C. R. Acad. Sci. Paris, Ser. I 353 (2015) 595–599

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Mathematical analysis/Dynamical systems

Ruelle operators with two complex parameters and applications

Opérateurs de Ruelle avec deux paramètres complexes et applications

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ARTICLE INFO

Article history:
Received 12 September 2014
Accepted after revision 9 April 2015
Available online 30 April 2015

Presented by Gilles Lebeau

ABSTRACT

For a C^2 weak-mixing Axiom-A flow $\phi_t : M \longrightarrow M$ on a Riemannian manifold M and a basic set Λ for ϕ_t, we consider the Ruelle transfer operator $L_{f-s\tau+zg}$, where f and g are real-valued Hölder functions on Λ, τ is the roof function and s, z are complex parameters. Under some assumptions about ϕ_t for arbitrary Hölder f, g, we establish estimates for the iterations of this Ruelle operator when $|\mathrm{Im}\, z| \le B|\mathrm{Im}\, s|^\nu$ for some constants $B > 0$, $0 < \nu < 1$ ($\nu = 1$ for Lipschitz f, g), in the spirit of the estimates for operators with one complex parameter (see [2,11,12]). Applying these estimates, we obtain a non-zero analytic extension of the zeta function $\zeta(s, z)$ for $P_f - \epsilon < \mathrm{Re}\,(s) \le P_f$ and $|z|$ small enough with a simple pole at $s = s(z)$. Two other applications are considered as well: the first concerns the Hannay–Ozorio de Almeida sum formula, while the second deals with the asymptotic of the counting function $\pi_F(T)$ for weighted primitive periods of the flow ϕ_t.

RÉSUMÉ

Soit $\phi_t : M \longrightarrow M$ un flot C^2, faiblement mélangeant, sur une variété riemannienne M. Soit Λ un ensemble basique pour ϕ_t. On considère l'opérateur de Ruelle de transfert $L_{f-s\tau+zg}$, où f et g sont des fonctions hölderiennes à valeurs réelles sur Λ, τ est la fonction roof et s, z sont des paramètres complexes. On suppose que ϕ_t satisfait quelques conditions et, pour des fonctions f, g arbitraires, on prouve des estimations pour les itérations de cet opérateur de Ruelle quand $|\mathrm{Im}\, z| \le B|\mathrm{Im}\, s|^\nu$ avec des constantes $B > 0$, $0 < \nu < 1$ ($\nu = 1$ si f, g sont des fonctions lipschitziennes) qui sont analogues aux estimations des opérateurs avec un paramètre complexe (cf. [2,11,12]). En appliquant ces estimations, on obtient un prolongement sans zéros de la fonction zêta $\zeta(s, z)$ pour $P_f - \epsilon < \mathrm{Re}\,(s) \le P_f$ et $|z|$ suffisamment petit avec un pôle simple en $s = s(z)$. Nous proposons aussi deux autres applications : la première concerne la formule de sommation de Hannay–Ozorio de Almeida, tandis que la seconde concerne l'asymptotique de la fonction de comptage $\pi_F(T)$ des périodes primitives du flot ϕ_t calculées avec des poids.

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[1] V.P. was partially supported by the project ANR 2011 BS01019 01.

http://dx.doi.org/10.1016/j.crma.2015.04.005
1631-073X/

1. Introduction

Let M be a C^2 complete (not necessarily compact) Riemannian manifold, and let $\phi_t : M \longrightarrow M$ be a C^2 weak-mixing Axiom-A flow (cf. [5]). Let Λ be a *basic set* for ϕ_t, i.e. Λ is a compact invariant subset of M, ϕ_t is hyperbolic and transitive on Λ and Λ is locally maximal, i.e. there exists an open neighborhood V of Λ in M such that $\Lambda = \cap_{t\in\mathbb{R}}\phi_t(V)$. It follows from the hyperbolicity of Λ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \epsilon_1$, then $W^s_{\epsilon_0}(x)$ and $\phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ intersect at exactly one point $[x, y] \in \Lambda$ (cf. [3]).

