Ordinary differential equations

Uniform simplification in a full neighborhood of a turning point

Simplification uniforme au voisinage d’un point tournant

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1. Introduction

Consider a singularly perturbed differential system of the form

\[ \varepsilon Y' = A(x, \varepsilon)Y, \quad \text{where} \quad A(x, \varepsilon) = \begin{pmatrix} \varepsilon a(x, \varepsilon) & x^\mu + \varepsilon c(x, \varepsilon) \\ x^{\mu+v} + \varepsilon b(x, \varepsilon) & -\varepsilon a(x, \varepsilon) \end{pmatrix}. \]  

(1)

\( x \) is a complex variable, \( \varepsilon \) is a small complex parameter, \( a, b, c \) are bounded holomorphic functions on \( D(0, r_0) \times D(0, \varepsilon_0) \), with \( \mu, v \) nonnegative integers such that \( \mu + v \neq 0 \) and \( r_0, \varepsilon_0 > 0 \).

The point \( x = 0 \) is a so-called turning point for system (1).

In [3], R.J. Hanson et D.L. Russell proved the following formal result:

**Theorem 1.1.** There exists \( \hat{T}(x, \varepsilon) = \sum_{n=0}^{\infty} T_n(x)\varepsilon^n \), a formal power series in \( \varepsilon \), where the \( T_n(x) \) are square matrices of bounded holomorphic functions on \( D(0, r_0) \) and \( \det T_0(x) = 1 \) on \( D(0, r_0) \), such that the change of variables \( Y = \hat{T}(x, \varepsilon)Z \) reduces the system (1) to

\[ Z' = T(x, \varepsilon)Z, \]

where \( T(x, \varepsilon) \) is a non-degenerate formal differential operator of order \( \mu + v < 2 \).

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Theorem and note.

Theorem 1.2. If one of the two following assumptions is satisfied:

(\(H_1\)) \(v\) is even and \(c(x, 0) = O\left(x^{1/2(v-2)}\right)\);  \(\(H_2\)\) \(v\) is odd and \(c(x, 0) = O\left(x^{1/2(v-1)}\right)\),

then, for every \(r \in [0, r_0]\) and every sufficiently small sector \(S\), there exists a square matrix \(T(x, \varepsilon)\) of bounded holomorphic functions on \(D(0, r) \times S\), having a 1-Gevrey asymptotic expansion, such that the change of variables \(Y = T(x, \varepsilon)Z\) reduces the system (1) to

\[
\varepsilon Z' = B(x, \varepsilon)Z, \quad \text{where} \quad B(x, \varepsilon) = \begin{pmatrix}
\varepsilon \sum_{k=0}^{\mu-1} b_k^{11}(\varepsilon)x^k & x^{\mu} + \varepsilon \sum_{k=0}^{\mu-1} b_k^{12}(\varepsilon)x^k \\
\varepsilon + \varepsilon \sum_{k=0}^{\mu+v-1} b_k^{21}(\varepsilon)x^k & -\varepsilon \sum_{k=0}^{\mu-1} b_k^{11}(\varepsilon)x^k
\end{pmatrix}
\]

and the \(b_k^{ij}(\varepsilon)\) admit 1-Gevrey asymptotic expansions.

Remark 1. The case \(\mu = 0\) was proved by Y. Sibuya in [6].

The assumptions \((H_1)\) and \((H_2)\) imply that the differential system (1) does not have secondary turning points, which appear after some rescaling \(x = \varepsilon^qX\) with a suitable rational \(q\). As these conditions are still satisfied by the reduced system \(\varepsilon Z' = B(x, \varepsilon)Z\), we can rescale this system to a regularly perturbed one.

Remark 2. Note that this theorem is a global result. The differential system (1) can be analytically reduced on every disk strictly included in \(D(0, r_0)\). We obtain this global simplification using the Gevrey composite asymptotic expansions of [1,2] and accurate tracking of their Gevrey type. This aspect is a technical part of the proof that will not be presented in this note. The reader interested in the details is referred to [4].

