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A refinement of a conjecture of Quillen

Un raffinement d'une conjecture de Quillen

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ABSTRACT

We present some new results on the cohomology of a large scope of SL_2 groups in degrees above the virtual cohomological dimension, yielding some partial positive results for the Quillen conjecture in rank one. We combine these results with the known partial positive results and the known types of counterexamples to the Quillen conjecture, in order to formulate a refined variant of the conjecture.

RÉSUMÉ

Nous présentons de nouveaux résultats sur la cohomologie d'un grand échantillon de groupes SL_2, en degrés au-dessus de la dimension cohomologique virtuelle. Ceci donne quelques résultats affirmatifs de caractère partiel pour la conjecture de Quillen en rang 1. Nous combinons ces résultats avec les résultats connus affirmant une partie de la conjecture de Quillen et avec les types connus de contre-exemples à cette conjecture, afin de formuler une variante raffinée de cette dernière.

1. Quillen's conjecture – formulation and history

In his fundamental work on the structure of equivariant cohomology rings, cf. [10], Quillen formulated a conjecture on the structure of cohomology rings of certain S-arithmetic groups. In the time that has passed since the formulation of the conjecture, it has been proved in some cases and disproved in others, but the exact nature of the conjecture and an explicit description of the cases where it holds has not yet been found. Our goal in the present note is to discuss some recent examples that shed new light on Quillen's conjecture. Guided by these examples, we attempt a refined formulation of the original conjecture.

We first state Quillen's original conjecture, cf. [10, Conjecture 14.7, p. 591]. For any number field K, and any set of places S of K, one of the natural embeddings $\mathrm{GL}_n(\mathcal{O}_{K,S}) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ induces a restriction map in cohomology:

$$\mathrm{res}_{K,S} : \mathrm{H}^\bullet(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell) \to \mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell).$$

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Moreover, denoting by c_i the i-th Chern class in $\mathrm{H}^\bullet_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell)$, there is a change-of-topology map

$$\delta : \mathbb{F}_\ell[c_1, \ldots, c_n] \cong \mathrm{H}^\bullet_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell) \to \mathrm{H}^\bullet(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell).$$

The conjecture of Quillen can now be stated as follows:

Conjecture 1 *(Quillen). Let ℓ be a prime number. Let K be a number field with $\zeta_\ell \in K$, and S a finite set of places containing the infinite places and the places over ℓ. Then the composition $\mathrm{res}_{K,S} \circ \delta$ makes $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ a free module over the cohomology ring $\mathrm{H}^\bullet_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell) \cong \mathbb{F}_\ell[c_1, \ldots, c_n]$.*

The range of validity of the conjecture has not yet been decided. Positive cases in which the conjecture has been established are $n = \ell = 2$ by Mitchell [9], $n = 3$, $\ell = 2$ by Henn [4], and $n = 2$, $\ell = 3$ by Anton [2]. Using [6, Remark on p. 51], counterexamples to Quillen's conjecture have been established by Dwyer [3] for $n \geq 32$ and $\ell = 2$, Henn and Lannes [5] for $n \geq 14$ and $\ell = 2$, and by Anton [2] for $n \geq 27$ and $\ell = 3$.

Via the remark in [6, p. 51], the Quillen conjecture has been viewed as closely related to the following question, to which we will refer as "detection question" in the sequel.

Question 2 *(Detection question). For which number fields K, place sets S of K, primes ℓ and natural numbers n is the restriction morphism $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \to \mathrm{H}^\bullet(\mathrm{T}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ injective, where T_n is the group of diagonal matrices in GL_n?*

However, while the remark in [6] concerns only the case $\mathrm{GL}_n(\mathbb{Z}[1/2])$ with $\mathbb{F}_2$-coefficients, the nature of the relation between Quillen's conjecture and detection questions has not been made precise yet. All we found in the literature was the following sentence on p. 13 of [8]: "This [the Quillen conjecture] implies the following conjecture [the detection question] in many cases." Unfortunately, the "many cases" are left unspecified. A secondary objective of the paper is to clarify the relation between Quillen's conjecture and detection questions.

