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Ordinary differential equations

Uniform simplification in a full neighborhood of a turning point

Simplification uniforme au voisinage d'un point tournant

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ABSTRACT

We give an analytic version of a formal theorem due to R.J. Hanson and D.L. Russell. This version is a result of uniform simplification in a full neighborhood of a turning point for linear singularly perturbed differential equations of the second order, which generalizes a well-known theorem of Y. Sibuya.

RÉSUMÉ

Nous donnons une version analytique d'un théorème formel dû à R.J. Hanson et D.L. Russell. Il s'agit d'un résultat de simplification uniforme au voisinage d'un point tournant pour des équations différentielles linéaires singulièrement perturbées du second ordre, qui généralise un théorème connu de Y. Sibuya.

1. Introduction

Consider a singularly perturbed differential system of the form

$$\varepsilon Y' = A(x,\varepsilon)Y, \quad \text{where} \quad A(x,\varepsilon) = \begin{pmatrix} \varepsilon \mathbf{a}(x,\varepsilon) & x^{\mu} + \varepsilon \mathbf{b}(x,\varepsilon) \\ x^{\mu+\nu} + \varepsilon \mathbf{c}(x,\varepsilon) & -\varepsilon \mathbf{a}(x,\varepsilon) \end{pmatrix}, \tag{1}$$

x is a complex variable, ε is a small complex parameter, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are bounded holomorphic functions on $D(0,r_0) \times D(0,\varepsilon_0)$, with μ, ν nonnegative integers such that $\mu + \nu \neq 0$ and $r_0, \varepsilon_0 > 0$.

The point $x = 0$ is a so-called *turning point* for system (1).

In [3], R.J. Hanson et D.L. Russell proved the following formal result:

Theorem 1.1. *There exists $\hat{T}(x,\varepsilon) = \sum_{n=0}^{\infty} T_n(x)\varepsilon^n$, a formal power series in ε, where the $T_n(x)$ are square matrices of bounded holomorphic functions on $D(0,r_0)$ and $\det T_0(x) = 1$ on $D(0,r_0)$, such that the change of variables $Y = \hat{T}(x,\varepsilon)Z$ reduces the system (1) to*

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$$\varepsilon Z' = \hat{B}(x,\varepsilon) Z, \quad \textit{where} \quad \hat{B}(x,\varepsilon) = \begin{pmatrix} \varepsilon \sum_{k=0}^{\mu-1} \hat{b}_k^{11}(\varepsilon) x^k & x^\mu + \varepsilon \sum_{k=0}^{\mu-1} \hat{b}_k^{12}(\varepsilon) x^k \\ x^{\mu+\nu} + \varepsilon \sum_{k=0}^{\mu+\nu-1} \hat{b}_k^{21}(\varepsilon) x^k & \varepsilon \sum_{k=0}^{\mu-1} \hat{b}_k^{22}(\varepsilon) x^k \end{pmatrix}$$

and the $\hat{b}_k^{ij}(\varepsilon)$ *are formal power series in* ε.

In the present note, we prove the following theorem using the Gevrey theory of composite asymptotic expansions [1,2].

Theorem 1.2. *If one of the two following assumptions is satisfied:*

$$(\mathcal{H}_1) \quad \nu \textit{ is even and } \mathbf{c}(x,0) = \mathcal{O}\left(x^{\frac{1}{2}(\nu-2)}\right); \quad (\mathcal{H}_2) \quad \nu \textit{ is odd and } \mathbf{c}(x,0) = \mathcal{O}\left(x^{\frac{1}{2}(\nu-1)}\right),$$

then, for every $r \in]0, r_0[$ *and every sufficiently small sector* S, *there exists a square matrix* $T(x,\varepsilon)$ *of bounded holomorphic functions on* $D(0,r) \times S$, *having a 1-Gevrey asymptotic expansion, such that the change of variables* $Y = T(x,\varepsilon)Z$ *reduces the system (1) to*

$$\varepsilon Z' = B(x,\varepsilon) Z, \quad \textit{where} \quad B(x,\varepsilon) = \begin{pmatrix} \varepsilon \sum_{k=0}^{\mu-1} b_k^{11}(\varepsilon) x^k & x^\mu + \varepsilon \sum_{k=0}^{\mu-1} b_k^{12}(\varepsilon) x^k \\ x^{\mu+\nu} + \varepsilon \sum_{k=0}^{\mu+\nu-1} b_k^{21}(\varepsilon) x^k & -\varepsilon \sum_{k=0}^{\mu-1} b_k^{11}(\varepsilon) x^k \end{pmatrix}$$

and the $b_k^{ij}(\varepsilon)$ *admit 1-Gevrey asymptotic expansions.*

Remark 1. The case $\mu = 0$ was proved by Y. Sibuya in [6].

The assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ imply that the differential system (1) does not have *secondary turning points*, which appear after some rescaling $x = \varepsilon^q X$ with a suitable rational q. As these conditions are still satisfied by the reduced system $\varepsilon Z' = B(x,\varepsilon)Z$, we can rescale this system to a regularly perturbed one.

Remark 2. Note that this theorem is a global result. The differential system (1) can be analytically reduced on every disk strictly included in $D(0, r_0)$. We obtain this global simplification using the Gevrey composite asymptotic expansions of [1,2] and accurate tracking of their Gevrey type. This aspect is a technical part of the proof that will not be presented in this note. The reader interested in the details is referred to [4].

2. Notations

Given $\alpha < \beta \le \alpha + 2\pi$, $\eta_0 > 0$ and $\mu > 0$, the sector $S = S(\alpha, \beta, \eta_0)$ is the set of all $\eta \in \mathbb{C}$ satisfying $0 < |\eta| < \eta_0$, $\alpha < \arg \eta < \beta$, and we call *infinite quasi-sector* the set $V = V(\alpha, \beta, \mu)$ of all $X \in \mathbb{C}$ such that $\mu < |X|$ and $\alpha < \arg X < \beta$.

Let $\mathcal{G}_n(V)$ denote the vector space of square matrices $G(X)$ of bounded holomorphic functions on V, having an asymptotic expansion in the Poincaré sense at infinity, $G(X) \sim \sum_{k\ge0} G_k X^{-k}$, $V \ni X \to \infty$, and let $\mathcal{H}_n(r_0)$ denote the vector space of square matrices of bounded holomorphic functions on the disk $D(0, r_0)$.

In this note, we use the definitions of *Gevrey composite asymptotic expansion* (and the associated notation $\sim_{\frac{1}{p}}$) and *consistent good covering* $S_l, V^j, V_l^j(\eta)$ introduced in [1,2].

We say that $L(x,\eta)$ is a *slow matrix* if $L(x,\eta) \sim_{\frac{1}{p}} \sum_{k\ge0} A_k(x)\eta^k$, as $S \ni \eta \to 0$ and $x \in D(0, r_0)$, with $(A_k)_{k\in\mathbb{N}} \in \mathcal{H}_n(r_0)^{\mathbb{N}}$, and that $R(x,\eta)$ is a *fast matrix* if there exists a sequence $(G_k)_{k\in\mathbb{N}}$ in $\mathcal{G}_n(V)^{\mathbb{N}}$ such that $R(x,\eta) \sim_{\frac{1}{p}} \sum_{k\ge0} G_k\left(\frac{x}{\eta}\right)\eta^k$, as $S \ni \eta \to 0$ and $x \in V(\eta)$. In the sequel, we use the variables η and ε simultaneously, but they are always connected by $\eta^p = \varepsilon$, with a suitable integer p.

Let $\mathrm{diag}(a_1, a_2)$ denote the diagonal matrix $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$.

3. Preliminaries

In this section we prove that a family $(Y_l^j)_{j,l}$ of matrices defined on a consistent good covering, whose differences are exponentially small, can be written as a slow–fast product.

Let V^j, S_l, $V_l^j(\eta)$, $j = 1, \dots, J$, $l = 1, \dots, L$, be a consistent good covering and $A, B > 0$. We define two spaces, $\mathbf{E}_n(A, B)$ and $\mathbf{L}_n(A)$.

Definition 3.1. Let $\mathbf{E}_n(A,B)$ denote the vector space of families $\left(Y_l^j(x,\eta)\right)_{j,l}$, $j=1,\dots,J$, $l=1,\dots,L$, of square matrices of size n of holomorphic functions defined for $\eta\in S_l$ and $x\in V_l^j(\eta)$, satisfying the three following assumptions: there exists a constant $C>0$ such that

$$\|Y_l^j(x,\eta)\|\le C,\quad \text{if } \eta\in S_l \text{ and } x\in V_l^j(\eta), \tag{2}$$

$$\|Y_{l+1}^j(x,\eta)-Y_l^j(x,\eta)\|\le C\exp\left(-\frac{A}{|\eta|^p}\right),\quad \text{if } \eta\in S_l\cap S_{l+1} \text{ and } x\in V_l^j(\eta)\cap V_{l+1}^j(\eta), \tag{3}$$

$$\|Y_l^{j+1}(x,\eta)-Y_l^j(x,\eta)\|\le C\exp\left(-B\left|\frac{x}{\eta}\right|^p\right),\quad \text{if } \eta\in S_l \text{ and } x\in V_l^j(\eta)\cap V_l^{j+1}(\eta). \tag{4}$$