2. Notations

Given \(\alpha < \beta < \alpha + 2\pi, \eta_0 > 0\) and \(\mu > 0\), the sector \(S = S(\alpha, \beta, \eta_0)\) is the set of all \(\eta \in \mathbb{C}\) satisfying \(0 < |\eta| < \eta_0, \alpha < \arg \eta < \beta\), and we call infinite quasi-sector the set \(V = V(\alpha, \beta, \mu)\) of all \(X \in \mathbb{C}\) such that \(\mu < |X|\) and \(\alpha < \arg X < \beta\).

Let \(G_{\eta}(V)\) denote the vector space of square matrices \(G(X)\) of bounded holomorphic functions on \(V\), having an asymptotic expansion in the Poincaré sense at infinity, \(G(X) \sim \sum_{k \geq 0} G_k X^{-k}, V \ni X \to \infty\), and let \(H_{\eta}(r_0)\) denote the vector space of square matrices of bounded holomorphic functions on the disk \(D(0, r_0)\).

In this note, we use the definitions of Gevrey composite asymptotic expansion (and the associated notation \(\sim_1\)) and consistent good covering \(S_l, V^j, V^j(\eta)\) introduced in [1,2].

We say that \(L(x, \eta)\) is a slow matrix if \(L(x, \eta) \sim_1 \sum_{k \geq 0} A_k(x)\eta^k\), as \(S \ni \eta \to 0\) and \(x \in D(0, r_0)\), with \((A_k)_{k \in \mathbb{N}} \in H_{\eta}(r_0)^{\mathbb{N}}\), and that \(R(x, \eta)\) is a fast matrix if there exists a sequence \((G_k)_{k \in \mathbb{N}}\) in \(G_{\eta}(V)^{\mathbb{N}}\) such that \(R(x, \eta) \sim_1 \sum_{k \geq 0} G_k \left(\frac{x}{\eta}\right)^{\eta^k}\), as \(S \ni \eta \to 0\) and \(x \in V(\eta)\). In the sequel, we use the variables \(\eta\) and \(\varepsilon\) simultaneously, but they are always connected by \(\eta^p = \varepsilon\), with a suitable integer \(p\).

Let \(\text{diag}(a_1, a_2)\) denote the diagonal matrix \(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\).

3. Preliminaries

In this section we prove that a family \((Y^j)_{j,l}\) of matrices defined on a consistent good covering, whose differences are exponentially small, can be written as a slow-fast product.

Let \(V, S_l, V^j(\eta), j = 1, \ldots, J, l = 1, \ldots, L\), be a consistent good covering and \(A, B > 0\). We define two spaces, \(\mathbb{E}_0(A, B)\) and \(\mathbb{L}_0(A)\).
Proposition

4.1. parity

Definition 3.1. Let $E_n(A, B)$ denote the vector space of families $(Y^j_l(x, \eta))_{j,l}$, $j = 1, \ldots, J$, $l = 1, \ldots, L$, of square matrices of size $n$ of holomorphic functions defined for $\eta \in S_l$ and $x \in V^j_l(\eta)$, satisfying the three following assumptions: there exists a constant $C > 0$ such that

$$
\|Y^j_l(x, \eta)\| \leq C, \quad \text{if } \eta \in S_l \text{ and } x \in V^j_l(\eta),
$$

$$
\|Y^j_{l+1}(x, \eta) - Y^j_l(x, \eta)\| \leq C \exp\left(-\frac{A}{10|\eta|^p}\right), \quad \text{if } \eta \in S_l \cap S_{l+1} \text{ and } x \in V^j_l(\eta) \cap V^j_{l+1}(\eta),
$$