2. Subgroup structure in high rank – negative results

We first discuss the known counterexamples to the Quillen conjecture. As mentioned above, these are built on a remark in [6] together with examples of the failure of detection (due to the non-triviality of class groups of group rings for sufficiently complicated finite subgroups) as found by Dwyer [3], Henn–Lannes [5] and Anton [2]. We provide a precise formulation of the result of Henn–Lannes–Schwartz. The proof given below is mostly contained in [6], details missing in [6] were explained to us by Hans-Werner Henn—we claim no originality except for mistakes we might have introduced.

Proposition 2.1 *(Henn–Lannes–Schwartz). Let ℓ be a prime number. Let K be a number field with $\zeta_\ell \in K$, and S a finite set of places containing the infinite places and the places over ℓ. Assume that all elementary Abelian ℓ-groups in $\mathrm{GL}_n(\mathcal{O}_{K,S})$ are conjugate to subgroups of the diagonal matrices, and that Quillen's conjecture holds for K, S and ℓ. Then detection holds for K, S and ℓ.*

Proof. We assume that detection does not hold, and we want to derive a contradiction.

(i) Let E_0 be the group of diagonal matrices of order ℓ. This is a maximal elementary Abelian ℓ-subgroup of $\mathrm{GL}_n(\mathcal{O}_{K,S})$. If detection fails, then the restriction map

$$\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \to \mathrm{H}^\bullet(E_0, \mathbb{F}_\ell)$$

is not injective. This follows from functoriality of group cohomology, because we have an inclusion $E_0 \leq \mathrm{T}_n(\mathcal{O}_{K,S}) \leq \mathrm{GL}_n(\mathcal{O}_{K,S})$ and the restriction map associated with the second map is not injective by assumption. For $g \in \mathrm{GL}_n(\mathcal{O}_{K,S})$, the homomorphism $c \mapsto gcg^{-1} : \mathrm{GL}_n(\mathcal{O}_{K,S}) \to \mathrm{GL}_n(\mathcal{O}_{K,S})$ induces the identity on $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$, cf. e.g. [8, Proposition A.1.11]. Together with the above argument, failure of detection implies that the following product of restriction maps is also not injective:

$$\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \to \prod_{E \in \mathcal{M}} \mathrm{H}^\bullet(E, \mathbb{F}_\ell), \tag{1}$$

where $\mathcal{M}$ is the set of all maximal elementary Abelian ℓ-subgroups $E \leq \mathrm{GL}_n(\mathcal{O}_{K,S})$.

(ii) Assume that there exists a class $x \in \mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ that is not a zero-divisor, and whose restriction to E_0 is not nilpotent but in the essential ideal. Then [6, Corollary 5.8] implies that the above product of restriction maps (1) is injective. The result is applicable since the group $\mathrm{GL}_n(\mathcal{O}_{K,S})$ is of finite virtual cohomological dimension and the cohomology ring $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ is Noetherian, cf. the discussion in [10]. Recall the collection $\mathcal{C}_x$ of [6, Corollary 5.8]: it is obtained as the collection of elementary Abelian ℓ-subgroups E of $\mathrm{GL}_n(\mathcal{O}_{K,S})$, such that the restriction $\mathrm{res}_E(x)$ is not nilpotent. The collection $\mathcal{C}_x$ is equal to $\mathcal{M}$: by assumption, the restriction of x to E_0 is not nilpotent, and since all maximal elementary Abelian ℓ-subgroups are conjugate, the same is true for all other $E \in \mathcal{M}$. On the other hand, since x is required to restrict to the essential ideal, its restriction to every proper subgroup of E_0 is trivial, so the same is true for all non-maximal elementary Abelian ℓ-subgroups.

(iii) It now suffices to find an element $x \in \mathrm{H}^\bullet_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell)$ whose restriction to $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ has the properties required in (ii): failure of detection in (i) and the assumption that x is not a zero-divisor in (ii) contradict each other. Therefore, x has to be a zero-divisor and hence $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ cannot be a free $\mathrm{H}^\bullet_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell)$-module.

(iv) The proof is completed by producing an element with the properties in (iii). For the structure of the essential ideal in the cohomology rings of elementary Abelian ℓ-groups, we refer to [1]. In the case $\ell = 2$, the product of all non-zero classes in $\mathrm{H}^1(E_0, \mathbb{F}_2)$ is an essential non-zero-divisor; its square is induced from the product of all non-zero classes in $\mathrm{H}^2_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_2)$. In the case of odd ℓ, the product of all non-zero classes in $\mathrm{H}^2(E_0, \mathbb{F}_\ell)$ is an essential non-zero-divisor that is Weyl-invariant and hence induced from continuous cohomology of $\mathrm{GL}_n(\mathbb{C})$. □

Remark 2.2. This proposition justifies the application of [6, p. 51] in [2]: all elementary Abelian 3-groups of maximal rank in $\mathrm{GL}_n(\mathbb{Z}[\zeta_3, 1/3])$ are conjugate.

