graph which is not a tree. Observe that if Γ has no vertex of degree one, then Γ is its own core. Note also that $|\Gamma|$ admits a deformation retraction to $|\mathrm{Core}(\Gamma)|$. The complement of $|\mathrm{Core}(\Gamma)|$ in $|\Gamma|$ is the set of points in $|\Gamma|$ that admit an open neighborhood whose closure is a tree and whose boundary is reduced to a vertex of Γ. We may thus think of Γ as its core with some disjoint trees attached to it.

We introduce now the equivalence relation in the set of graphs on which the characterization given in Theorem B relies on:

Definition 3.2. Two graphs Γ and Γ' are equivalent if either their cores are both empty or neither is empty and $|\mathrm{Core}(\Gamma)|$ is homeomorphic to $|\mathrm{Core}(\Gamma')|$.

Note that this equivalence relation is stricter than the homotopy equivalence. The three graphs consisting of two triangles sharing a vertex, two triangles sharing a side, and a line segment with a triangle attached to each endpoint, have all homotopy equivalent topological realizations, but are not pairwise equivalent.

4. Homeomorphism type in the normal surface singularity case

Let x be a singular point of a normal algebraic surface X defined over an algebraically closed field k. We say that a resolution of singularities $\pi_{X'} : X' \to X$ is a *good resolution* if the exceptional divisor $E_{X'} = \pi_{X'}^{-1}(x)$ is a divisor with normal crossing singularities such that its irreducible components are smooth and the intersection of any two of them is at most a point.

With any good resolution is associated its *dual graph* $\Gamma_{X'}$ whose vertices are in bijection with the irreducible components of $E_{X'}$ and where two vertices are adjacent if and only if the corresponding irreducible components of $E_{X'}$ intersect. As explained in [4, Section 1.1], the topological realization of any dual graph $\Gamma_{X'}$ can be embedded into $\mathrm{NL}(X,x)$ as a closed set and there exists a continuous retraction map $r_{X'} : \mathrm{NL}(X,x) \to |\Gamma_{X'}|$.

We now present a sketch of the proof of Theorem B.

(i) ⇒ *(ii)* The inverse image of a point $\nu \in \mathrm{NL}(X,x)$ by the canonical map $\mathrm{RZ}(X,x) \to \mathrm{NL}(X,x)$ consists of all valuations lying in the closure of a valuation associated with ν in a way that depends on its nature. This fact implies that $\mathrm{NL}(X,x)$ is the largest Hausdorff quotient of $\mathrm{RZ}(X,x)$ (see [8, Ch. V, 9, Proposition 2]), and proves that *(i)* implies *(ii)*.
(ii) ⇒ *(iii)* The key observation is the following:

Proposition 4.1. *Let $\pi_{X'} : X' \to X$ be a good resolution. Any fiber $r_{X'}^{-1}(\nu)$ under the natural retraction $r_{X'} : \mathrm{NL}(X,x) \to |\Gamma_{X'}|$ is a tree whose boundary is reduced to ν.*

A proof of this fact follows from [5, Theorem 6.51]. One can show that the fiber $r_{X'}^{-1}(\nu)$ is in fact an analytic disk when endowed with its canonical analytic structure (see [3, Proposition 9.5 (i)]).

We mean here by a *tree* a topological space which is homeomorphic to a rooted nonmetric tree in the sense of [5, Sections 3.1 and 7.2] (see also [9, Definition 3.1]). Roughly speaking, it is a topological space where any two different points are joined by a unique real line interval. The trees we defined in Section 3 are trees in this sense.

Definition 4.2. The core of $\mathrm{NL}(X,x)$ is defined to be the set of all points in $\mathrm{NL}(X,x)$ that do not admit an open neighborhood whose closure is a tree and whose boundary is reduced to a single point of $\mathrm{NL}(X,x)$. We denote it by $\mathrm{Core}\,(\mathrm{NL}(X,x))$.

In [1, p. 76] the core is referred to as the skeleton. Observe that by definition $\mathrm{Core}\,(\mathrm{NL}(X,x))$ is empty if and only if $\mathrm{NL}(X,x)$ is a tree. Proposition 4.1 and the fact that any arcwise connected subspace of a tree is also a tree imply:

Proposition 4.3. *Let $\pi_{X'}: X' \to X$ be a good resolution. The space $\mathrm{NL}(X,x)$ is a tree if and only if $\Gamma_{X'}$ is a tree. If neither is a tree, we have $\mathrm{Core}\,(\mathrm{NL}(X,x)) = |\mathrm{Core}\,(\Gamma_{X'})|$ as subspaces of $\mathrm{NL}(X,x)$.*

It directly follows from Proposition 4.3 that *(ii)* implies *(iii)*.

(iii) $\Rightarrow$ *(i)* This is the most delicate part of the proof. We start with two good resolutions $\pi_{X'}: X' \to X$ and $\pi_{Y'}: Y' \to Y$, and suppose that their dual graphs are equivalent in the sense of Definition 3.2. Our goal is to construct an homeomorphism from $\mathrm{RZ}(X,x)$ to $\mathrm{RZ}(Y,y)$. We first construct two good resolutions $\pi_{X''}: X'' \to X$ and $\pi_{Y''}: Y'' \to Y$ which factor through $\pi_{X'}$ and $\pi_{Y'}$ respectively and such that $\Gamma_{X''}$ and $\Gamma_{Y''}$ are isomorphic graphs. This isomorphism determines a natural bijection between the irreducible components $\{E_i\}_{i=1}^m$ of $E_{X''}$ and those, say $\{D_i\}_{i=1}^m$, of $E_{Y''}$. We map the divisorial valuation in $\mathrm{RZ}(X,x)$ defined by E_i to the divisorial valuation in $\mathrm{RZ}(Y,y)$ defined by D_i. Thus, in order to define a bijection from $\mathrm{RZ}(X,x)$ to $\mathrm{RZ}(Y,y)$, it suffices to concentrate on the valuations having as center in X'' a closed point. To do so we choose a bijection σ between the set of closed points of $E_{X''}$ and $E_{Y''}$ such that $\sigma(E_i \cap E_j) = D_i \cap D_j$ and $\sigma(E_i) \subseteq D_i$. The idea is to apply Theorem A to obtain an homeomorphism from $\mathrm{RZ}(X'',x'')$ to $\mathrm{RZ}(Y'',\sigma(x''))$. The construction of the bijection from $\mathrm{RZ}(X,x)$ to $\mathrm{RZ}(Y,y)$ using this idea requires a careful local study at the points of $E_{X''}$. The fact that it is an homeomorphism then follows by examination of the behaviors of sequences of centers and their images by σ.

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