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Complex analysis/Functional analysis

On holomorphic domination, II

Sur la majoration holomorphe, II

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ABSTRACT

Let X be a separable Banach space with the bounded approximation property, $\Omega \subset X$ pseudoconvex open, and $u: \Omega \to \mathbb{R}$ locally upper bounded. We show that there are a Banach space Z and a holomorphic function $h: \Omega \to Z$ with $u(x) < \|h(x)\|$ for $x \in \Omega$.

RÉSUMÉ

Étant donnée une fonction localement bornée $u: \Omega \to \mathbb{R}$ sur un ouvert pseudoconvexe Ω dans un espace de Banach séparable jouissant de la propriété d'approximation bornée, on montre ici qu'il y a une majoration de la forme $u(x) < \|h(x)\|$ pour $x \in \Omega$, où $h: \Omega \to Z$ est une fonction holomorphe convenable à valeurs dans un espace de Banach convenable Z. Une majoration holomorphe comme celle ci-dessus est une propriété de convexité holomorphe qui joue un rôle profitable en analyse complexe sur des variétés de Banach.

The ideas of plurisubharmonic domination and holomorphic domination along with some of their applications appeared in [6] by Lempert. Following him we say that *plurisubharmonic domination* is possible on a complex Banach manifold M if for every locally upper bounded function $u: M \to \mathbb{R}$ there is a continuous plurisubharmonic function $\psi: M \to \mathbb{R}$ with $u(x) < \psi(x)$ for all $x \in M$. If ψ can be taken in the form $\psi(x) = \|h(x)\|$ for $x \in M$, where $h: M \to Z$ is a holomorphic function to a Banach space Z, then we say that *holomorphic domination* is possible in M. Clearly, holomorphic domination on M implies plurisubharmonic domination on M.

The notions of holomorphic domination and plurisubharmonic domination relate to holomorphic convexity and can be used similarly to holomorphic convexity in Stein theory. See [7–9] for some applications of plurisubharmonic domination and holomorphic domination to complex Banach manifolds, e.g., for analytic cohomology. Holomorphic domination is also one of the axioms in [12] for Stein Banach manifolds.

Theorem 1 below summarizes some facts about holomorphic domination and plurisubharmonic domination.

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Theorem 1.

(a) *(Lempert, [6, Thm. 1.1]) If X is a Banach space with a countable unconditional basis, then holomorphic domination is possible in any pseudoconvex open $\Omega \subset X$.*
(b) *[10] Holomorphic domination is possible on the whole space X if X is a separable Banach space.*
(c) *[11] If X is a separable Banach space with the bounded approximation property (e.g., X has a Schauder basis), then plurisubharmonic domination is possible in any pseudoconvex open $\Omega \subset X$.*

In this paper we adapt the argument of Lempert in [6] and apply Theorem 1(b) to prove Theorem 2 below that augments Theorem 1(c).

Theorem 2. *If X is a separable Banach space with the bounded approximation property and $\Omega \subset X$ is pseudoconvex open, then holomorphic domination is possible in Ω.*

We use Theorem 3 below in the proof of Theorem 2 above, which is just the special case of Theorem 2 for a special function $u(x) = 1/\operatorname{dist}(x, X \setminus \Omega)$ to be holomorphically dominated.

Theorem 3. *If X is a separable Banach space with the bounded approximation property, $\Omega \subset X$ is pseudoconvex open, and $\Omega \neq X$, then there are a Banach space Z and a holomorphic function $h: \Omega \to Z$ with $1/\operatorname{dist}(x, X \setminus \Omega) \le \|h(x)\|$ for all $x \in \Omega$.*

Proof that Theorem 3 implies Theorem 2. If $\Omega = X$, then we are done by Theorem 1(b). Assume that $\Omega \neq X$, and let Z and h be as given by Theorem 3. We employ the graph method as in [6, Thm. 1.3]. The closed linear span of all the points $h(x) \in Z$ as x runs through Ω (or which is the same, through a countable dense subset of Ω) is a separable closed linear subspace Z_0 of Z. We may replace Z with the separable Banach space Z_0, i.e., we may assume that Z itself is a separable Banach space. The map $g: \Omega \to X \times Z$, $g(x) = (x, h(x))$, embeds Ω into the graph $g(\Omega)$ of h, and $g(\Omega)$ is a closed subset of the separable Banach space $X \times Z$.