3. Subgroup structure in rank one – positive results

We next discuss the rank one case, i.e., the groups $\mathrm{SL}_2(\mathcal{O}_{K,S})$. In this case, the subgroup structure (and consequently the cohomology above the virtual cohomological dimension) is under good control. This allows us to establish variants and partial positive results related to the Quillen conjecture.

3.1. Quillen conjecture for Farrell–Tate cohomology

The following is one of the main results of [11]. It provides a complete computation of the Farrell–Tate cohomology for $\mathrm{SL}_2(\mathcal{O}_{K,S})$ based on an explicit description of conjugacy classes of finite cyclic subgroups and their normalizers in $\mathrm{SL}_2(\mathcal{O}_{K,S})$. Similar results can be established for $\mathrm{(P)GL}_2(\mathcal{O}_{K,S})$.

We first explain some notation. We will consider global fields K, place sets S and primes ℓ, and $\mathcal{O}_{K,S}$ denotes the relevant ring of S-integers. In the situation where $\zeta_\ell + \zeta_\ell^{-1} \in K$, we will abuse notation and write $\mathcal{O}_{K,S}[\zeta_\ell]$ to mean the ring $\mathcal{O}_{K,S}[T]/(T^2 - (\zeta_\ell + \zeta_\ell^{-1})T + 1)$. Moreover, we denote the norm maps for class groups and units by

$$\mathrm{Nm}_0 : \widetilde{\mathrm{K}_0}(\mathcal{O}_{K,S}[\zeta_\ell]) \to \widetilde{\mathrm{K}_0}(\mathcal{O}_{K,S}) \quad \text{and} \quad \mathrm{Nm}_1 : \mathcal{O}_{K,S}[\zeta_\ell]^\times \to \mathcal{O}_{K,S}^\times.$$

Theorem 3.1. *Let K be a global field, let S be a non-empty finite set of places of K containing the infinite places, and let ℓ be an odd prime different from the characteristic of K. Assume that $\zeta_\ell + \zeta_\ell^{-1} \in K$ and $\ell \in S$.*

(1) *The set $\mathcal{C}_\ell$ of conjugacy classes of order ℓ elements in $\mathrm{SL}_2(\mathcal{O}_{K,S})$ sits in an extension*

$$1 \to \operatorname{coker} \mathrm{Nm}_1 \to \mathcal{C}_\ell \to \ker \mathrm{Nm}_0 \to 0.$$

The set $\mathcal{K}_\ell$ of conjugacy classes of order ℓ subgroups of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ can be identified with the quotient $\mathcal{K}_\ell = \mathcal{C}_\ell / \mathrm{Gal}(K(\zeta_\ell)/K)$. There is a direct sum decomposition

$$\widehat{\mathrm{H}}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell) \cong \bigoplus_{[\Gamma] \in \mathcal{K}_\ell} \widehat{\mathrm{H}}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma), \mathbb{F}_\ell)$$

which is compatible with the ring structure, i.e., the Farrell–Tate cohomology ring of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ is a direct sum of the sub-rings for the subgroups $N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma)$.

(2) *If the class of Γ is not $\mathrm{Gal}(K(\zeta_\ell)/K)$-invariant, then $N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma) \cong \ker \mathrm{Nm}_1$. There is a ring isomorphism*

$$\widehat{\mathrm{H}}^\bullet(\ker \mathrm{Nm}_1, \mathbb{Z})_{(\ell)} \cong \mathbb{F}_\ell[a_2, a_2^{-1}] \otimes_{\mathbb{F}_\ell} \bigwedge (\ker \mathrm{Nm}_1)\,.$$

In particular, this is a free module over the subring $\mathbb{F}_\ell[a_2^2, a_2^{-2}]$.