As in [11], fix a *(pseudo-)Markov partition* $\mathcal{R} = \{R_i\}_{i=1}^k$ of *pseudo-rectangles* $R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}$. Set $R = \cup_{i=1}^k R_i$, $U = \cup_{i=1}^k U_i$. Consider the *Poincaré map* $\mathcal{P} : R \longrightarrow R$, defined by $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\phi_{\tau(x)}(x) \in R$. The function τ is the so-called *first return time* associated with $\mathcal{R}$. Let $\sigma : U \longrightarrow U$ be the *shift map* given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \longrightarrow U$ is the *projection* along stable leaves. Let $\widehat{U}$ be the set of those points $x \in U$ such that $\mathcal{P}^m(x)$ is not a boundary point of a rectangle for any integer m. In a similar way, define $\widehat{R}$. Clearly, in general, τ is not continuous on U; however, under the assumption that the holonomy maps are Lipschitz (see Section 2), τ is *essentially Lipschitz* on U in the sense that there exists a constant $L > 0$ such that if $x, y \in U_i \cap \sigma^{-1}(U_j)$ for some i, j, then $|\tau(x) - \tau(y)| \le L\,d(x, y)$. The same applies to $\sigma : U \longrightarrow U$.

Define a $k \times k$ matrix $A = \{A(i, j)\}_{i,j=1}^k$ by $A(i, j) = 1$ if $\mathcal{P}(\text{Int}\, R_i) \cap \text{Int}\, R_j \neq \emptyset$ and $A(i, j) = 0$ otherwise. It is possible to construct a Markov partition $\mathcal{R}$ so that A is irreducible and aperiodic. Set $R^\tau = \{(x, t) \in R \times \mathbb{R} : 0 \le t \le \tau(x)\}/\sim$, where we identify by $\sim$ the points $(x, \tau(x))$ and $(\sigma x, 0)$. Introduce the suspended flow $\sigma^\tau_t(x, s) = (x, s + t)$ on R^τ taking into account the identification $\sim$. For a Hölder continuous function f on R, let $\text{Pr}(f)$ denote the topological pressure of f with respect to σ (see, e.g., [5]). We say that f and g are cohomologous and we denote this by $f \sim g$ if there exists a continuous function w such that $f = g + w \circ \sigma - w$. Set $v^n(x) = v(x) + v(\sigma(x)) + \ldots + v(\sigma^{n-1}(x))$ for a function v on R.

Let γ denote a primitive periodic orbit of ϕ_t and let $\lambda(\gamma)$ denote its least period. Given a Hölder function $F : \Lambda \longrightarrow \mathbb{R}$, introduce the weighted period $\lambda_F(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t(x_\gamma))\,dt$, where $x_\gamma \in \gamma$. Consider the weighted version of the dynamical zeta function (see Section 9 in [5])

$$\zeta_\phi(s, F) := \prod_\gamma \left(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma)}\right)^{-1}.$$

Then a closed σ-orbit $\{x, \sigma x, \ldots, \sigma^{n-1}x\}$ in U corresponds to a closed orbit γ in Λ with a least period $\lambda(\gamma) = \tau^n(x) := \tau(x) + \tau(\sigma(x)) + \ldots + \tau(\sigma^{n-1}(x))$. The analysis of $\zeta_\varphi(s, F)$ is reduced to that of the Dirichlet series

$$\eta(s) = \sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x)}$$

with a Hölder continuous function $f(x) = \int_0^{\tau(x)} F(\phi_t(x))\,dt : R \longrightarrow \mathbb{R}$. To deal with certain problems (see Chapter 9 in [5] and [10]), it is necessary to study a more general series:

$$\eta_g(s) = \sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} g^n(x)\, e^{f^n(x) - s\tau^n(x)}$$

with a Hölder continuous function $G : \Lambda \longrightarrow \mathbb{R}$ and $g(x) = \int_0^{\tau(x)} G(\phi_t(x))\,dt : R \longrightarrow \mathbb{R}$. For this purpose, it is convenient to examine the function:

$$\zeta(s, z) := \prod_\gamma \left(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma) + z\lambda_G(\gamma)}\right)^{-1} = \exp\Big(\sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x) + zg^n(x)}\Big) \tag{1.1}$$

depending on two complex variables $s, z \in \mathbb{C}$. Formally, we have $\eta_g(s) = \frac{\partial \log \zeta(s,z)}{\partial z}\Big|_{z=0}$.

Example 1. If $G = 0$ we obtain the classical Ruelle *dynamical zeta function* $\zeta_\phi(s) = \prod_\gamma \left(1 - e^{-s\lambda(\gamma)}\right)^{-1}$. Then $Pr(0) = h$, where $h > 0$ is the topological entropy of ϕ_t and $\zeta_\phi(s)$ is absolutely convergent for $\text{Re}\, s > h$ (see Chapter 6 in [5]).