The vector space $E_n(A, B)$ is equipped with the following norm:

$$
\|(Y^j_l)_{j,l}\|_E = \inf \{C \text{ satisfying (2), (3) and (4)}\}.
$$

Definition 3.2. Let $L_n(A)$ denote the vector space of families $(L_l(x, \eta))$, $l = 1, \ldots, L$, of square matrices of size $n$ of holomorphic functions, defined for $\eta \in S_l$ and $x \in D(0, r_0)$, such that there exists a constant $C > 0$ satisfying

$$
\|L_l(x, \eta)\| \leq C, \quad \text{if } \eta \in S_l \text{ and } x \in D(0, r_0),
$$

$$
\|L_{l+1}(x, \eta) - L_l(x, \eta)\| \leq C \exp\left(-\frac{A}{10|\eta|^p}\right), \quad \text{if } \eta \in S_l \cap S_{l+1} \text{ and } x \in D(0, r_0).
$$

The vector space $L_n(A)$ is equipped with the following norm:

$$
\|(L_l)\|_L = \inf \{C \text{ satisfying (5) and (6)}\}.
$$

Theorem 3.3. Let $(Y^j_l)_{j,l} \in E_n(A, B)$. If $\|(Y^j_l)_{j,l}\|_E$ is sufficiently small, then there exists a pair $((L_l), (R^j_l)) \in L_n(A) \times E_n(A, B)$ such that the $L_l$ are slow matrices, the $R^j_l$ are fast matrices and

$$
I + Y^j_l = (I + L_l)(I + R^j_l).
$$

Proof. In [2] Fruchard and Schäfe showed that every element of $E_n(A, B)$ can be written as a sum of a slow matrix and a fast one. We improve their result: there exists a bounded linear operator $T : E_n(A, B) \to L_n(A) \times E_n(A, B)$ such that $T(Y^j_l) = ((L_l), (R^j_l))$ implies that for all $(l, j)$, the $L_l$ are slow matrices, the $R^j_l$ are fast ones and $Y^j_l = L_l + R^j_l$.

The factorization (7) is equivalent to $L_l + R^j_l = Y^j_l - L_lR^j_l$. We can obtain such a factorization as a fixed point of the mapping $\Phi : L_n(A) \times E_n(A, B) \to L_n(A) \times E_n(A, B), \quad \Phi((L_l), (R^j_l)) = T\left((Y^j_l - L_lR^j_l)\right)$, provided that $\|(Y^j_l)\|_E$ is sufficiently small.

4. Proof of the main theorem

In this section, we present the key steps of the proof of Theorem 1.2. For the sake of simplicity, we assume that $\nu$ is an even natural number. The case $\nu$ odd is proved similarly but requires more precautions; in particular, we need to track the parity of functions at each step of the proof.

4.1. Fundamental matrices of solutions to (1)

First we describe the structure of a family of fundamental matrices of solutions to (1) defined on a suitable consistent good covering.

Proposition 4.1. For all $r \in ]0, r_0[$, there exist $\mu > 0$, $\eta_1 > 0$ and $S_l$, $V^j_l(\eta)_l = 0, \ldots, L - 1$, $j = 0, \ldots, p - 1$, a consistent good covering of the set \{(x, \eta) : 0 < |\eta| < \eta_1, \text{ and } |\eta| < |x| < r\}, and

$$
Y^j_l(x, \eta) = \text{diag} \left(1, x^\gamma \right) \cdot H^j_l(x, \eta) \cdot E^j_l(x, \eta)
$$

a family of fundamental matrices of solutions to (1) such that

(i) $p = \frac{\nu}{2} + 1$, 
(ii) the $H^j_l(x, \eta)$ are holomorphic matrices defined for $\eta \in S_l$ and $x \in V^j_l(\eta)$, having a composite asymptotic expansion of Gevrey order $\frac{1}{p}$, as $\eta \to 0$ in $S_l$ and $x \in V^j_l(\eta)$.
(iii) the matrices $E_l^j(x, \eta)$, defined for $\eta \in S_l$ and $x \in V_l^j(\eta)$, are of the form

$$E_l^j(x, \eta) = \exp \left( \text{diag} \left( -\frac{1}{p} x^p + R_l^{(1)}(\eta) \log x, -\frac{1}{p} x^p + R_l^{(2)}(\eta) \log x \right) \right),$$

where the $R_l^{(1)}(\eta)$ and the $R_l^{(2)}(\eta)$ are holomorphic functions defined on $S_l$, having an asymptotic expansion of Gevrey order 1 as $\epsilon \to 0$ in $S_l$.