(3) *If the class of Γ is $\mathrm{Gal}(K(\zeta_\ell)/K)$-invariant, then there is an extension*

$$0 \to \ker \mathrm{Nm}_1 \to N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma) \to \mathbb{Z}/2 \to 1.$$

There is a ring isomorphism

$$\widehat{\mathrm{H}}^\bullet(N_{\mathrm{SL}_2(\mathcal{O}_{K,S})}(\Gamma), \mathbb{Z})_{(\ell)} \cong \left(\mathbb{F}_\ell[a_2, a_2^{-1}] \otimes_{\mathbb{F}_\ell} \bigwedge (\ker \mathrm{Nm}_1)\right)^{\mathbb{Z}/2},$$

with the $\mathbb{Z}/2$-action given by multiplication with -1 on a_2 and $\ker \mathrm{Nm}_1$. In particular, this is a free module over the subring $\mathbb{F}_\ell[a_2^2, a_2^{-2}] \cong \widehat{\mathrm{H}}^\bullet(D_{2\ell}, \mathbb{Z})_{(\ell)}$.

(4) *The restriction map induced from the inclusion $\mathrm{SL}_2(\mathcal{O}_{K,S}) \to \mathrm{SL}_2(\mathbb{C})$ maps the second Chern class c_2 to the sum of the elements a_2^2 in all the components.*

Corollary 3.2. *Let ℓ be a prime number. Let K be a number field with $\zeta_\ell \in K$, and S a finite set of places containing the infinite places and the places over ℓ.*

(1) *The Quillen conjecture is true for the Farrell–Tate cohomology of* $\mathrm{SL}_2(\mathcal{O}_{K,S})$. *More precisely, the natural morphism* $\mathbb{F}_\ell[c_2] \cong \mathrm{H}^\bullet_{\mathrm{cts}}(\mathrm{SL}_2(\mathbb{C}), \mathbb{F}_\ell) \to \mathrm{H}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ *extends to a morphism*

$$\phi : \mathbb{F}_\ell[c_2, c_2^{-1}] \to \widehat{\mathrm{H}}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$$

which makes $\widehat{\mathrm{H}}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ *a free* $\mathbb{F}_\ell[c_2, c_2^{-1}]$*-module.*

(2) *The Quillen conjecture is true for* $\mathrm{SL}_2(\mathcal{O}_{K,S})$ *in cohomological degrees above the virtual cohomological dimension.*

Remark 3.3. Above the virtual cohomological dimension, this is an SL_2-analogue of the results of [2].

On the other hand, there are examples of the failure of detection for SL_2. In particular, the Quillen conjecture does not generally imply detection; some non-trivial hypothesis is necessary in Proposition 2.1. A rather simple example for the failure of detection is given by $K = \mathbb{Q}(\zeta_{23})$, $S = \{(23)\} \cup S_\infty$ and $\ell = 23$. In this case, there are three conjugacy classes of elements of order 23 (corresponding to $\mathbb{Q}(\zeta_{23})$ having class number three) and two conjugacy classes of cyclic subgroups of order 23 (the two non-trivial classes of elements forming one conjugacy orbit). Detection fails by a simple rank argument—the source of the restriction map has two copies of the cohomology of a dihedral extension of $\mathcal{O}_{K,S}^\times$, the target only one copy of the cohomology of $\mathcal{O}_{K,S}^\times$, cf. [11]. A similar class of examples is given by $K = \mathbb{Q}(\sqrt{-m}, \zeta_3)$ with $m \equiv 1 \bmod 3$, $S = \{(3)\} \cup S_\infty$, $\ell = 3$ for those (infinitely many) m for which K has class number ≥ 3. These examples for the failure of the detection apply to the Farrell–Tate cohomology as well as to group cohomology above the virtual cohomological dimension.

The computation of $\widehat{\mathrm{H}}^\bullet(\mathrm{SL}_2(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ is obtained by considering the action of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ on the associated symmetric space $\mathfrak{X}_{K,S}$ (which is a product of hyperbolic upper half spaces for the complex places, upper half planes for the real places, and Bruhat–Tits trees for the finite places). It is possible to describe completely the subspace of $\mathfrak{X}_{K,S}$ consisting of points fixed by some finite subgroup. The local structure of this subcomplex is determined by examining the representation theory of the relevant finite groups on the "tangent space" of $\mathfrak{X}_{K,S}$. The global structure only depends on number-theoretic data: the connected components are in bijection with conjugacy classes of finite cyclic subgroups, and the homotopy type of each connected component is (up to the prime 2) the classifying space of the normalizer of the corresponding finite subgroup. The conjugacy classification of finite cyclic subgroups and the description of the normalizers is an extension of the classical Latimer–MacDuffee theorem. After having obtained this precise description, the computation of the Farrell–Tate cohomology of $\mathrm{SL}_2(\mathcal{O}_{K,S})$ is immediate.