Example 2. Consider the expansion function $E : \Lambda \longrightarrow \mathbb{R}$ defined by $E(x) := \lim_{t\to 0} \frac{1}{t} \log |\text{Jac}\,(D\phi_t|_{E^u(x)})|$. Introduce the function $\lambda^u(\gamma) = \lambda_E(\gamma)$ and define $f : R \longrightarrow \mathbb{R}$ by $f(x) = -\int_0^{\tau(x)} E(\phi_t(x))\,dt$. Then we have $-\lambda^u(\gamma) = f^n(x)$, f is Hölder continuous and $Pr(f) = 0$ (see [1]). Consequently, the series

$$\sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{f^n(x) - s\tau^n(x)} \tag{1.2}$$

is absolutely convergent for $\mathrm{Re}\, s > 0$ and nowhere-vanishing and analytic for $\mathrm{Re}\, s \geq 0$, except for a simple pole at $\mathrm{Re}\, s = 0$ (see Theorem 9.2 in [5]). The roof function $\tau(x)$ is constant on stable leaves of rectangles R_i, and we can assume that $\tau(x)$ depends only on $x \in U$. By a standard argument (see [5]), we can replace f by a Hölder function $\hat{f}(x)$ that depends only on $x \in U$ so that $f \sim \hat{f}$. Thus the series (1.2) can be written using functions $\hat{f}$, τ depending only on $x \in U$. We keep the notation f below assuming that f depends only on $x \in U$. The analysis of the analytic continuation of (1.2) is based on spectral estimates for the iterations of the Ruelle operator

$$L_{f-s\tau} v(x) = \sum_{\sigma y = x} \mathrm{e}^{f(y) - s\tau(y)} v(y), \; v \in C^\alpha(U), \; s \in \mathbb{C}$$

(see, for more details, [2,9,11]).

Example 3. Let f, τ be real-valued Hölder functions and let $P_f > 0$ be the unique real number such that $Pr(f - P_f \tau) = 0$. Let $g(x) = \int_0^{\tau(x)} G(\phi_t(x))\, \mathrm{d}t$, where $G : \Lambda \longrightarrow \mathbb{R}$ is a Hölder function. Then if the suspended flow σ_t^τ is weak-mixing, the function (1.1) is a nowhere-vanishing analytic function for $\mathrm{Re}\, s > P_f$ and z in a neighborhood of 0 (depending on s) with a nowhere-vanishing analytic extension to $\mathrm{Re}\, s = P_f$ ($s \neq P_f$) for small $|z|$. This statement is just Theorem 6.4 in [5]. To examine the analytic continuation of $\zeta(s,z)$ for $P_f - \eta_0 \leq \mathrm{Re}\, s$ and small $|z|$, it is necessary to have some spectral estimates for the iterations of the Ruelle operator

$$L_{f-s\tau+zg} v(x) = \sum_{\sigma y = x} \mathrm{e}^{f(y) - s\tau(y) + zg(y)} v(y), \; v \in C^\alpha(U), \; s \in \mathbb{C}, z \in \mathbb{C}. \tag{1.3}$$

The estimates of the iterations of operator (1.3) are important for the analysis of large deviations [13,6].

The results that we will discuss in this Note are proved under some relatively general assumptions (see the Standing Assumptions in [7]), however here we will consider the following one.

Simplifying assumption: ϕ_t is a C^2 contact Anosov flow satisfying the following pinching condition:
(P): *there exist constants $C > 0$ and $\beta_0 \geq \alpha_0 > 0$ such that for every $x \in M$, we have*

$$\frac{1}{C}\, \mathrm{e}^{\alpha_x t}\, \|u\| \leq \|\mathrm{d}\phi_t(x) \cdot u\| \leq C\, \mathrm{e}^{\beta_x t}\, \|u\| \quad , \quad u \in E^u(x) \;, t > 0$$

for some constants $\alpha_x, \beta_x > 0$ with $\alpha_0 \leq \alpha_x \leq \beta_x \leq \beta_0$ and $2\alpha_x - \beta_x \geq \alpha_0$ for all $x \in M$.

The condition (P) implies that the local stable/unstable holonomy maps are Lipschitz (see, e.g., [12]). We should note that (P) holds for the geodesic flow on a manifold with strictly negative sectional curvature satisfying the so called $\frac{1}{4}$-pinching condition. (P) always holds when $\dim(M) = 3$.

