C. R. Acad. Sci. Paris, Ser. I 353 (2015) 773–778

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Combinatorics

An identity on pairs of Appell-type polynomials

Une identité sur des paires de polynômes de type Appell

Miloud Mihoubi, Yamina Saidi

RECITS Laboratory, Faculty of Mathematics, USTHB, P.O. Box 32, El Alia 16111, Bab-Ezzouar, Algiers, Algeria

ARTICLE INFO

Article history:
Received 28 December 2014
Accepted after revision 16 June 2015
Available online 22 July 2015

Presented by the Editorial Board

ABSTRACT

In this paper, we define a sequence of polynomials $P_n^{(\alpha)}(x \mid A, H)$ depending only on the choice of two analytic functions A and H in a neighborhood of zero. For a pair of compositional inverses A and B, we will show the identity $P_n^{(\alpha)}(x \mid B, H \circ B) = P_n^{(n+1-\alpha)}(1 - x \mid A, A'H)$, which generalize the Carlitz's identity on Bernoulli polynomials.

RÉSUMÉ

Dans ce papier, on définit une suite de polynômes $P_n^{(\alpha)}(x \mid A, H)$ dépendant seulement du choix de deux fonctions analytiques dans un voisinage de zéro. Pour une paire de fonctions réciproques A et B, on montre l'identité $P_n^{(\alpha)}(x \mid B, H \circ B) = P_n^{(n+1-\alpha)}(1 - x \mid A, A'H)$, qui généralise l'identité de Carlitz sur les polynômes de Bernoulli.

1. Introduction

Let A and H be two analytic functions around zero with $A(0) = 0$, $A'(0) \neq 0$ and let $P_n^{(\alpha)}(A, H)$ be defined by

$$\left(\frac{t}{A(t)}\right)^{\alpha} H(t) = \sum_{n \geq 0} P_n^{(\alpha)}(A, H) \frac{t^n}{n!}. \tag{1}$$

This definition is motivated by the works of Tempesta [8] and [10] on the generalized higher-order Bernoulli polynomials. The higher-order Bernoulli polynomials of the first kind $B_n^{(\alpha)}(x)$ [4,9] correspond to the choice $A(t) = \exp(t) - 1$, $H(t) = \exp(xt)$, the higher-order Bernoulli polynomials of the second kind $b_n^{(\alpha)}(x)$ [7,6] correspond to the choice $A(t) = \ln(1+t)$, $H(t) = (1+t)^x$ and the degenerate Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x; \beta, \lambda)$ correspond to the choice $A(t) = \frac{(1+\lambda t)^{\beta/\lambda} - 1}{\beta}$ and $H(t) = (1 + \beta t)^{(\lambda/\beta - 1)x}$, from which we have in particular $B_n^{(\alpha)}(x) = \lim_{\lambda \to 0} \mathcal{B}_n^{(\alpha)}(x; 1, \lambda)$ and $b_n^{(\alpha)}(x) = \lim_{\beta \to 0} \mathcal{B}_n^{(\alpha)}(x; \beta, 1)$.

In this paper, we show that the Carlitz's identity $B_n^{(n+1-\alpha)}(x) = n! b_n^{(\alpha)}(x - 1)$ [2, Eqs. (2.11), (2.12)] can be generalized to a large class of polynomials $P_n^{(\alpha)}(x \mid A, H)$ introduced below.

E-mail addresses: mmihoubi@usthb.dz (M. Mihoubi), y_saidi34@yahoo.com (Y. Saidi).

http://dx.doi.org/10.1016/j.crma.2015.06.013
1631-073X/

2. The main identity

The main result of this paper is given by the following theorem.

Theorem 2.1. *Let H, A, B be analytic functions around zero with $(A\circ B)(t)=(B\circ A)(t)=t$. Then*

$$P_n^{(\alpha)}(B, H\circ B) = P_n^{(n+1-\alpha)}\left(A, A'H\right),$$

where $\left(A'H\right)(z) := H(z)\, D_z A(z)$ and $D_z = \frac{\mathrm{d}}{\mathrm{d}z}$.

