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Geometry

The reverse Hlawka inequality in a Minkowski space

L'inégalité de Hlawka inverse dans un espace de Minkowski

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A B S T R A C T

We show that in the future cone of the Minkowski space, the pseudo-norm satisfies a Hlawka-type inequality:

$$\ell(x)+\ell(y)+\ell(z)+\ell(x+y+z)\le\ell(x+y)+\ell(y+z)+\ell(z+x).$$

The inequality is opposite to that in the Euclidean case, exactly as in the situation of the Cauchy–Schwarz inequality.

R É S U M É

Dans le cône du futur de l'espace de Minkowski, la pseudo-norme associée à la métrique lorentzienne satisfait une inégalité du type de Hlawka :

$$\ell(x)+\ell(y)+\ell(z)+\ell(x+y+z)\le\ell(x+y)+\ell(y+z)+\ell(z+x).$$

Le signe est l'opposé de celui du cas euclidien, tout comme dans l'inégalité « à la Cauchy–Schwarz ».

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1. Motivation and statement

In a Euclidean space E, the Hlawka inequality (see [3] or [4]) reads:

$$\|x+y\|+\|y+z\|+\|z+x\|\le\|x\|+\|y\|+\|z\|+\|x+y+z\|,\qquad\forall x,y,z\in E. \tag{1}$$

This inequality is sharp in three ways:

- the equality holds true if one of the vectors is 0,
- the equality holds true if x, y, z are positively collinear,
- the equality holds true if $x+y+z=0$.

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The two first equality cases are tightly related to the fact that the norm is positively homogeneous of degree one. Therefore, let us say that a continuous function $f : K \to \mathbb{R}$, defined over a closed convex cone in $\mathbb{R}^n$, which is positively homogeneous of degree one, *satisfies a Hlawka-type inequality* if

$$f(x+y)+f(y+z)+f(z+x) \le f(x)+f(y)+f(z)+f(x+y+z), \qquad \forall x,y,z \in K. \tag{2}$$

A necessary condition for this to happen is obtained by taking $z = y$:

$$2f(x+y) \le f(x)+f(x+2y), \qquad \forall x,y \in K.$$

This tells us that if $u, v \in K$ are ordered, that is if $v-u \in K$, then the convexity inequality

$$f(u+v) \le f(u)+f(v)$$

holds true. Actually, we have a bit more

Proposition 1.1. *If f is $\mathcal{C}^2$ in the interior of K and f satisfies the Hlawka-type inequality, then f is convex.*

Proof. It is enough to prove that the Hessian $\mathrm{D}^2 f$ is non-negative at interior points. For this, let us denote

$$\phi(x,y,z) = f(x)+f(y)+f(z)+f(x+y+z)-f(x+y)-f(y+z)-f(z+x),$$

which is non-negative by assumption. Since $\phi(x,x,x)=0$, the Hessian of ϕ at (x,x,x) must be non-negative if x is interior (one finds easily that the gradient is zero). Let us just compute the Hessian with respect to x:

$$\mathrm{D}_x^2\phi_{(x,x,x)} = \mathrm{D}^2 f_x + \mathrm{D}^2 f_{3x} - 2\mathrm{D}^2 f_{2x}.$$

Because $\mathrm{D}^2 f$ is homogeneous of degree -1, one deduces

$$0 \le \mathrm{D}_x^2\phi_{(x,x,x)} = \frac{1}{3}\mathrm{D}^2 f_x \qquad \square$$

The proposition above suggests to investigate which among the convex functions, positively homogenenous of degree one, satisfy a Hlawka inequality. The first natural candidates are norms, where $K = \mathbb{R}^n$. However it is known that the Hlawka inequality is not always true: Witsenhausen [5] proved that a finite-dimensional normed space whose unit ball is a polytope satisfies (1) if and only if it is L^1-embeddable. See also Theorem 8.3.2 in [1].

Other candidates are given in terms of *hyperbolic polynomials*. Recall that a polynomial p over $\mathbb{R}^n$, homogeneous of degree d, is hyperbolic in the direction of some vector $\mathbf{e}$ if $p(\mathbf{e}) > 0$ and if for every $x \in \mathbb{R}^n$, the roots of the univariate polynomial $t \mapsto p(x+t\mathbf{e})$ are real. Gårding [2] introduced this notion in connection with the well-posedness theory of the Cauchy problem for hyperbolic differential operators; the vector $\mathbf{e}$ is time-like. He proved two important facts:

- the connected component of $\mathbf{e}$ in $\{p > 0\}$ is a convex cone. Its elements are time-like vectors too;
- if we denote K the closure of this cone, so that $p \ge 0$ over K, the function $x \mapsto p(x)^{1/d}$ is concave over K.

