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Group theory/Algebraic geometry

Automorphisms of $\overline{T}$

Automorphismes de $\overline{T}$

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A R T I C L E I N F O

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A B S T R A C T

Let $\overline{G}$ be the wonderful compactification of a simple affine algebraic group G defined over $\mathbb{C}$ such that its center is trivial and $G \neq \mathrm{PSL}(2,\mathbb{C})$. Take a maximal torus $T \subset G$, and denote by $\overline{T}$ its closure in $\overline{G}$. We prove that T coincides with the connected component, containing the identity element, of the group of automorphisms of the variety $\overline{T}$.

R É S U M É

Soit $\overline{G}$ la compactification magnifique d'un groupe algébrique affine G défini sur $\mathbb{C}$, dont le centre est trivial et tel que $G \neq \mathrm{PSL}(2,\mathbb{C})$. Soit $T \subset G$ un tore maximal, et soit $\overline{T}$ son adhérence dans $\overline{G}$. Nous montrons que T est égal à la composante connexe contenant l'élément neutre du groupe d'automorphismes de la variété $\overline{T}$.

1. Introduction

Let G be a simple affine algebraic group defined over the complex numbers such that the center of G is trivial. De Concini and Procesi constructed a very interesting compactification of G, which is known as the wonderful compactification [2, p. 14, 3.1, THEOREM]. The wonderful compactification of G will be denoted by $\overline{G}$. Fix a maximal torus T of G. Let $\overline{T}$ denote the closure of T in $\overline{G}$. The connected component, containing the identity element, of the group of all automorphisms of the variety $\overline{T}$ will be denoted by $\mathrm{Aut}^0(\overline{T})$. For more details about the variety $\overline{T}$, we refer to [1, § 1]. Our aim here is to compute $\mathrm{Aut}^0(\overline{T})$.

Using the action of G on $\overline{G}$, we have $T \subset \mathrm{Aut}^0(\overline{T})$; this inclusion does not depend on whether the right or the left action is chosen. We prove that $T = \mathrm{Aut}^0(\overline{T})$, provided that $G \neq \mathrm{PSL}(2,\mathbb{C})$; see Theorem 3.1.

Note that $\mathrm{Aut}(\overline{T})$ is not connected since $\overline{T}$ is stable under the conjugation of the normalizer $N_G(T)$ of T in G.

If $G = \mathrm{PSL}(2,\mathbb{C})$, then $\overline{T} = \mathbb{P}^1$, and hence $\mathrm{Aut}^0(\overline{T}) = \mathrm{PSL}(2,\mathbb{C})$.

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2. Lie algebra and algebraic groups

In this section, we recall some basic facts and notation on Lie algebra and algebraic groups (see [5,6] for details). Throughout G denotes an affine algebraic group over $\mathbb{C}$ which is simple and of adjoint type. We also assume that the rank of G is at least two, equivalently $G \neq \mathrm{PSL}(2, \mathbb{C})$.

For a maximal torus T of G, the group of all characters of T will be denoted by $X(T)$. The Weyl group of G with respect to T is defined as $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. By $R \subset X(T)$ we denote the root system of G with respect to T. For a Borel subgroup B of G containing T, let $R^+(B)$ denote the set of positive roots determined by T and B. Let

$$S = \{\alpha_1, \cdots, \alpha_n\}$$

be the set of simple roots in $R^+(B)$. Let B^- denote the opposite Borel subgroup of G determined by B and T. For $\alpha \in R^+(B)$, let $s_\alpha \in W$ be the reflection corresponding to α. The Lie algebras of G, T and B will be denoted by $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{b}$, respectively. The dual of the real form $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{t}$ is $X(T) \otimes \mathbb{R} = Hom_{\mathbb{R}}(\mathfrak{t}_{\mathbb{R}}, \mathbb{R})$.

The positive definite W-invariant form on $Hom_{\mathbb{R}}(\mathfrak{t}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form on $\mathfrak{g}$ is denoted by $(\, , \,)$. We use the notation

$$\langle \nu , \alpha \rangle := \frac{2(\nu, \alpha)}{(\alpha, \alpha)} .$$

In this setting, one has the Chevalley basis

$$\{x_\alpha, h_\beta \mid \alpha \in R, \ \beta \in S\}$$

of $\mathfrak{g}$ determined by T. For a root α, we denote by U_α (respectively, $\mathfrak{g}_\alpha$) the one-dimensional T stable root subgroup of G (respectively, the subspace of $\mathfrak{g}$) on which T acts through the character α.

Now, let σ be the involution of $G \times G$ defined by $\sigma(x, y) = (y, x)$. Note that the diagonal subgroup $\Delta(G)$ of $G \times G$ is the subgroup of fixed points, while $T \times T$ is a σ-stable maximal torus of $G \times G$ and $B \times B^-$ is a Borel subgroup having the property that $\sigma(\alpha) \in -R^+(B \times B^-)$ for every $\alpha \in R^+(B \times B^-)$.