3.2. Quillen conjecture in function field situations

The Quillen conjecture can also be asked in function field situations. Let p and ℓ be distinct primes. By Quillen's computations, there is a natural element $c_2 \in \mathrm{H}^4(\mathrm{SL}_2(\overline{\mathbb{F}_p}), \mathbb{F}_\ell)$ such that we have an identification $\mathrm{H}^\bullet(\mathrm{SL}_2(\overline{\mathbb{F}_p}), \mathbb{F}_\ell) \cong \mathbb{F}_\ell[c_2]$. This element comes from the roots of unity, hence exists over any algebraically closed field of characteristic p. In particular, there is a natural summand $\mathbb{F}_\ell[c_2]$ in $\mathrm{H}^\bullet(\mathrm{SL}_2(K), \mathbb{F}_\ell)$ for any algebraically closed field K of characteristic p. Note that Friedlander's generalized isomorphism conjecture predicts that this summand is the whole cohomology.

It is then possible to ask if the natural map

$$\phi : \mathbb{F}_\ell[c_2] \to \mathrm{H}^\bullet(\mathrm{SL}_2(k[C]), \mathbb{F}_\ell)$$

makes the cohomology ring a free module over the image of ϕ, when $k = \mathbb{F}_q$ such that $\ell \mid q-1$ or k is an algebraically closed field. The answer is similar to the number field case discussed above, which follows from (a slight reformulation of) the results of [13].

Theorem 3.4. *Let* $k = \mathbb{F}_q$ *be a finite field, let* ℓ *be a prime with* $\ell \mid q-1$. *Let* $\overline{C}$ *be a smooth projective curve over k, let* $P_1, \dots, P_s \in \overline{C}$ *be closed points, and set* $C = \overline{C} \setminus \{P_1, \dots, P_s\}$. *Then the parabolic cohomology (as defined in [13]) has a direct sum decomposition* $\widehat{\mathrm{H}}^\bullet(\mathrm{SL}_2(k[C]), \mathbb{F}_\ell) \cong \bigoplus_{[\mathcal{L}] \in \mathcal{K}(C)} \widehat{\mathrm{H}}^\bullet(\Gamma_C(\mathcal{L}), \mathbb{F}_\ell)$, *where the index set* $\mathcal{K}(C)$ *is the quotient set* $\mathcal{K}(C) = \mathrm{Pic}(C)/\iota$ *of the Picard group of C modulo the involution* $\iota : \mathcal{L} \mapsto \mathcal{L}^{-1}$. *The components of this direct sum are:*

(1) *If* $\mathcal{L}|_C \not\cong \mathcal{L}|_C^{-1}$, *then* $\widehat{\mathrm{H}}^\bullet(\Gamma_C(\mathcal{L}), \mathbb{F}_\ell) \cong \mathrm{H}^\bullet(k[C]^\times, \mathbb{F}_\ell)$.

(2) *If* $\mathcal{L}|_C \cong \mathcal{L}|_C^{-1}$, *then* $\widehat{\mathrm{H}}^\bullet(\Gamma_C(\mathcal{L}), \mathbb{F}_\ell) \cong \mathrm{H}^\bullet(\widetilde{\mathcal{SN}}, \mathbb{F}_\ell)$, *where* $\widetilde{\mathcal{SN}}$ *denotes the group of monomial matrices in* $\mathrm{SL}_2(k[C])$.

Since $\widehat{\mathrm{H}}^i(\mathrm{SL}_2(k[C]), \mathbb{F}_\ell) \cong \mathrm{H}^i(\mathrm{SL}_2(k[C]), \mathbb{F}_\ell)$ *for i greater than the virtual* p'*-cohomological dimension of* $\mathrm{SL}_2(k[C])$, *the above function field analogue of Quillen's conjecture holds in those degrees.*

The proof strategy is similar to the number field case: consider the action of $\mathrm{SL}_2(k[C])$ on the associated symmetric space (which is a product of Bruhat–Tits trees for the places at infinity). It is then possible to work out explicitly the structure of the subcomplex of cells that are fixed by a non-unipotent non-central subgroup. The components of this "parabolic subcomplex" are in bijection with a quotient of the Picard group, and each component (up to the prime 2) has the homotopy type of the classifying space of its setwise stabilizer. From this, again, the computation of the relevant cohomology is immediate.

