



Topology/Computer science

## Digital homotopy fixed point theory

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## ABSTRACT

In this paper, we construct a framework which is called the digital homotopy fixed point theory. We get new results associating digital homotopy and fixed point theory. We also give an application on this theory.

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## R É S U M É

Nous démontrons de nouveaux résultats sur les images digitales dont les homotopies digitales entre deux transformations continues de l'image possèdent un chemin de points fixes. Ceci conduit à une théorie du point fixe des homotopies digitales, dont nous donnons une application sur une image digitale.

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## 1. Introduction

Digital topology plays a key role in image processing and computer graphics. In this field, the general aim is to obtain significant results on digital images in  $\mathbb{Z}^n$  by using methods of geometric and algebraic topology. Fixed point theory interacts with several areas of mathematics such as mathematical analysis, general topology, and functional analysis. There are various applications of fixed point theory in mathematics, topology, game theory, computer science, engineering, and image processing. Fixed point theorems are used to solve some problems in mathematics and engineering.

In recent years, there have been many developments in digital topology. Boxer [1–6] studied digital images. Some results and characteristic properties on the digital homology groups of 2D digital images are given in [7] and [14]. Ege and Karaca [8] construct Lefschetz fixed point theory for digital images and study the fixed point properties of digital images. Ege and Karaca [9] give relative and reduced Lefschetz fixed point theorems for digital images. They also calculate degree of the antipodal map for sphere-like digital images using fixed point properties. Ege and Karaca [10] prove Banach's fixed point theorem for digital images and give an application to image processing.

This work is organized as follows. In the first part, we give the required background about the digital topology and digital homotopy. Then, we state and prove some results on digital retractions and digital fixed point theory. We give an application of digital fixed point theory to a digital image.

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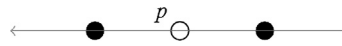


Fig. 1. 2-Adjacency.

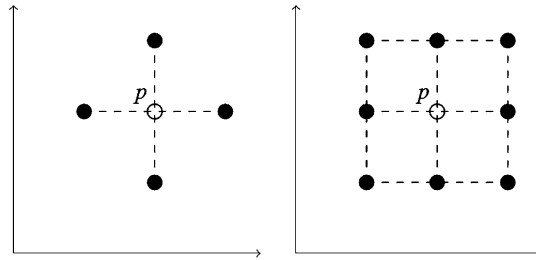


Fig. 2. 4-Adjacency and 8-adjacency.

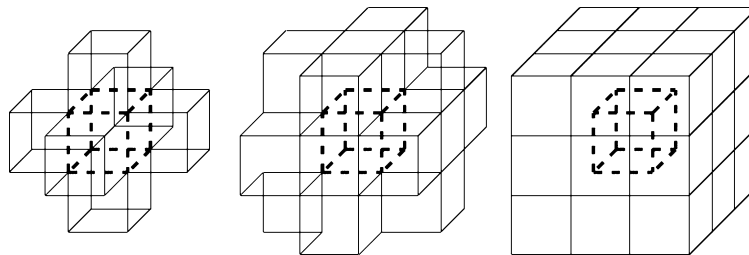


Fig. 3. 6-Adjacency, 18-adjacency, and 26-adjacency.

## 2. Preliminaries

A digital image is a pair  $(X, \kappa)$ , where  $X \subseteq \mathbb{Z}^n$  for some positive integer  $n$  and  $\kappa$  is an adjacency relation for the members of  $X$ . There are various adjacency relations [12,13].

**Definition 2.1.** (See [4].) For a positive integer  $l$  with  $1 \leq l \leq n$  and two distinct points  $p = (p_1, p_2, \dots, p_n)$ ,  $q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$ ,  $p$  and  $q$  are  $c_l$ -adjacent if

- (1) there are at most  $l$  indices  $i$  such that  $|p_i - q_i| = 1$ , and
- (2) for all other indices  $j$  such that  $|p_j - q_j| \neq 1$ ,  $p_j = q_j$ .