In particular, there exist two positive constants $A$ and $B$ such that $(H_l^j(x, \eta))_{l,l} \in E_2(A, B)$.

Using the implicit function theorem, we prove the existence of a family $(q_l(x, \eta))_{l}$ of polynomials in $x$ of degree $\frac{p}{2} - 1$ without constant term, such that the $q_l$ have asymptotic expansions of Gevrey order 1 and the coefficient $(1, 2)$ of the matrices $L_l$ introduced later satisfies $\langle L_l \rangle_1(x, \epsilon) = \mathcal{O}(x^2)$. Now applying Theorem 3.3, we separate the asymptotic of the matrices $H_l^j(x, \eta) \cdot \exp \left( \text{diag}(q_l, -q_l) \right)$:

$$H_l^j(x, \eta) \cdot \exp \left( \text{diag}(q_l(x, \epsilon), -q_l(x, \epsilon)) \right) = L_l(x, \epsilon) \cdot R_l^j(x, \eta),$$

where the $L_l$ are slow matrices defined on $D(0, r) \times \Sigma_l$, with $\Sigma_l = \{ \eta^p \mid \eta \in S_l \}$, and the $R_l^j$ are fast matrices defined on the set of $(x, \eta)$ such that $\eta \in S_l$ and $x \in V_l^j(\eta)$. Let $P_l = \text{diag} \left( (1, x^2), \cdots, \text{diag} \left( 1, x^{2l} \right) \right)$. As $(L_l)_{12}(x, \epsilon) = \mathcal{O}(x^2)$, the coefficient $(1, 2)$ of the matrices $P_l$ does not have any poles at $x = 0$. The matrices $P_l$ are also slow matrices; they have an asymptotic expansion of Gevrey order 1: $P_l(x, \epsilon) \sim_1 P(x, \epsilon)$ as $\epsilon \to 0$ in $S_l$ and $x \in D(0, r)$. We can rewrite the fundamental matrices of solutions to system (1):

$$Y_l^j(x, \eta) = P_l(x, \epsilon) \cdot \text{diag} \left( 1, x^2 \right) \cdot R_l^j(x, \eta) \cdot \tilde{E}_l^j(x, \eta),$$

where $\tilde{E}_l^j = \exp \left( \text{diag}(-q_l, q_l) \right) \cdot E_l^j$.

4.2 Uniform simplification

In the sequel, the variable $\eta$ lies in the sector $S_l$, $\epsilon$ in the sector $\Sigma_l$ and $x$ in the quasi-sector $V_l^j(\eta)$ (or in the disk $D(0, r)$ if the variable $x$ is associated with a slow matrix). From now on, we omit the indices $l$ and $j$.

Applying the change of variables $Y = P(x, \epsilon)U$, we reduce the differential system (1) to

$$\epsilon U' = C(x, \epsilon)U, \quad \text{where} \quad C = P^{-1}AP - \epsilon P^{-1}P'.$$

(8)

On the one hand, the expression of $C$ implies that $C$ is a slow matrix. On the other hand, we know a fundamental matrix of solutions to equation (8), $U(x, \eta) = \text{diag} \left( 1, x^2 \right) R(x, \eta) \tilde{E}(x, \eta)$, and we have $C = \epsilon U' U^{-1}$. We use this second expression to give an upper bound on the degree of each coefficient of the matrix $\tilde{C}(x, \epsilon)$.