An especially interesting example is that of $p(A) = \det A$ over the space $\mathbf{Sym}_d(\mathbb{R})$ (here $n = \frac{d(d+1)}{2}$), which is hyperbolic in the direction of I_d. The future cone K is made of the positive semi-definite matrices, and the concavity property bears the name of Minkovski's determinantal inequality:

$$(\det A)^{1/d} + (\det B)^{1/d} \le (\det(A+B))^{1/d}.$$

It is therefore natural to consider $f_p = -p^{1/d}$, where p is a homogeneous hyperbolic polynomial of degree d, and ask whether f_p satisfies the Hlawka inequality, that is whether

$$p(x)^{1/d}+p(y)^{1/d}+p(z)^{1/d}+p(x+y+z)^{1/d} \le p(x+y)^{1/d}+p(y+z)^{1/d}+p(z+x)^{1/d}, \quad \forall x,y,z \in K. \tag{3}$$

The following example shows that this turns out to be false in general. Take again for p the determinant over symmetric matrices, where $d \ge 3$. One can write $I_d = P+Q+R$ as the sum of non-trivial mutually orthogonal projectors. Then $\det P = \cdots = \det(Q+R) = 0$, but $\det(P+Q+R) = 1$, so that (3) is violated. This flaw looks to be caused by the fact that the boundary of K has flat parts.

The above counter-example leaves open the case $d = 2$, where the determinant is a non-degenerate quadratic form. In degree 2, the determinant becomes actually a paradigm, because of the following observations:

- the Hlawka inequality involves only three vectors. By restricting to the space spanned by x, y and z, it is therefore enough to consider forms in 2 or 3 space variables;
- a quadratic form q is hyperbolic if and only if its signature is $(1, n-1)$; in other words, when $(\mathbb{R}^n, q)$ is a Minkowski space. In particular, there is only one hyperbolic quadratic form in $\mathbb{R}^n$, up to a change of variable.

Since $\mathbf{Sym}_2(\mathbb{R})$, equipped with the determinant, is a Minkowski space of dimension 3, we deduce that the status of the Hlawka inequality for $-\sqrt{\det}$ is the same as the status for any Minkowski metric. Our main result is as follows.

Theorem 1.1. *The reverse Hlawka inequality is true in Minkowski spaces: if q is a quadratic form on $\mathbb{R}^n$, with signature $(1, n-1)$, then the "length" $\ell = \sqrt{q}$ satisfies*

$$\ell(x) + \ell(y) + \ell(z) + \ell(x+y+z) \leq \ell(x+y) + \ell(y+z) + \ell(z+x) \tag{4}$$

for every vectors x, y, z in the future cone.

Remarks.

- The fact that the sign in this inequality is opposite to the sign in the Euclidean Hlawka inequality (1) is all but a surprise. The same flip occurs in the Cauchy–Schwarz inequality, whose Lorentzian counterpart is $\ell(x)\ell(y) \leq x \cdot y$, for every $x, y \in K$.
- We do not exclude the possibility that some $p^{1/d}$ satisfy the reverse Hlawka inequality in the future cone, when p is hyperbolic homogeneous of higher degree $d \geq 2$ over $\mathbb{R}^n$. For instance, this is true when $p = q^m$ for $m \geq 2$ and q is a Lorentz quadratic form, because then $p^{1/d} = \sqrt{q}$. We leave open the case when $p_3(x) = \sigma_1(x)\sigma_2(x)$, where σ_j are the elementary symmetric polynomials, hyperbolic in the direction $\mathbf{1} = (1, \dots, 1)$. Because of the formula $(n-1)p_3 = (\mathbf{1} \cdot \nabla)(\sigma_2^2)$, this raises the question whether the Hlawka inequality transfers from a hyperbolic polynomial p to its derivative $(\mathbf{e} \cdot \nabla)p$ in a time-like direction.

Outline of the paper. According to the observations made above, it is enough to consider the cases

- $n = 2$ and $q(x) = x_1 x_2$,
- $n = 3$ and $q(A) = \det A$, with $\mathbb{R}^3 \sim \mathbf{Sym}_2(\mathbb{R})$.

We treat the first case in Section 2. We prove in Section 3 that it implies the second one. We study the equality case in the last section.