Further explicit computations exhibit function field cases where Quillen's conjecture holds in all cohomological degrees. The function field analogue of Quillen's conjecture is true for $\mathrm{SL}_2(\mathbb{F}_q[C])$ in the following cases, which in some sense can be considered function field analogues of the results of Mitchell [9] and Anton [2]:

(1) $C = \mathbb{P}^1 \setminus \{\infty\}$ (Soulé),
(2) $C = \mathbb{P}^1 \setminus \{0, \infty\}$ and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ ([13], but see also [8, Section 4.4] and [7]),
(3) $C = \overline{E} \setminus \{P\}$ with $\overline{E}$ an elliptic curve with a k-rational point P ([8, Section 4.5]).

4. Non-detectable cohomology classes

In the previous section, we have seen some positive results concerning the Quillen conjecture in rank one, and we have seen that the results for the number field and function field cases are close analogues. In the function field case, however, it is possible to get new examples of cases where the Quillen conjecture fails badly, cf. [14].

Theorem 4.1. *Let $k = \mathbb{F}_q$ with $q \geq 11$, and let $\ell \nmid q$ be a prime. For $C = \mathbb{P}^1 \setminus \{0, 1, \infty, P\}$ for some k-rational point P, there exist cohomology classes in $\mathrm{H}^4(\mathrm{GL}_2(\mathbb{F}_q[C]), \mathbb{F}_\ell)$ that cannot be detected on any maximal torus or any finite subgroup.*

This result is proved by considering the action of $\mathrm{GL}_2(k[C])$ on the associated building $\mathfrak{X}_C$, which is a product of four Bruhat–Tits trees corresponding to the four points 0, 1, ∞ and P on $\mathbb{P}^1$. The existence of many non-trivial cells in the quotient $\mathrm{GL}_2(k[C])\backslash\mathfrak{X}_C$ can basically be traced to the fact that the configuration space of 4 points on $\mathbb{P}^1$ is positive-dimensional. Similar results can be obtained for $\mathbb{P}^1 \setminus \{P_1, \ldots, P_s\}$ with $s \geq 5$.

These counterexamples to the Quillen conjecture are of a different nature than those discussed in Section 2—they are not related to finite subgroups, in fact they cannot be detected on any finite subgroup. This is a new obstruction to the Quillen conjecture, which instead is (somehow) related to cusp forms.

In the spirit of the analogy between number fields and function fields, it makes sense to expect that the Quillen conjecture fails for $\mathrm{GL}_2(\mathbb{Z}[1/n])$ where n has at least three prime factors (and hence the curve $\mathbb{Z}[1/n]$ has at least four places "at infinity").

5. Refinement of Quillen's conjecture

With the results outlined in the previous sections, we now have some more positive and negative cases of the Quillen conjecture at our disposal. Assuming that all reasons for potential counterexamples have been accounted for, we arrive at the following refinement of Quillen's conjecture.

Conjecture 3. *Let K be a number field. Fix a prime ℓ such that $\zeta_\ell \in K$, and an integer $n < \ell$. Assume that S is a set of places containing the infinite places and the places lying over ℓ. If each cohomology class of $\mathrm{GL}_n(\mathcal{O}_{K,S})$ is detected on some finite subgroup, then $\mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$ is a free module over the image of the restriction map $\mathrm{H}^\bullet_{\mathrm{cts}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{F}_\ell) \to \mathrm{H}^\bullet(\mathrm{GL}_n(\mathcal{O}_{K,S}), \mathbb{F}_\ell)$.*

We now discuss how the above refinement of Quillen's conjecture fits in the landscape of known examples and counterexamples.

(1) Conjecture 3 is true for SL_2, as follows from Theorem 3.1 and Theorem 3.4.
(2) Requiring $\ell > n$ implies that ℓ does not divide the order of the Weyl group. All counterexamples of Section 2 are excluded by this requirement; the known counterexamples are for primes 2 and 3 in high-enough rank. Generally, finite subgroups in Lie groups are fairly complicated to handle. However, the special case of normalizers of elementary Abelian subgroups for ℓ not dividing the order of the Weyl group is substantially simpler, cf. [12]; it is much closer to the rank one case of Section 3. One could hope that the groups appearing do not give rise to counterexamples coming from non-triviality of class groups of representation rings as in Section 2.
(3) Requiring that all cohomology classes are detected on some finite subgroup excludes counterexamples of the type discussed in Section 4 (and allows the passage from SL_2 to GL_2 in Section 3). However, the results of Section 4 show that this requirement (at least in function field situations) is only rarely satisfied.