Let $\overline{G}$ denote the wonderful compactification of the group G, where G is identified with the symmetric space $(G \times G)/\Delta(G)$ (see [2, p. 14, 3.1. THEOREM]). Let $\overline{T}$ be the closure of T in $\overline{G}$.

3. The connected component of the automorphism group

Recall that if X is a smooth projective variety over $\mathbb{C}$, the connected component of the group of all automorphisms of X containing the identity automorphism is an algebraic group (see [8, p. 17, Theorem 3.7] and [4, p. 268] (which deals also with the case when X is singular or is defined over any field)). Further, the Lie algebra of this automorphism group is isomorphic to the space of all vector fields on X, that is the space $H^0(X, \Theta_X)$ of all global sections of the tangent bundle Θ_X of X (see [8, p. 13, Lemma 3.4]).

Let $\mathrm{Aut}(\overline{T})$ denote the group of all algebraic automorphisms of the variety $\overline{T}$. Let

$$\mathrm{Aut}^0(\overline{T}) \subset \mathrm{Aut}(\overline{T})$$

be the connected component containing the identity element. We note that $\mathrm{Aut}^0(\overline{T})$ is an algebraic group with Lie algebra $\mathrm{H}^0(\overline{T}, \Theta_{\overline{T}})$, where $\Theta_{\overline{T}}$ is the tangent bundle of the variety $\overline{T}$; the Lie algebra structure on $\mathrm{H}^0(\overline{T}, \Theta_{\overline{T}})$ is given by the Lie bracket of vector fields.

The subvariety $\overline{T} \subset \overline{G}$ is stable under the action of $T \times T$. Further, the subgroup $T \times 1 \subset T \times T$ acts faithfully on $\overline{T}$, and $T \subset \overline{T}$ is a stable Zariski open dense subset for this action of T. Hence, we get an injective homomorphism:

$$\rho : T \longrightarrow \mathrm{Aut}^0(\overline{T}) .$$

Theorem 3.1. *The above homomorphism ρ is an isomorphism.*

Proof. We know that T is a maximal torus of $\mathrm{Aut}^0(\overline{T})$ [3, p. 521, Corollaire 1]. Choose a Borel subgroup $B' \subset \mathrm{Aut}^0(\overline{T})$ containing the maximal torus T of $\mathrm{Aut}^0(\overline{T})$. The action of B' on $\overline{T}$ fixes a point because $\overline{T}$ is a projective variety (see [6, p. 134, 21.2, Theorem]). Let $x \in \overline{T}$ be a point fixed by B'. Clearly, $n\overline{T}n^{-1} = \overline{T}$ for $n \in N_G(T)$, and the diagonal subgroup of $T \times T$ acts trivially on $\overline{T}$. Hence $W = N_G(T)/T$ is a subgroup of $\mathrm{Aut}(\overline{T})$. The diagonal subgroup of $T \times T$ acts trivially on $\overline{T}$. So we see that $T \times T$ fixes the point x. Therefore, by [1, p. 477, (1.2.7)] and [1, p. 478, (1.3.8)] we have that $x = w(z)$ for some $w \in W$, where z is the unique $B \times B^-$ fixed point in $\overline{G}$. Using conjugation by w^{-1}, we may assume that B' fixes z. Let

$$Q \subset \mathrm{Aut}^0(\overline{T})$$

be the stabilizer subgroup for the point z. As $B' \subset Q$, it follows that Q is in fact a parabolic subgroup of $\mathrm{Aut}^0(\overline{T})$.

We first show that $\mathrm{Aut}^0(\overline{T})$ is reductive. Let R_u be the unipotent radical of $\mathrm{Aut}^0(\overline{T})$. Therefore, R_u is also the unipotent radical of $\mathrm{Aut}(\overline{T})$. Hence $wR_uw^{-1} = R_u$ for all $w \in W$. Consequently, $R_u \subset B'$ fixes $w(z)$ for every $w \in W$.

For $\chi \in X(B) = X(T)$, let $\mathcal{L}_\chi$ be the line bundle on $\overline{G}$ associated with χ (see [2, p. 26, 8.1, Proposition]). Take any $w \in W$. The action of R_u fixes $w(z)$, so the fiber $(\mathcal{L}_\chi)_{w(z)}$ of $\mathcal{L}_\chi$ over $w(z)$ is a one-dimensional representation of R_u. This R_u-module $(\mathcal{L}_\chi)_{w(z)}$ is trivial because the group R_u is unipotent.

Let $\mathbb{C}[T]$ be the coordinate ring of the affine algebraic group T. We note that $\mathbb{C}[T]$ is a unique factorization domain, and therefore any line bundle on T is trivial. As $T \subset \overline{T}$ is a T stable open dense subset for the left translation action, we see that the T module $H^0(\overline{T}, \mathcal{L}_\chi)$ is a submodule of $\mathbb{C}[T]$. If χ is a dominant character of T, and $w \in W$, then the weight space $\mathbb{C}[T]$ of weight $-w(\chi)$ is one dimensional and spanned by $t^{-w(\chi)}$. Moreover, we have $t^{-w(\chi)} \in H^0(\overline{T}, \mathcal{L}_\chi)$, because it is the unique section of weight $-w(\chi)$ not vanishing at $w(z)$. Thus, from the above it follows that $t^{-w(\chi)}$ is fixed by R_u for every dominant character χ of T and every $w \in W$.