Proof. Since $(A\circ B)(t)=t$, then the equation $t=A(z)$ gives $z=B(t)$. So, the equation $z=t\left(\frac{z}{A(z)}\right)$ admits the unique solution $z=B(t)$, and on using the Lagrange inversion formula, for any function F analytic around $z=0$, we get

$$F(z) = F(0) + \sum_{n\geq 1}\frac{t^n}{n!}D_{z=0}^{n-1}\left(F'(z)\left(\frac{z}{A(z)}\right)^n\right).$$

So, for the choice $F(z)=\left(\frac{z}{A(z)}\right)^{-\alpha}H(z)$ and since $f\,Dg = D(fg) - gDf$, we can write

$$\begin{aligned}
F'(z)\left(\frac{z}{A(z)}\right)^n &= D_z\left(\left(\frac{z}{A(z)}\right)^{n-\alpha}H(z)\right) - n\left(\frac{z}{A(z)}\right)^{n-\alpha-1}\left(\frac{1}{A(z)} - z\frac{A'(z)}{(A(z))^2}\right)H(z)\\
&= D_z\left(\left(\frac{z}{A(z)}\right)^{n-\alpha}H(z)\right) - \frac{n}{z}\left(\left(\frac{z}{A(z)}\right)^{n-\alpha}H(z) - \left(\frac{z}{A(z)}\right)^{n-\alpha+1}A'(z)\,H(z)\right)\\
&= D_z\left(\left(\frac{z}{A(z)}\right)^{n-\alpha}H(z)\right) - n\sum_{j\geq 0}\left(\frac{P_{j+1}^{(n-\alpha)}(A,H) - P_{j+1}^{(n-\alpha+1)}\left(A,A'H\right)}{j+1}\right)\frac{z^j}{j!},
\end{aligned}$$

thus, upon using (1), the number $D_{z=0}^{n-1}\left[F'(z)\left(\frac{z}{A(z)}\right)^n\right]$ can be written as

$$\begin{aligned}
D_{z=0}^{n-1}\left(\left(\frac{z}{A(z)}\right)^n F'(z)\right) &= D_{z=0}^{n}\left(\left(\frac{z}{A(z)}\right)^{n-\alpha}H(z)\right) - nD_{z=0}^{n-1}\sum_{j\geq 0}\left(\frac{P_{j+1}^{(n-\alpha)}(A,H) - P_{j+1}^{(n-\alpha+1)}\left(A,A'H\right)}{j+1}\right)\frac{z^j}{j!}\\
&= P_n^{(n-\alpha)}(A,H) - n\left(\frac{P_n^{(n-\alpha)}(A,H) - P_n^{(n-\alpha+1)}\left(A,A'H\right)}{n}\right)\\
&= P_n^{(n-\alpha+1)}\left(A,A'H\right).
\end{aligned}$$

Then

$$\left(\frac{z}{A(z)}\right)^{-\alpha}H(z) = \sum_{n\geq 0}P_n^{(n-\alpha+1)}\left(A,A'H\right)\frac{t^n}{n!}. \tag{2}$$

On the other hand, since $z=B(t)$ we also get

$$\left(\frac{z}{A(z)}\right)^{-\alpha}H(z) = \left(\frac{t}{B(t)}\right)^{\alpha}H(B(t)) = \sum_{n\geq 0}P_n^{(\alpha)}(B, H\circ B)\frac{t^n}{n!}. \tag{3}$$

So, the desired relation follows from (2) and (3). □

To present some applications of Theorem 2.1, we give the following definition.