Let $u: \Omega \to \mathbb{R}$ be the locally upper bounded function for which we need to find a Banach space W and a holomorphic function $H: \Omega \to W$ with $u(x) \le \|H(x)\|$ for all $x \in \Omega$. The zero extension $u_0: X \times Z \to \mathbb{R}$ of u given by $u_0(x, z) = 0$ if $(x, z) \notin g(\Omega)$ and $u_0(x, z) = u(x)$ if $(x, z) \in g(\Omega)$ is a locally upper bounded function $u_0: X \times Z \to \mathbb{R}$ since $g(\Omega)$ is a closed subset of $X \times Z$. As $X \times Z$ is a separable Banach space, Theorem 1(b) yields a Banach space W and holomorphic function $k: X \times Z \to W$ with $u_0(x, z) \le \|k(x, z)\|$ for all $(x, z) \in X \times Z$. Then the function $H: \Omega \to W$, $H(x) = k(x, h(x))$, is holomorphic and satisfies that $u(x) = u_0(x, h(x)) \le \|k(x, h(x))\| = \|H(x)\|$ for all $x \in \Omega$. □

Proof of Theorem 3. Recall that any separable Banach space with the bounded approximation property is a direct summand of a Banach space with a Schauder basis; see [13] by Pełczyński or [2] by Johnson, Rosenthal, and Zippin. Using this fact it is easy to see that it is enough to prove Theorem 3 when the Banach space X has a Schauder basis.

Look at the function $u(x) = 1/\operatorname{dist}(x, X \setminus \Omega)$ for $x \in \Omega$. The proof to dominate this function u is similar to standard arguments of Runge-type approximation. Since this function is bounded on bounded sets that are bounded away from the boundary of Ω, it is an easy one to dominate. In the process of domination the only type of holomorphic functions that need to be used can be taken to be of the form $f(\zeta, z) \in Z$ for $(\zeta, z) \in D \times Y$, where ζ runs through a bounded pseudoconvex open set D of $\mathbb{C}^N$ (of finite dimension $N \to \infty$), Y, Z are Banach spaces, and $f(\zeta, z)$ is a polynomial in $z \in Y$ for each $\zeta \in D$. The relevant uniform Runge approximation is very similar to the usual Runge process in a pseudoconvex open subset of complex Euclidean space, and the required modifications can be seen in Lempert's papers [4, Thm. 0.1, 5, Thm. 6.1, 7, Thm. 4.5].

The function u is bounded on any of the sets $\Omega_N\langle\alpha\rangle$, etc, used by Lempert in [6] to exhaust Ω in his proof of [6, Thm. 1.6]. The Hypothesis 1.5 of [6] on Runge-type approximation of unbounded holomorphic functions is not necessary to envoke in Lempert's proof of [6, Thm. 1.6] in order to holomorphically dominate the special function u, and so his proof goes through almost verbatim and gives a Banach space Z and a holomorphic function $h: \Omega \to Z$ with $u(x) \le \|h(x)\|$ for all $x \in \Omega$. In Lemma 4.1 and Proposition 4.2 of [6] the entire function g that appears there should be taken to be arbitrary constant values $g = 1, 2, 3, \ldots$. In Proposition 2.1 of [6] the Banach spaces V_B and the holomorphic functions f_B should be taken as $V_B = \mathbb{C}$ and f_B constant equal to $f_B(x) = |\sup_B u|$. In the proof of [6, Thm. 1.6] for the special function u it is not necessary to derive a contradiction, just apply [6, Prop. 2.1] once directly. □

Recall the definition of a Stein Banach manifold given in [12]. Then Theorem 2 clearly implies Theorem 4 below.

Theorem 4. *If X is a separable Banach space with the bounded approximation property and $\Omega \subset X$ is pseudoconvex open, then Ω is a Stein Banach manifold modeled on X.*

Dineen showed in his paper [1] on bounding sets, see also [3] by Josefson, that if $e_n = (\delta_{nk})_{k=1}^{\infty}$, $n \ge 1$, is the standard basis of the space ℓ_1 of summable sequences and $f: \ell_\infty \to \mathbb{C}$ is an entire holomorphic function on the space ℓ_∞ of bounded sequences, then $f(e_n) \in \mathbb{C}$ is bounded for $n \geq 1$. This implies that holomorphic domination does not hold on the nonseparable space ℓ_∞, since a continuous function $u: \ell_\infty \to \mathbb{R}$ can be constructed, e.g., using the extension theorem of Tietze and Urysohn, such that $u(e_n) = n$ for $n \geq 1$. Such a function u cannot be holomorphically dominated on ℓ_∞, due to the theorem of Dineen mentioned above, valid also for entire holomorphic functions with values in a Banach space.

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