The vector space $\mathbf{E}_n(A,B)$ is equipped with the following norm:

$$\|(Y_l^j)_{j,l}\|_{\mathbf{E}}=\inf\{C \text{ satisfying (2), (3) and (4)}\}.$$

Definition 3.2. Let $\mathbf{L}_n(A)$ denote the vector space of families $\left(L_l(x,\eta)\right)_l$, $l=1,\dots,L$, of square matrices of size n of holomorphic functions, defined for $\eta\in S_l$ and $x\in D(0,r_0)$, such that there exists a constant $C>0$ satisfying

$$\|L_l(x,\eta)\|\le C,\quad \text{if } \eta\in S_l \text{ and } x\in D(0,r_0), \tag{5}$$

$$\|L_{l+1}(x,\eta)-L_l(x,\eta)\|\le C\exp\left(-\frac{A}{|\eta|^p}\right),\quad \text{if } \eta\in S_l\cap S_{l+1} \text{ and } x\in D(0,r_0). \tag{6}$$

The vector space $\mathbf{L}_n(A)$ is equipped with the following norm:

$$\|(L_l)_l\|_{\mathbf{L}}=\inf\{C \text{ satisfying (5) and (6)}\}.$$

Theorem 3.3. *Let $(Y_l^j)_{j,l}\in\mathbf{E}_n(A,B)$. If $\|(Y_l^j)_{j,l}\|_{\mathbf{E}}$ is sufficiently small, then there exists a pair $\left((L_l)_l,(R_l^j)_{j,l}\right)\in\mathbf{L}_n(A)\times\mathbf{E}_n(A,B)$ such that the L_l are slow matrices, the R_l^j are fast matrices and*

$$I+Y_l^j=(I+L_l)(I+R_l^j). \tag{7}$$

Proof. In [2] Fruchard and Schäfke showed that every element of $\mathbf{E}_n(A,B)$ can be written as a sum of a slow matrix and a fast one. We improve their result: there exists a bounded linear operator $\mathbf{T}:\mathbf{E}_n(A,B)\to\mathbf{L}_n(A)\times\mathbf{E}_n(A,B)$ such that $\mathbf{T}\left((Y_l^j)\right)=\left((L_l),(R_l^j)\right)$ implies that for all (l,j), the L_l are slow matrices, the R_l^j are fast ones and $Y_l^j=L_l+R_l^j$.

The factorization (7) is equivalent to $L_l+R_l^j=Y_l^j-L_lR_l^j$. We can obtain such a factorization as a fixed point of the mapping $\Phi:\mathbf{L}_n(A)\times\mathbf{E}_n(A,B)\to\mathbf{L}_n(A)\times\mathbf{E}_n(A,B)$, $\quad\Phi\left((L_l),(R_l^j)\right)=\mathbf{T}\left((Y_l^j-L_lR_l^j)\right)$, provided that $\|(Y_l^j)\|_{\mathbf{E}}$ is sufficiently small. □

4. Proof of the main theorem

In this section, we present the key steps of the proof of Theorem 1.2. For the sake of simplicity, we assume that ν is an even natural number. The case ν odd is proved similarly but requires more precautions; in particular, we need to track the parity of functions at each step of the proof.

4.1. Fundamental matrices of solutions to (1)

First we describe the structure of a family of fundamental matrices of solutions to (1) defined on a suitable consistent good covering.