2. The two-dimensional case

We consider the form $q(x) = x_1 x_2$, whose future cone is $K = (\mathbb{R}^+)^2$. The corresponding bilinear form is

$$x \cdot y = \frac{1}{2}(x_1 y_2 + x_2 y_1).$$

Let g denote $\sqrt{q}$ (the opposite of f). One seeks for the inequality

$$g(x) + g(y) + g(z) + g(x+y+z) \leq g(x+y) + g(y+z) + g(z+x), \qquad \forall x, y, z \in K \tag{5}$$

Because both sides of (5) are non-negative, and because of the identity

$$q(x) + q(y) + q(z) + q(x+y+z) = q(x+y) + q(y+z) + q(z+x),$$

the inequality is equivalent to

$$\begin{aligned}(g(x) + g(y) + g(z))g(x+y+z) + g(x)g(y) + g(y)g(z) + g(z)g(x) \\ \leq g(x+y)g(y+z) + g(y+z)g(z+x) + g(z+x)g(x+y), \qquad \forall x, y, z \in K.\end{aligned} \tag{6}$$

The latter can be written as

$$\theta(x, y, z) + \theta(z, x, y) + \theta(y, z, x) \leq 0,$$

where

$$\theta(x, y, z) := g(x)g(x+y+z) + g(y)g(z) - g(x+y)g(x+z).$$

It is therefore enough to prove that

$$\theta(x, y, z) \leq 0, \qquad \forall x, y, z \in K. \tag{7}$$

Because g is non-negative, (7) is equivalent to

$$(g(x)g(x+y+z) + g(y)g(z))^2 \leq (g(x+y)g(x+z))^2,$$

that is to

$$2g(x)g(y)g(z)g(x+y+z) \leq \pi(x,y,z) \qquad \forall x,y,z \in K$$
$$:= q(x+y)q(x+z) - q(x)q(x+y+z) - q(y)q(z). \tag{8}$$

One verifies

$$\pi(x,y,z) = 4(x\cdot y)(x\cdot z) + 2(x\cdot y)q(z) + 2(x\cdot z)q(y) - 2(y\cdot z)q(x),$$

or equivalently

$$\pi = x_1^2 y_2 z_2 + x_2^2 y_1 z_1 + 2(x\cdot y)q(z) + 2(x\cdot z)q(y),$$

which is obviously non-negative for $x, y, z \in K$. From this, we infer that (8) holds true if and only if

$$4q(x)q(y)q(z)q(x+y+z) \leq \pi(x,y,z)^2, \qquad \forall x,y,z \in K. \tag{9}$$

The latter inequality turns out to hold true in an even more generality, because of the identity

$$\pi(x,y,z)^2 - 4q(x)q(y)q(z)q(x+y+z) = Q^2 \geq 0,$$

where $Q := x_1 y_2 z_2 (x_1 + y_1 + z_1) - x_2 y_1 z_1 (x_2 + y_2 + z_2)$. This follows from the factorization

$$\pi = x_1 y_2 z_2 (x+y+z)_1 + x_2 y_1 z_1 (x+y+z)_2.$$

The correctness of (9) is that of (7), which implies the correctness of (6), which amounts to the truth of (5). This ends the proof of the two-dimensional case.

3. The end of the proof

We now turn to the three-dimensional case where $K = \mathbf{Sym}_2^+$ and $q(A) = \det A$. Again, we write $g = \sqrt{q}$. By a continuity argument, we may assume that the three elements, denoted here A, B, C, are positive definite.

Defining $A' = C^{-1/2} A C^{-1/2}$ and $B' = C^{-1/2} B C^{-1/2}$, we see that (5) amounts to

$$g(I_2 + A' + B') + g(A') + g(B') + 1 \leq g(I_2 + A') + g(I_2 + B') + g(A' + B').$$

In other words, it is enough to consider the case where $C = I_2$.

Let us denote $a_1 \leq a_2$ and $b_1 \leq b_2$ the eigenvalues of A and B, and λ, μ those of $A+B$. We know $\lambda + \mu = T := \operatorname{Tr} A + \operatorname{Tr} B$. By Weyl's inequalities, we have

$$a_1 + b_1 \leq \lambda, \mu \leq a_2 + b_2.$$

We therefore have the constraints $\bar{s} := (a_1 + b_1)(a_2 + b_2) \leq \lambda\mu \leq T^2/4$. Let us estimate

$$\sqrt{\det(I_2 + A + B)} - \sqrt{\det(A+B)} = \sqrt{1 + T + \lambda\mu} - \sqrt{\lambda\mu}.$$