Definition 2.1. Let A and H be two analytic functions around zero with $A(0)=0$, $A'(0)\neq 0$ and let α, x be real numbers. A sequence of polynomials $\left(P_n^{(\alpha)}(x\mid A,H)\right)$ is said to be of Appell type if

$$\left(\frac{t}{A(t)}\right)^{\alpha}\left(A'(t)\right)^x H(t) = \sum_{n\geq 0}P_n^{(\alpha)}(x\mid A,H)\frac{t^n}{n!}.$$

When we replace $H(t)$ with $\left(A'(t)\right)^x H(t)$ in Theorem 2.1, we get:

Corollary 2.2. *Let α, x be real numbers and let A, B be analytic functions around zero with $(A\circ B)(t) = (B\circ A)(t) = t$. Then, it holds*

$$P_n^{(\alpha)}(x \mid B, H\circ B) = P_n^{(n+1-\alpha)}(1-x \mid A, H), \quad n \geq 1.$$

In particular, for $H(t) = 1$ in Corollary 2.2, the polynomials $P_n^{(\alpha)}(x \mid A)$ defined by

$$\left(\frac{t}{A(t)}\right)^{\alpha} \left(A'(t)\right)^x = \sum_{n\geq 0} P_n^{(\alpha)}(x \mid A)\frac{t^n}{n!} \tag{4}$$

satisfy

$$P_n^{(\alpha)}(x \mid B) = P_n^{(n+1-\alpha)}(1-x \mid A), \tag{5}$$

and for $H(t) = \left(\frac{t}{B(t)}\right)^{\beta}\left(B'(t)\right)^y$ in Corollary 2.2 with β, y real numbers, the polynomials $P_n^{(\alpha,\beta)}(x, y \mid A, B)$ defined by

$$\left(\frac{t}{A(t)}\right)^{\alpha}\left(A'(t)\right)^x \left(\frac{t}{B(t)}\right)^{\beta}\left(B'(t)\right)^y = \sum_{n\geq 0} P_n^{(\alpha,\beta)}(x, y \mid A, B)\frac{t^n}{n!}$$

satisfy

$$P_n^{(\alpha,\beta)}(x, y \mid A, B) = P_n^{(n+1-\beta,-\alpha)}(1-y, -x \mid A, B) = P_n^{(-\beta,n+1-\alpha)}(-y, 1-x \mid A, B).$$

3. Connection to the partial Bell polynomials

Let $B_{n,k}(x_1, x_2, \ldots) := B_{n,k}\left(x_j\right)$ be the partial Bell polynomials (see for instance [1,3,5]) defined by

$$\sum_{n\geq k} B_{n,k}(x_1, x_2, \ldots)\frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{j\geq 1} x_j \frac{t^j}{j!}\right)^k.$$

Proposition 3.1. *The sequence $\left(P_k^{(\alpha)}(A, H)\right)$ satisfies the following recurrence relations*

$$P_n^{(n+1+\alpha)}\left(A, A'H\right) = \sum_{k=0}^{n} B_{n,k}(b_1, b_2, \ldots) P_k^{(\alpha)}(A, H),$$

$$P_n^{(\alpha)}\left(A, A'H\right) = \sum_{k=0}^{n}\binom{n}{k} a_{n-k+1} P_k^{(\alpha)}(A, H),$$

where the sequences $(a_n; n \geq 1)$ and $(b_n; n \geq 1)$ are defined by

$$\sum_{n\geq 1} a_n \frac{t^n}{n!} = A(t) \quad \text{and} \quad \sum_{n\geq 1} b_n \frac{t^n}{n!} = B(t), \quad a_1 b_1 = 1.$$

Proof. With the above notation, from (1) and (3), we have:

$$\left(\frac{z}{A(z)}\right)^{\alpha} H(z) = \sum_{k\geq 0} P_k^{(\alpha)}(A, H)\frac{z^k}{k!} = \sum_{n\geq 0} P_n^{(-\alpha)}(B, H\circ B)\frac{t^n}{n!} \quad \text{with } z = B(t).$$

Then

$$\sum_{n\geq 0} P_n^{(-\alpha)}(B, H\circ B)\frac{t^n}{n!} = \sum_{k\geq 0} P_k^{(\alpha)}(A, H)\frac{(B(t))^k}{k!} = \sum_{n\geq 0}\frac{t^n}{n!}\sum_{k=0}^{n} B_{n,k}(b_1, b_2, \ldots) P_k^{(\alpha)}(A, H),$$

which gives $P_n^{(-\alpha)}(B, H\circ B) = \sum_{k=0}^{n} B_{n,k}(b_1, b_2, \ldots) P_k^{(\alpha)}(A, H)$. So, the first relation follows on using Theorem 2.1. The second one follows from the definition of $P_n^{(\alpha)}(A, H)$. □

For $H(t) = 1$ in Proposition 3.1 we get the following corollary.