**Proposition 4.2.** The change of variables $Y = P(x, \epsilon)U$ transforms the differential system (1) into $\epsilon U' = C(x, \epsilon)U$, where $C(x, \epsilon) \sim_1 \tilde{C}(x, \epsilon)$ as $\epsilon \to 0$ in $\Sigma_l$ and $x \in D(0, r)$, and

$$\tilde{C}(x, \epsilon) = \left( \begin{array}{cc} \sum_{k=0}^{\mu + v/2} \tilde{c}_{k}^{11}(\epsilon) x^k & \sum_{k=0}^{\mu} \tilde{c}_{k}^{12}(\epsilon) x^k \\ \sum_{k=0}^{\mu + v/2} \tilde{c}_{k}^{21}(\epsilon) x^k & -\sum_{k=0}^{\mu + v/2} \tilde{c}_{k}^{11}(\epsilon) x^k \end{array} \right),$$

where the $\tilde{c}_{k}^{ij}(\epsilon)$ are formal power series of Gevrey order 1, such that $\tilde{c}_{k}^{12}(\epsilon) = 1 + \mathcal{O}(\epsilon), \tilde{c}_{k}^{21}(\epsilon) = 1 + \mathcal{O}(\epsilon)$ and $\tilde{c}_{k}^{11}(\epsilon) = \mathcal{O}(\epsilon)$ if $k \neq \mu, \mu + v$.

We truncate the Taylor series of the coefficients of the matrix $C$ to define a new matrix $\tilde{C}$:

$$\tilde{C}(x, \epsilon) = \left( \begin{array}{cc} \sum_{k=0}^{\mu + v/2} c_{k}^{11}(\epsilon) x^k & \sum_{k=0}^{\mu} c_{k}^{12}(\epsilon) x^k \\ \sum_{k=0}^{\mu + v/2} c_{k}^{21}(\epsilon) x^k & -\sum_{k=0}^{\mu + v/2} c_{k}^{11}(\epsilon) x^k \end{array} \right).$$
This matrix $\tilde{C}$ still satisfies $\tilde{C}(x, \varepsilon) \sim_1 \tilde{C}(x, \varepsilon)$, as $\varepsilon \to 0$ in $\Sigma_1$ and $x \in D(0, r)$. Now using the exponential estimate of the differences $\tilde{C}_{i+1} - \tilde{C}_i$ and the result of [5], we prove the existence of a matrix $\Delta(x, \varepsilon)$, solution to $\varepsilon \Delta' = A\Delta - \Delta \tilde{C} + R$, with $R = P(C - \tilde{C})$, which remains exponentially small on $D(0, r) \times \Sigma_i$. Then we set $\tilde{P} = P + \Delta$. This step can be summarized as follows.

**Proposition 4.3.** For all $r \in [0, r_0]$, there exists a bounded holomorphic matrix $\tilde{P}(x, \varepsilon)$ on $D(0, r) \times \Sigma_i$, having the same asymptotic expansion as $P(x, \varepsilon)$, $\tilde{P}(x, \varepsilon) \sim_1 \tilde{P}(x, \varepsilon)$, such that the transformation $Y = \tilde{P}(x, \varepsilon)U$ reduces the differential system (1) to $\varepsilon U' = \tilde{C}(x, \varepsilon)U$.

Up to now, we transformed the first differential system into a polynomial differential system, but the condition on the degree of the polynomials that appears in Theorem 1.2 is not yet satisfied. To obtain the enunciated simplification, we need to apply a last change of variables, $U = \Psi(x, \varepsilon)Z$, where $\Psi(x, \varepsilon)$ is a matrix of polynomials in $x$, which allows us to reduce the degree of the diagonal coefficients of the matrix $C$. Applying the transformation $U = \Psi(x, \varepsilon)Z$, we reduce the system $\varepsilon U' = C(x, \varepsilon)U$ to a polynomial differential system, $\varepsilon Z' = B(x, \varepsilon)Z$, where the matrix $B$ satisfies the conditions of Theorem 1.2.

**References**