Theorem 1. *Under the assumptions above for a basic set Λ, for any Hölder continuous functions $F, G : \Lambda \longrightarrow \mathbb{R}$ there exists $\eta_0 > 0$ such that the function $\zeta(s,z)$ admits a non-vanishing analytic continuation for*

$$(s,z) \in \{(s,z) \in \mathbb{C}^2 : \; P_f - \eta_0 \leq \mathrm{Re}\, s, \; s \neq s(z), \; |z| \leq \eta_0\}$$

with a simple pole at $s(z)$. The pole $s(z)$ is determined as the root of the equation $Pr(f - s\tau + zg) = 0$ with respect to s for $|z| \leq \eta_0$.

As an application of Theorem 1, we obtain the so-called Hannay–Ozorio de Almeida sum formula (see [10] and the references there). Let $\phi_t : M \longrightarrow M$ be the geodesic flow on the unit-tangent bundle over a compact negatively curved surface M. In [10] it was proved that there exists $\epsilon > 0$ such that if $(\delta(T))^{-1} = \mathcal{O}(e^{\epsilon T})$, for every Hölder continuous function $G : M \longrightarrow \mathbb{R}$, we have

$$\lim_{T \to +\infty} \frac{1}{\delta(T)} \sum_{T - \frac{\delta(T)}{2} \leq \lambda(\gamma) \leq T + \frac{\delta(T)}{2}} \lambda_G(\gamma)\, \mathrm{e}^{-\lambda^u(\gamma)} = \int G\, \mathrm{d}\mu, \tag{1.4}$$

where the notation $\lambda(\gamma)$ and $\lambda^u(\gamma)$ for a primitive periodic orbit γ are introduced above, while μ is the unique ϕ_t-invariant probability measure, which is absolutely continuous with respect to the volume measure on M. The measure μ is called SRB (Sinai–Ruelle–Bowen) measure (see [1]). Notice that in the above case, the Anosov flow ϕ_t is weak mixing and M is an attractor. Applying Theorem 1 and the arguments in [10], we prove the following.

Theorem 2. *Let Λ be an attractor, i.e. there exists an open neighborhood V of Λ such that $\Lambda = \cap_{t \geq 0} \phi_t(V)$. Under the assumptions above for the basic set Λ, there exists $\epsilon > 0$ such that if $(\delta(T))^{-1} = \mathcal{O}(e^{\epsilon T})$, then for every Hölder function $G : \Lambda \longrightarrow \mathbb{R}$, the formula (1.4) holds with the SRB measure μ for ϕ_t.*

Our next application concerns the counting function $\pi_F(T) = \sum_{\lambda(\gamma) \leq T} e^{\lambda_F(\gamma)}$, where γ is a primitive period orbit for $\phi_t : \Lambda \longrightarrow \Lambda$, $\lambda(\gamma)$ is the least period and $\lambda_F(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t(x_\gamma))\,dt$, $x_\gamma \in \gamma$. For $F = 0$, we obtain the counting function $\pi_0(T) = \#\{\gamma : \lambda(\gamma) \leq T\}$. These counting functions have been studied in many works (see [9] for references concerning $\pi_0(T)$ and [5,8] for $\pi_F(T)$). The study of $\pi_F(T)$ is based on the analytic continuation of the function

$$\zeta_F(s) = \prod_\gamma \left(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma)}\right)^{-1}, \; s \in \mathbb{C}$$

which is just the function $\zeta(s, 0)$ defined above. We then have the following.

Theorem 3. *Let Λ be a basic set and let $F : \Lambda \longrightarrow \mathbb{R}$ be a Hölder function. Under the assumptions above for Λ, there exists $\epsilon > 0$ such that*

$$\pi_F(T) = li(e^{Pr(F)T})(1 + \mathcal{O}(e^{-\epsilon T})), \; T \to \infty,$$

where $li(x) := \int_2^x \frac{1}{\log y} dy \sim \frac{x}{\log x}$, $x \to +\infty$.

In the case when $\phi_t : T^1(M) \longrightarrow T^1(M)$ is the geodesic flow on the unit tangent bundle $T^1(M)$ of a compact C^2 manifold M of negative section curvature satisfying the $\frac{1}{4}$-pinching condition, the above result has been established in [8]. It follows from [11,12] that the special case of a geodesic flow in [8] is covered by Theorem 3.