Corollary 3.1. *Let A be an analytic function around zero with $A(0)=0$, $A'(0)\neq 0$ and*

$$\left(\frac{t}{A(t)}\right)^{\alpha}\left(A'(t)\right)^{x}=\sum_{n\geq 0}P_n^{(\alpha)}(x\mid A)\frac{t^n}{n!}.$$

Then, if we denote by B for the compositional inverse of A, the sequence of polynomials $\left(P_n^{(\alpha)}(x\mid A)\right)$ satisfies the following recurrence relations

$$P_n^{(n+1+\alpha)}(x+1\mid A)=\sum_{k=0}^{n}B_{n,k}(b_1,b_2,\ldots)P_k^{(\alpha)}(x\mid A),$$

$$P_n^{(\alpha)}(x+1\mid A)=\sum_{k=0}^{n}\binom{n}{k}a_{n-k-1}P_k^{(\alpha)}(x\mid A).$$

4. Connection to the successive derivatives of a function

We show in this section some connections of Theorem 2.1 to the successive derivatives of a function.

Proposition 4.1. *Let B and H be two analytic functions around zero with $B(0)\neq 0$ and let α, x be real numbers. We have:*

$$D_{t=0}^{n}\left((B(t))^{\alpha}H(xtB(t))\right)=\sum_{k=0}^{n}\binom{n}{k}D_{t=0}^{n-k}\left((B(t))^{k+\alpha}\right)D_{t=0}^{k}(H(t))x^k.$$

Proof. Let $V(t)=tB(t)$ and U be the compositional inverse of V. We can write, using Theorem 2.1:

$$\begin{aligned}
D_{t=0}^{n}\left((B(t))^{\alpha}H(xtB(t))\right)&=D_{t=0}^{n}\left(\left(\frac{t}{V(t)}\right)^{-\alpha}H(xV(t))\right)\\
&=D_{t=0}^{n}\left(\left(\frac{t}{U(t)}\right)^{n+1+\alpha}U'(t)H(xt)\right)\\
&=\sum_{k=0}^{n}\binom{n}{k}D_{t=0}^{n-k}\left(\left(\frac{t}{U(t)}\right)^{n+1+\alpha}U'(t)\right)D_{t=0}^{k}(H(xt))\\
&=\sum_{k=0}^{n}\binom{n}{k}D_{t=0}^{n-k}\left(\left(\frac{t}{U(t)}\right)^{(n-k)+1+(\alpha+k)}U'(t)\right)D_{t=0}^{k}(H(t))x^k\\
&=\sum_{k=0}^{n}\binom{n}{k}D_{t=0}^{n-k}\left(\left(\frac{t}{V(t)}\right)^{-\alpha-k}\right)D_{t=0}^{k}(H(t))x^k\\
&=\sum_{k=0}^{n}\binom{n}{k}D_{t=0}^{n-k}\left(\left(\frac{t}{V(t)}\right)^{-\alpha-k}\right)D_{t=0}^{k}(H(t))x^k\\
&=\sum_{k=0}^{n}\binom{n}{k}D_{t=0}^{n-k}\left((B(t))^{k+\alpha}\right)D_{t=0}^{k}(H(t))x^k,
\end{aligned}$$

which completes the proof. □

For different choices for H and B in Proposition 4.1, we get:

Corollary 4.1. *Let B and H be two analytic functions around zero with $B(0)\neq 0$. We have:*

$$D_{t=0}^{n}(\exp(\alpha t)H(xt\exp(t)))=\sum_{k=0}^{n}\binom{n}{k}(k+\alpha)^{n-k}D_{t=0}^{k}(H(t))x^k,$$

$$D_{t=0}^{n}\left((1-t)^{-\alpha}H\left(\frac{xt}{1-t}\right)\right)=\sum_{k=0}^{n}\frac{n!}{k!}\binom{\alpha+n-1}{n-k}D_{t=0}^{k}(H(t))x^k,$$

$$D_{t=0}^{n}\left((1+t)^{\alpha}H(xt(1+t))\right)=\sum_{k=0}^{n}\frac{n!}{k!}\binom{\alpha+k}{n-k}D_{t=0}^{k}(H(t))x^{k},$$

$$D_{t=0}^{n}\left((B(t))^{\alpha}\exp(xtB(t))\right)=\sum_{k=0}^{n}\binom{n}{k}D_{t=0}^{n-k}\left((B(t))^{k+\alpha}\right)x^{k},$$

$$D_{t=0}^{n}\left(\frac{(B(t))^{\alpha}}{1-xtB(t)}\right)=\sum_{k=0}^{n}\frac{n!}{(n-k)!}D_{t=0}^{n-k}\left((B(t))^{k+\alpha}\right)x^{k},$$

where $\binom{x}{k}:=\frac{(x)_k}{k!}$, *and,* $(x)_k:=x(x-1)\cdots(x-k+1)$ *for* $k\geq 1$ *and* $(x)_0:=1$.

5. Some applications

For the following examples, let $s(n,k)$ and $S(n,k)$ be, respectively, the Stirling numbers of the first and second kinds.

Example 1. For $A(t)=t+\frac{t^2}{2}$ we get $B(t)=\sqrt{1+2t}-1$. Then, from (4) we get:

$$\left(1+\frac{t}{2}\right)^{-\alpha}(1+t)^{x}=\sum_{n\geq 0}P_n^{(\alpha)}(x\mid A)\frac{t^n}{n!},$$

for which we can verify that $P_n^{(\alpha)}(x\mid A)=n!\sum_{k=0}^{n}\binom{-\alpha}{k}\binom{x}{n-k}2^{-k}$ and show upon using (5) that we have

$$\left(\frac{t}{\sqrt{1+2t}-1}\right)^{\alpha}(1+2t)^{-\frac{x}{2}}=\sum_{n\geq 0}P_n^{(\alpha)}(x\mid B)\frac{t^n}{n!},$$

with $P_n^{(\alpha)}(x\mid B)=P_n^{(n+1-\alpha)}(1-x\mid A)=n!\sum_{k=0}^{n}\binom{n+1-\alpha}{k}\binom{1-x}{n-k}2^{-k}$.

Example 2. For $A(t)=\frac{t}{1+t^2}$ we get $B(t)=\frac{1-\sqrt{1-4t^2}}{2t}$. Then, from (4) we get

$$\left(1+t^2\right)^{\alpha}\left(1-t^2\right)^{x}=\left(\frac{t}{A(t)}\right)^{\alpha+2x}\left(A'(t)\right)^{x}=\sum_{n\geq 0}P_n^{(\alpha+2x)}(x\mid A)\frac{t^n}{n!},$$

for which we can verify that $P_{2n}^{(\alpha+2x)}(x\mid A)=\frac{(2n)!}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\alpha)_k(x)_{n-k}$ and $P_{2n+1}^{(\alpha+2x)}(x\mid A)=0$, and show upon using (5) that we have

$$\left(\frac{1-\sqrt{1-4t^2}}{2t^2}\right)^{\alpha}\left(1-4t^2\right)^{x}=\left(\frac{t}{B(t)}\right)^{\alpha-2x}\left(B'(t)\right)^{-2x}=\sum_{n\geq 0}P_n^{(\alpha-2x)}(-2x\mid B)\frac{t^n}{n!},$$

with $P_n^{(\alpha-2x)}(-2x\mid B)=P_n^{(n+1-\alpha-2x)}(1+2x\mid A)$.