Because the function $s \mapsto \sqrt{1+T+s} - \sqrt{s}$ is monotone decreasing, its maximum under the conditions $\bar{s} \leq s \leq T^2/4$ is achieved at $\bar{s}$. We deduce

$$\sqrt{\det(I_2 + A + B)} - \sqrt{\det(A+B)} \leq \sqrt{(1 + a_1 + b_1)(1 + a_2 + b_2)} - \sqrt{(a_1 + b_1)(a_2 + b_2)}.$$

Since

$$g(I_2 + A) + g(I_2 + B) - g(A) - g(B) = \sqrt{(1+a_1)(1+a_2)} + \sqrt{(1+b_1)(1+b_2)} - \sqrt{a_1 a_2} - \sqrt{b_1 b_2},$$

there remains to prove

$$\sqrt{(1 + a_1 + b_1)(1 + a_2 + b_2)} + \sqrt{a_1 a_2} + \sqrt{b_1 b_2} + 1 \leq \sqrt{(1+a_1)(1+a_2)} + \sqrt{(1+b_1)(1+b_2)}$$
$$+ \sqrt{(a_1 + b_1)(a_2 + b_2)},$$

which is a consequence of the two-D case studied in Section 2.

Remark. One might have tried to prove the Theorem in every dimensions by following the same strategy as in the two-dimensional case, that is by proving that the corresponding function

$$\theta(x,y,z) := g(x)g(x+y+z) + g(y)g(z) - g(x+y)g(x+z)$$

remains non-positive. This is how the Euclidean Hlawka inequality was proved in [4]. This approach fails here because θ does not keep a constant sign in dimension ≥ 3.

4. The equality case

Proposition 4.1. *The equality holds in (4) if and only if*

- *either one vector among x, y or z is 0,*
- *or x, y and z are collinear.*

Proof.

Case $n=2$. If equality happens in (4), then we have $\theta(x,y,z)=\theta(y,z,x)=\theta(z,x,y)=0$. We may assume that none of the vectors be 0.

If $x_1=0$, we thus have $x_2>0$ and

$$0=y_1z_1=y_1z_2(x_1+y_1+z_1)=z_1y_2(x_1+y_1+z_1).$$

If $y_1=z_1=0$, then x,y,z are collinear. If not, there remains $0=y_1z_1=y_1z_2=z_1y_2$, which implies that either y or z is 0. The same analysis works if any of the five other coordinates vanishes.

Now, if all coordinates are positive, we obtain

$$\frac{x_2+y_2+z_2}{x_1+y_1+z_1}=\frac{x_1y_2z_2}{x_2y_1z_1}=\frac{x_2y_1z_2}{x_1y_2z_1}=\frac{x_2y_2z_1}{x_1y_1z_2},$$

which implies

$$\frac{x_2}{x_1}=\frac{y_2}{y_1}=\frac{z_2}{z_1}.$$

Therefore the vectors are collinear.

Case $n=3$. We first assume $C=I_2$. We keep the notations of Section 3. On the one hand, the equality in (4) implies

$$\sqrt{\det(I_2+A+B)}-\sqrt{\det(A+B)}=\sqrt{(1+a_1+b_1)(1+a_2+b_2)}-\sqrt{(a_1+b_1)(a_2+b_2)},$$

which amounts to

$$\lambda=a_1+b_1,\quad \mu=a_2+b_2.$$

This equality case in Weyl's inequality implies that A and B commute with each other. Going back to the general situation where C is positive definite, we obtain that A,B and C are diagonal in the same orthogonal basis. Finally, the vectors of eigenvalues must satisfy the two-dimensional equality case, meaning that either one matrix is 0_2, or that A,B and C are collinear.

There remains the sub-case where all of A,B and C are rank-one, say $A=aa^{\mathrm{T}}$, $B=bb^{\mathrm{T}}$ and $C=cc^{\mathrm{T}}$. Then $\det A=\det B=\det C=0$. Denoting $u_j=(a_j,b_j,c_j)$, we also have

$$\det(B+C)=(u_1\times u_2)_1^2,\quad \det(C+A)=(u_1\times u_2)_2^2,\quad \det(A+B)=(u_1\times u_2)_3^2$$

and

$$\det(A+B+C)=\|u_1\times u_2\|^2.$$

The equality in (4) tells us therefore

$$\|u_1\times u_2\|=\sum_{\alpha=1}^{3}|(u_1\times u_2)_\alpha|.$$

This implies that two coordinates of $u_1\times u_2$ vanish. This can happen only if either one of the vectors a, b or c is 0, or if all of them are collinear.

The case where $n\geq 4$ reduces to the cases $n\leq 3$ by restriction to the subspace spanned by x, y and z. □

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