2. Idea of the proof of Theorem 1

First, we prove a generalization of the so-called Ruelle's lemma that yields a link between the convergence by packets of a Dirichlet series like (1.2) and $\log \zeta(s, z)$ and the estimates of the iterations of the corresponding Ruelle operator. The reader may consult [14] for the precise result in this direction and the previous works (see [4] and the comments there), treating this question. For our needs in this paper, we prove an analogue of this lemma for Dirichlet series with two complex parameters following the approach in [14].

Next under the assumptions in this paper (or the more general Standing Assumptions in [7]) we prove spectral estimates for the iterations of Ruelle operators with two complex parameters.

Theorem 4. *Let $0 < \beta < \alpha$. Then for any real-valued functions $f, g \in C^\alpha(\widehat{U})$, for any constants $\epsilon > 0$, $B > 0$ and $\nu \in (0, 1)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a| \leq a_0, |c| \leq a_0$, then $\|L^m_{f-(P_f+a+\mathbf{i}b)\tau+(c+\mathbf{i}w)g} h\|_{\beta,b} \leq C\rho^m |b|^\epsilon \|h\|_{\beta,b}$ for all $h \in C^\beta(\widehat{U})$, all integers $m \geq 1$ and all $b, w \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B|b|^\nu$.*

Next, consider the function $\zeta(s, z)$ introduced in Section 1. Let $s = a + \mathbf{i}b$, $z = c + \mathbf{i}w$ with real $a, b, c, w \in \mathbb{R}$. Assume that f and g are functions in $C^\alpha(\Lambda)$ for some $0 < \alpha < 1$. Taking their restrictions[2] to R, we obtain functions in $C^\alpha(R)$ that we denote again by f and g. We assume that $Pr(f - P_f\tau) = 0$ and we set $s = P_f + a + \mathbf{i}b$. The functions f, g depend on $x \in R$. A second reduction is to replace f and g by functions $\hat{f}, \hat{g} \in C^{\alpha/2}(U)$ depending only on $x \in U$ so that $f = \hat{f} + h_1 - h_1 \circ \sigma$, $g = \hat{g} + h_2 - h_2 \circ \sigma$ (see Proposition 1.2 in [5]). Since for periodic points with $\sigma^n x = x$ we have $f^n(x) = \hat{f}^n(x)$, $g^n(x) = \hat{g}^n(x)$, we obtain the representation

$$\zeta(s, z) = \exp\Big(\sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{\hat{f}^n(x) - (P_f + a + \mathbf{i}b)\tau^n(x) + (c + \mathbf{i}w)\hat{g}^n(x)}\Big)$$
$$= \exp\Big(\sum_{n=1}^\infty \frac{1}{n} Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)\Big).$$

We will now prove that there exist $\epsilon > 0$ and $\epsilon_0 > 0$ such that the function $\zeta(s, z)$ has a non-vanishing analytic continuation for $-\epsilon \leq a \leq 0$ and $|z| \leq \epsilon_0$ with a simple pole at $s = s(z)$, $s(0) = P_f$. Here $s(z)$ is determined from the equation $Pr(f - s\tau + zg) = 0$ with respect to s. For simplicity of the notation we denote below $\hat{f}$ and $\hat{g}$ again by f, g.

First consider the case $0 < \delta \leq |b| \leq b_0$. Since our standing assumptions imply that the flow ϕ_t is weak mixing, Theorem 6.4 in [5] says that for every fixed b lying in the compact interval $[\delta, b_0]$ there exists $\epsilon(b) > 0$ so that the function $\zeta(s, z)$ is analytic for $|s - P_f + \mathbf{i}b| \leq \epsilon(b)$, $|z| \leq \epsilon(b)$. This implies that there exists $\eta_0 = \eta_0(\delta, b_0) > 0$ such that $\zeta(s, z)$ is analytic for $P_f - \eta_0 \leq \mathrm{Re}\, s \leq P_f + \eta_0, \delta \leq |\mathrm{Im}\, s| \leq b_0$, $|z| \leq \eta_0$. Decreasing $\delta > 0$ and η_0, if necessary, we apply once more Theorem 6.4

[2] In fact, one has to define first f and g as functions in $C^\alpha(\hat{R})$ and then extend them as α-Hölder functions on R. In the same way one should proceed with Hölder functions on U.

in [5], to conclude that $\zeta(s,z)(1-\mathrm{e}^{Pr(f-s\tau+zg)})$ is analytic for $s \in \{s \in \mathbb{C}: |\mathrm{Re}\, s - P_f| \le \eta, |\mathrm{Im}\, s| \le \delta\}$ and $|z| \le \eta_0$. Consequently, the singularities of $\zeta(s,z)$ are given by (s,z) for which we have $Pr(f - s\tau + zg) = 0$ and, solving this equation, we get $s = s(z)$ with $s(0) = P_f$. It is clear that we have a simple pole at $s(z)$ since $\frac{\mathrm{d}}{\mathrm{d}s}Pr(f - s\tau + zg) \neq 0$ for $|z|$ small enough.