Example 3. For the pair of compositional inverse functions $A(t)=\exp(t)-1$, $B(t)=\ln(1+t)$, we get from (4) and (5) $P_n^{(\alpha)}(x\mid A)=B_n^{(\alpha)}(x)$, $P_n^{(\alpha)}(x\mid B)=n!b_n^{(\alpha)}(-x)$ and

$$n!b_n^{(\alpha)}(x)=B_n^{(n+1-\alpha)}(x+1),\qquad B_n^{(\alpha)}(x)=n!b_n^{(n+1-\alpha)}(x-1).$$

For $A(t)=\exp(t)-1$ and $H(t)=\left(A'(t)\right)^{x-1}$ in Proposition 3.1, we get

$$B_n^{(n+1+\alpha)}(x)=\sum_{k=0}^{n}(-1)^{n-k}|s(n,k)|B_k^{(\alpha)}(x-1),\qquad B_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(x-1).$$

For $A(t)=\ln(1+t)$ and $H(t)=(1+t)^{x+1}$ in Proposition 3.1, we get

$$n!b_n^{(n+1+\alpha)}(x)=\sum_{k=0}^{n}k!S(n,k)b_k^{(\alpha)}(x-1),\qquad b_n^{(\alpha)}(x)=\sum_{k=0}^{n}(-1)^{n-k}b_k^{(\alpha)}(x-1).$$

Example 4. Let the pair of compositional inverse functions

$$A(t)=\frac{(1+\lambda t)^{\beta/\lambda}-1}{\beta}=\sum_{n\geq 1}(\beta\mid\lambda)_n\frac{t^n}{n!},\qquad B(t)=\frac{(1+\beta t)^{\lambda/\beta}-1}{\lambda}.$$

Then, from (4) and (5), we get $P_n^{(\alpha)}(x\mid A)=\mathcal{B}_n^{(\alpha)}(x;\beta,\lambda)$ and $\mathcal{B}_n^{(\alpha)}(x;\beta,\lambda)=\mathcal{B}_n^{(n+1-\alpha)}(1-x;\lambda,\beta)$. From Proposition 3.1 we get

$$\mathcal{B}_n^{(n+1+\alpha)}(x+1;\lambda,\beta)=\sum_{k=0}^{n}B_{n,k}\left((\beta\mid\lambda)_j\right)\mathcal{B}_k^{(\alpha)}(x;\lambda,\beta),$$

$$\mathcal{B}_n^{(\alpha)}(x+1;\lambda,\beta)=\sum_{k=0}^{n}\binom{n}{k}(\beta\mid\lambda)_{n-k+1}\mathcal{B}_k^{(\alpha)}(x;\lambda,\beta).$$

References

[1] E.T. Bell, Exponential polynomials, Ann. Math. 35 (1934) 258–277.
[2] L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, Util. Math. 15 (1979) 51–88.
[3] L. Comtet, Advanced Combinatorics, D. Reidel Publishing Company, Dordrecht, the Netherlands/Boston, USA, 1974.
[4] D.C. Kurtz, A note on concavity properties of triangular arrays of numbers, J. Comb. Theory, Ser. A 13 (1972) 135–139.
[5] M. Mihoubi, Bell polynomials and binomial type sequences, Discrete Math. 308 (2008) 2450–2459.
[6] T.R. Prabhakar, Sharda Gupta, Bernoulli polynomials of the second kind and general order, Indian J. Pure Appl. Math. 11 (10) (1980) 1361–1368.
[7] S. Roman, The Umbral Calculus, Dover Publ. Inc., New York, 2005.
[8] P. Tempesta, On Appell sequences of polynomials of Bernoulli and Euler type, J. Math. Anal. Appl. 341 (2008) 1295–1310.
[9] R. Tremblay, S. Gaboury, B.-J. Fugère, A new class of generalized Apostol–Bernoulli polynomials and some analogues of the Srivastava–Pintér addition theorem, Appl. Math. Lett. 24 (2011) 1888–1893.
[10] W. Wang, Generalized higher order Bernoulli number pairs and generalized Stirling number pairs, J. Math. Anal. Appl. 364 (2010) 255–274.