Proposition 4.1. *For all $r\in\,]0,r_0[$, there exist $\mu>0$, $\eta_1>0$ and S_l, V^j, $V_l^j(\eta)$, $l=0,\dots,L-1$, $j=0,\dots,p-1$, a consistent good covering of the set $\{(x,\eta);\ 0<|\eta|<\eta_1$ and $\mu|\eta|<|x|<r\}$, and*

$$Y_l^j(x,\eta)=\operatorname{diag}\left(1,x^{\frac{\nu}{2}}\right)\cdot H_l^j(x,\eta)\cdot E_l^j(x,\eta)$$

a family of fundamental matrices of solutions to (1) such that

(i) $p=\mu+\frac{\nu}{2}+1$,

(ii) *the $H_l^j(x,\eta)$ are holomorphic matrices defined for $\eta\in S_l$ and $x\in V_l^j(\eta)$, having a composite asymptotic expansion of Gevrey order $\frac{1}{p}$, as $\eta\to 0$ in S_l and $x\in V_l^j(\eta)$,*

(iii) *the matrices $E_l^j(x,\eta)$, defined for $\eta \in S_l$ and $x \in V_l^j(\eta)$, are of the form*

$$E_l^j(x,\eta) = \exp\left(\operatorname{diag}\left(-\frac{1}{p}\frac{x^p}{\varepsilon} + R_l^-(\varepsilon)\log x\,,\ \frac{1}{p}\frac{x^p}{\varepsilon} + R_l^+(\varepsilon)\log x\right)\right),$$

where the $R_l^-(\varepsilon)$ and the $R_l^+(\varepsilon)$ are holomorphic functions defined on S_l, having an asymptotic expansion of Gevrey order 1 as $\varepsilon \to 0$ in S_l.

In particular, there exist two positive constants A and B such that $(H_l^j(x,\eta))_{j,l} \in \mathbf{E}_2(A,B)$.

Using the implicit function theorem, we prove the existence of a family $(q_l(x,\varepsilon))_l$ of polynomials in x of degree $\frac{\nu}{2}-1$ without constant term, such that the q_l have asymptotic expansions of Gevrey order 1 and the coefficient $(1,2)$ of the matrices L_l introduced later satisfies $(L_l)_{12}(x,\varepsilon) = \mathcal{O}(x^{\frac{\nu}{2}})$. Now applying Theorem 3.3, we separate the asymptotic of the matrices $H_l^j(x,\eta)\cdot\exp\left(\operatorname{diag}(q_l,-q_l)\right)$:

$$H_l^j(x,\eta)\cdot\exp\left(\operatorname{diag}\left(q_l(x,\varepsilon),-q_l(x,\varepsilon)\right)\right) = L_l(x,\varepsilon)\cdot R_l^j(x,\eta),$$

where the L_l are slow matrices defined on $D(0,r)\times\Sigma_l$, with $\Sigma_l = \left\{\eta^p \mid \eta\in S_l\right\}$, and the R_l^j are fast matrices defined on the set of (x,η) such that $\eta\in S_l$ and $x\in V_l^j(\eta)$. Let $P_l = \operatorname{diag}\left(1,x^{\frac{\nu}{2}}\right)\cdot L_l\cdot\operatorname{diag}\left(1,x^{-\frac{\nu}{2}}\right)$. As $(L_l)_{12}(x,\varepsilon)=\mathcal{O}(x^{\frac{\nu}{2}})$, the coefficient $(1,2)$ of the matrices P_l does not have any poles at $x=0$. The matrices P_l are also slow matrices; they have an asymptotic expansion of Gevrey order 1: $P_l(x,\varepsilon)\sim_1 \hat{P}(x,\varepsilon)$ as $\varepsilon\to 0$ in Σ_l and $x\in D(0,r)$. We can rewrite the fundamental matrices of solutions to system (1):

$$Y_l^j(x,\eta) = P_l(x,\varepsilon)\cdot\operatorname{diag}\left(1,x^{\frac{\nu}{2}}\right)\cdot R_l^j(x,\eta)\cdot\tilde{E}_l^j(x,\eta),$$

where $\tilde{E}_l^j = \exp\left(\operatorname{diag}(-q_l,q_l)\right)\cdot E_l^j$.

4.2. Uniform simplification

In the sequel, the variable η lies in the sector S_l, ε in the sector Σ_l and x in the quasi-sector $V_l^j(\eta)$ (or in the disk $D(0,r)$ if the variable x is associated with a slow matrix). From now on, we omit the indices l and j.

Applying the change of variables $Y=P(x,\varepsilon)U$, we reduce the differential system (1) to

$$\varepsilon U' = C(x,\varepsilon)U, \quad \text{where} \quad C = P^{-1}AP - \varepsilon P^{-1}P'. \tag{8}$$

On the one hand, the expression of C implies that C is a slow matrix. On the other hand, we know a fundamental matrix of solutions to equation (8), $U(x,\eta)=\operatorname{diag}\left(1,x^{\frac{\nu}{2}}\right)R(x,\eta)\tilde{E}(x,\eta)$, and we have $C=\varepsilon U'U^{-1}$. We use this second expression to give an upper bound on the degree of each coefficient of the matrix $\hat{C}(x,\varepsilon)$.