Now we pass to the case when $|\mathrm{Im}\, s| = |b| \ge b_0 > 0$, $|z| \le \eta_0$. Then we fix a $\beta \in (0, \alpha/2)$ and we get with $0 < \mu < 1$ the inequality $|\mathrm{Im}\, b| \ge B_0|z|^\mu$ with $B_0 = \frac{b_0}{\eta_0^\mu}$. Thus we are in position to apply the estimates of Theorem 4 saying that for every $\epsilon > 0$ there exist $0 < \rho < 1$ and $C_\epsilon > 0$ so that

$$\|L^m_{f-(P_f+a+\mathbf{i}b)\tau+zg}\|_{\beta,b} \le C_\epsilon \rho^m |b|^\epsilon, \ \forall m \in \mathbb{N} \tag{2.1}$$

for $|a| \le a_0$, $|b| \ge b_0$, $|z| \le \eta_0$. Next we apply the generalized Ruelle's Lemma (cf. [7]) with functions $f, g \in C^\beta(U)$. For $|\mathrm{Re}\, s - P_f| \le \eta_0$, $|\mathrm{Im}\, s| \ge b_0$ and $|z| \le \eta_0$ we deduce with $\gamma_0 > 1$ the estimate

$$\begin{aligned}|Z_n(f-(P_f+a+\mathbf{i}b)\tau+zg)| &\le \sum_{i=1}^k |L^n_{f-(P_f+\mathbf{i}a+b)\tau+zg}(\chi_i)(x_i)| \\ &\quad + C(1+|b|)\sum_{m=2}^n \|L^{n-m}_{f-(P_f+a+\mathbf{i}b)\tau+zg}\|_\beta \gamma_0^{-m\beta} e^{mPr(f-(P_f+a)\tau+(\mathrm{Re}\, z)g)} \\ &\le k\|L^n_{f-(P_f+a+\mathbf{i}b)\tau+zg}\|_\beta + C_\epsilon(1+|b|)|b|^\epsilon \sum_{m=2}^n \rho^{n-m}\gamma_0^{-m\beta} e^{m(\epsilon+Pr(f-(P_f+a)\tau+cg))}.\end{aligned}$$

Taking η_0 and ϵ small, we arrange $\gamma_0^{-\beta}\mathrm{e}^{\epsilon+Pr(f-(P_f+a)\tau+cg)} \le \gamma_2 < 1$ for $|a| \le \eta_0$, $|c| \le \eta_0$, since $Pr(f - P_f\tau) = 0$ and $\gamma_0^{-\beta} < 1$. Next, increasing $0 < \rho < 1$, if it is necessary, we get $\frac{\gamma_2}{\rho} < 1$. Thus, the sum above will be bounded by $C_\epsilon(1 + |b|)|b|^\epsilon \rho^n \sum_{m=2}^\infty \left(\frac{\gamma_2}{\rho}\right)^m \le C'_\epsilon |b|^{1+\epsilon}\rho^n$ for $|a| \le \eta_0$, $|z| \le \eta_0$. The analysis of the term $\|L^n_{f-(P_f+a+\mathbf{i}b)+zg}\|_\beta$ follows the same argument, in fact it is simpler. Finally, we get $|Z_n(f - (P_f + a + \mathbf{i}b)\tau + zg)| \le B_\epsilon |b|^{1+\epsilon}\rho^n$ for all $n \in \mathbb{N}$, and the series $\sum_{n=1}^\infty \frac{1}{n} Z_n(f-(P_f+a+\mathbf{i}b)\tau+zg)$ is absolutely convergent for $|a| \le \eta_0$, $|b| \ge b_0$, $|z| \le \eta_0$. This implies the analytic continuation of $\zeta(s,z)$ for $|\mathrm{Re}\, s - P_f| \le \eta_0$, $|\mathrm{Im}\, s| \ge b_0$, $|z| \le \eta_0$, thus completing the proof of Theorem 1. Detailed proofs of our results can be found in [7].

Acknowledgement

Thanks are due to the referee for his/her useful comments.

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