Finally, we should note that there is an implicit leap of faith in the above conjecture lying in the passage from torsion-free modules to free modules. Showing that the module is torsion-free is easier, we only need to show that the second Chern class is not a zero-divisor. The passage from torsion-free modules to free modules is automatic in the case SL_2, because the cohomology ring is a polynomial ring in one variable; but this may be much more subtle in higher-rank cases.

Certainly, the work done for the results in Section 3 shows the way how to investigate Conjecture 3, cf. [11]: away from the Weyl group, it is possible to work out the classification of finite subgroups much more easily, cf. [12]. Then one can consider the action of an S-arithmetic group $G(\mathcal{O}_{K,S})$ on the corresponding symmetric space. The structure of the subcomplex fixed by finite-order elements can be understood locally in terms of the representation of the finite subgroups on the tangent spaces of their fixed points. Conjugacy classification of finite subgroups in S-arithmetic groups can be reduced to number theory by counting conjugacy classes in terms of ideal classes in suitable ring extensions. The normalizers of finite subgroups of arithmetic groups can be understood in terms of parabolic subgroups in algebraic groups. The final hurdle is the evaluation of the spectral sequence and the description of the differentials. At least the case SL_3 can still be understood, and will be investigated in a forthcoming paper.

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References

[1] F. Altunbulak Aksu, D.J. Green, Essential cohomology for elementary Abelian p-groups, J. Pure Appl. Algebra 213 (2009) 2238–2243.
[2] M.F. Anton, On a conjecture of Quillen at the prime 3, J. Pure Appl. Algebra 144 (1) (1999) 1–20.
[3] W.G. Dwyer, Exotic cohomology for $GL_n(\mathbb{Z}[1/2])$, Proc. Amer. Math. Soc. 126 (7) (1998) 2159–2167.
[4] H.-W. Henn, The cohomology of $SL(3,\mathbb{Z}[1/2])$, K-Theory 16 (4) (1999) 299–359.
[5] H.-W. Henn, J. Lannes, Exotic classes in the mod 2 cohomology of $GL_n(\mathbb{Z}[1/2])$, in: Special Issue "Guido's Book of Conjectures", Enseign. Math. 54 (2008) 107–108.
[6] H.-W. Henn, J. Lannes, L. Schwartz, Localization of unstable $\mathcal{A}$-modules and equivariant mod p cohomology, Math. Ann. 301 (1995) 23–68.
[7] K. Hutchinson, On the low-dimensional homology of $SL_2(k[t,t^{-1}])$, J. Algebra 425 (2015) 324–366, http://dx.doi.org/10.1016/j.jalgebra.2014.11.021.
[8] K.P. Knudson, Homology of Linear Groups, Progress in Mathematics, vol. 193, Birkhäuser Verlag, Basel, Switzerland, 2001.
[9] S.A. Mitchell, On the plus construction for $BGL(\mathbb{Z}[1/2])$ at the prime 2, Math. Z. 209 (2) (1992) 205–222.
[10] D. Quillen, The spectrum of an equivariant cohomology ring. I, Ann. of Math. (2) 94 (1971) 549–572;
D. Quillen, The spectrum of an equivariant cohomology ring. II, Ann. of Math. (2) 94 (1971) 573–602.
[11] A.D. Rahm, M. Wendt, On Farrell–Tate cohomology of SL_2 over S-integers, Preprint, arXiv:1411.3542 [math.KT], https://hal.archives-ouvertes.fr/hal-01081081.
[12] J.-P. Serre, Sous-groupes finis des groupes de Lie, Séminaire Bourbaki, vol. 864, 1999.
[13] M. Wendt, Homology of SL_2 over function fields I: parabolic subcomplexes, to appear in J. Reine Angew. Math. (Crelle J.), Preprint, arXiv:1404.5825.
[14] M. Wendt, Homology of SL_2 over function fields II: rational function fields, in preparation.