Proposition 4.2. *The change of variables $Y=P(x,\varepsilon)U$ transforms the differential system (1) into $\varepsilon U' = C(x,\varepsilon)U$, where $C(x,\varepsilon)\sim_1 \hat{C}(x,\varepsilon)$ as $\varepsilon\to 0$ in Σ_l and $x\in D(0,r)$, and*

$$\hat{C}(x,\varepsilon) = \begin{pmatrix} \displaystyle\sum_{k=0}^{\mu+\nu/2} \hat{c}_k^{11}(\varepsilon)x^k & \displaystyle\sum_{k=0}^{\mu} \hat{c}_k^{12}(\varepsilon)x^k \\ \displaystyle\sum_{k=0}^{\mu+\nu} \hat{c}_k^{21}(\varepsilon)x^k & -\displaystyle\sum_{k=0}^{\mu+\nu/2} \hat{c}_k^{11}(\varepsilon)x^k \end{pmatrix},$$

where the $\hat{c}_k^{ij}(\varepsilon)$ are formal power series of Gevrey order 1, such that $\hat{c}_\mu^{12}(\varepsilon) = 1+\mathcal{O}(\varepsilon)$, $\hat{c}_{\mu+\nu}^{21}(\varepsilon)=1+\mathcal{O}(\varepsilon)$ and $\hat{c}_k^{ij}(\varepsilon)=\mathcal{O}(\varepsilon)$ if $k\neq \mu,\mu+\nu$.

We truncate the Taylor series of the coefficients of the matrix C to define a new matrix $\tilde{C}$:

$$\tilde{C}(x,\varepsilon) = \begin{pmatrix} \displaystyle\sum_{k=0}^{\mu+\nu/2} c_k^{11}(\varepsilon)x^k & \displaystyle\sum_{k=0}^{\mu} c_k^{12}(\varepsilon)x^k \\ \displaystyle\sum_{k=0}^{\mu+\nu} c_k^{21}(\varepsilon)x^k & -\displaystyle\sum_{k=0}^{\mu+\nu/2} c_k^{11}(\varepsilon)x^k \end{pmatrix}.$$

This matrix $\tilde{C}$ still satisfies $\tilde{C}(x,\varepsilon) \sim_1 \hat{C}(x,\varepsilon)$, as $\varepsilon \to 0$ in Σ_l and $x \in D(0,r)$. Now using the exponential estimate of the differences $\tilde{C}_{l+1} - \tilde{C}_l$ and the result of [5], we prove the existence of a matrix $\Delta(x,\varepsilon)$, solution to $\varepsilon\Delta' = A\Delta - \Delta\tilde{C} + R$, with $R = P(C - \tilde{C})$, which remains exponentially small on $D(0,r) \times \Sigma_l$. Then we set $\tilde{P} = P + \Delta$. This step can be summarized as follows.

Proposition 4.3. *For all $r \in]0, r_0[$, there exists a bounded holomorphic matrix $\tilde{P}(x,\varepsilon)$ on $D(0,r) \times \Sigma_l$, having the same asymptotic expansion as $P(x,\varepsilon)$, $\tilde{P}(x,\varepsilon) \sim_1 \hat{P}(x,\varepsilon)$, such that the transformation $Y = \tilde{P}(x,\varepsilon)U$ reduces the differential system (1) to $\varepsilon U' = \tilde{C}(x,\varepsilon)U$.*

Up to now, we transformed the first differential system into a polynomial differential system, but the condition on the degree of the polynomials that appears in Theorem 1.2 is not yet satisfied. To obtain the enunciated simplification, we need to apply a last change of variables, $U = \Psi(x,\varepsilon)Z$, where $\Psi(x,\varepsilon)$ is a matrix of polynomials in x, which allows us to reduce the degree of the diagonal coefficients of the matrix $\tilde{C}$. Applying the transformation $U = \Psi(x,\varepsilon)Z$, we reduce the system $\varepsilon U' = \tilde{C}(x,\varepsilon)U$ to a polynomial differential system, $\varepsilon Z' = B(x,\varepsilon)Z$, where the matrix B satisfies the conditions of Theorem 1.2.

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