The set $\{t^\chi \mid \chi \in X(T)\}$ is a basis for the complex vector space $\mathbb{C}[T]$. Therefore, the action of R_u on $H^0(\overline{T}, \mathcal{L}_\chi)$ is trivial for every regular dominant character χ of T. We have

$$\overline{T} \subset \mathbb{P}(H^0(\overline{T}, \mathcal{L}_\chi)),$$

and hence it follows that the action of R_u on $\overline{T}$ is trivial, implying that R_u is trivial. Thus, the group $\mathrm{Aut}^0(\overline{T})$ is reductive.

Next we will show that $Q = B'$. Fix a dominant character χ of $T \subset B$. As $\mathrm{Aut}^0(\overline{T})$ is reductive, $\mathrm{Aut}^0(\overline{T})/Z$ is semisimple, where $Z(\subset Q)$ is the center of $\mathrm{Aut}^0(\overline{T})$. Note that Q/Z is a parabolic subgroup of $\mathrm{Aut}^0(\overline{T})/Z$ and that it fixes z; by the arguments in the proofs of [7, p. 81–82, 3.2, Proposition and 3.3, Corollary] there is a positive integer a such that Q/Z acts linearly on the fiber of the line bundle $\mathcal{L}_\chi^{\otimes a}$ over z through some character χ' of Q/Z. Pulling back χ' to Q, we see that Q acts on the fiber of the line bundle $\mathcal{L}_\chi^{\otimes a}$ over z by a character. The group $X(T)$ is finitely generated and Abelian, and hence the image of the restriction map

$$X(Q) \longrightarrow X(B') = X(T)$$

is of finite index. This implies that the rank of $X(Q)$ is equal to the rank of $X(B')$. Thus, we have $Q = B'$.

We will now show that $\mathrm{Aut}^0(\overline{T})$ is not semisimple. If $\mathrm{Aut}^0(\overline{T})$ is semisimple, then

$$\dim \overline{T} = \dim T \le \dim B'_u = \dim(\mathrm{Aut}^0(\overline{T})/B'), \tag{1}$$

where B'_u is the unipotent radical of B'. Note that by the above observation, B' is the stabilizer of z in $\mathrm{Aut}^0(\overline{T})$. Since B' is a Borel subgroup of $\mathrm{Aut}^0(\overline{T})$, $\mathrm{Aut}^0(\overline{T})/B'$ is a closed subvariety of $\overline{T}$. Thus from (1), we get that $\mathrm{Aut}^0(\overline{T})/B' = \overline{T}$. This implies that $\overline{T} = (\mathbb{P}^1)^n$, and $\mathrm{Lie}(\mathrm{Aut}^0(\overline{T})) = \mathrm{sl}(2,\mathbb{C})^n$, where $n = \dim T$. The T-fixed points of $(\mathbb{P}^1)^n$ are indexed by the elements of the Weyl group of $\mathrm{PSL}(2,\mathbb{C})^n$. Therefore, $\overline{T}$ has 2^n fixed points for the action of T. On the other hand, by [1, p. 477, (1.2.7) and p. 478, (1.3.8)], all $w(z) \in \overline{T}$, $w \in W$, are fixed by T, and $w'(z) = w(z)$ only if $w' = w$. Consequently, the order of W is at most 2^n. As $n = \dim T$, this is possible only if $W = S_2^n$. Hence it follows that $G = \mathrm{PSL}(2,\mathbb{C})^n$. But this contradicts the assumption that G is simple of rank $n \ge 2$. So $\mathrm{Aut}^0(\overline{T})$ is not semisimple.

The group $\mathrm{Aut}^0(\overline{T})$ is reductive but not semisimple, and this implies that the connected component Z^0, containing the identity element, of the center of $\mathrm{Aut}^0(\overline{T})$ is a positive dimensional sub-torus of T. Further, since

$$w\mathrm{Aut}^0(\overline{T})w^{-1} = \mathrm{Aut}^0(\overline{T}),$$

it follows that $wZ^0w^{-1} = Z^0$ for every $w \in W$. Thus, the restriction map

$$r : X(T) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow X(Z^0) \otimes_{\mathbb{Z}} \mathbb{R} \tag{2}$$

is a nonzero homomorphism of W modules. Note that $X(T) \otimes \mathbb{R}$ is an irreducible W module (this is because G is simple). So we conclude that the homomorphism r in (2) is an isomorphism. Consequently, we have $T = Z^0$ and $T = \mathrm{Aut}^0(\overline{T})$. □

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