The notation  $c_l$  represents the number of points  $q \in \mathbb{Z}^n$  that are adjacent to a given point  $p \in \mathbb{Z}^n$ . Thus, in  $\mathbb{Z}$ , we have  $c_1 = 2$ -adjacency (see Fig. 1); in  $\mathbb{Z}^2$ , we have  $c_1 = 4$ -adjacency and  $c_2 = 8$ -adjacency (see Fig. 2); in  $\mathbb{Z}^3$ , we have  $c_1 = 6$ -adjacency,  $c_2 = 18$ -adjacency, and  $c_3 = 26$ -adjacency [4] (see Fig. 3).

Given a natural number  $l$  in conditions (1) and (2) with  $1 \leq l \leq n$ ,  $l$  determines each one of the  $\kappa$ -adjacency relations of  $\mathbb{Z}^n$  in terms of (1) and (2) [11] as follows:

$$\kappa \in \left\{ 2n \ (n \geq 1), \ 3^n - 1 \ (n \geq 2), \ 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 \ (2 \leq r \leq n - 1, n \geq 3) \right\}$$

where  $C_t^n = \frac{n!}{(n-t)!t!}$ .

**Definition 2.2.** (See [1].) The set  $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$  is called a digital interval where  $a, b \in \mathbb{Z}$  and  $a < b$ .

**Definition 2.3.** (See [12].) Given two points  $x_i, y_i \in (X_i, \kappa_i)$ ,  $i \in \{0, 1\}$ ,  $(x_0, x_1)$  and  $(y_0, y_1)$  are adjacent in  $X_0 \times X_1$  if and only if one of the following is satisfied:

- (i)  $x_0 = y_0$  and  $x_1$  and  $y_1$  are  $\kappa_1$ -adjacent; or
- (ii)  $x_0$  and  $y_0$  are  $\kappa_0$ -adjacent and  $x_1 = y_1$ ; or
- (iii)  $x_0$  and  $y_0$  are  $\kappa_0$ -adjacent and  $x_1$  and  $y_1$  are  $\kappa_1$ -adjacent.

The adjacency of the Cartesian product of digital images  $(X_0, \kappa_0)$  and  $(X_1, \kappa_1)$  is denoted by  $\kappa_*$ .

A  $\kappa$ -neighbor of  $p \in \mathbb{Z}^n$  [2] is a point of  $\mathbb{Z}^n$  that is  $\kappa$ -adjacent to  $p$ . A digital image  $X \subset \mathbb{Z}^n$  is  $\kappa$ -connected [13] if and only if for every pair of different points  $x, y \in X$ , there is a set  $\{x_0, x_1, \dots, x_r\}$  of points of a digital image  $X$  such that  $x = x_0$ ,  $y = x_r$  and  $x_i$  and  $x_{i+1}$  are  $\kappa$ -neighbors where  $i = 0, 1, \dots, r - 1$ .

**Definition 2.4.** (See [2].) Let  $X \subset \mathbb{Z}^{n_0}$  and  $Y \subset \mathbb{Z}^{n_1}$  be digital images with  $\kappa_0$ -adjacency and  $\kappa_1$ -adjacency, respectively. A function  $f : X \rightarrow Y$  is said to be  $(\kappa_0, \kappa_1)$ -continuous if, for every  $\kappa_0$ -connected subset  $U$  of  $X$ ,  $f(U)$  is a  $\kappa_1$ -connected subset of  $Y$ . We say that such a function is digitally continuous.

**Proposition 2.1.** (See [2].) Let  $X \subset \mathbb{Z}^{n_0}$  and  $Y \subset \mathbb{Z}^{n_1}$  be digital images with  $\kappa_0$ -adjacency and  $\kappa_1$ -adjacency, respectively. The function  $f : X \rightarrow Y$  is  $(\kappa_0, \kappa_1)$ -continuous if and only if for every  $\kappa_0$ -adjacent points  $\{x_0, x_1\}$  of  $X$ , either  $f(x_0) = f(x_1)$  or  $f(x_0)$  and  $f(x_1)$  are  $\kappa_1$ -adjacent in  $Y$ .

**Definition 2.5.** (See [5].) Let  $X \subset \mathbb{Z}^{n_0}$  and  $Y \subset \mathbb{Z}^{n_1}$  be digital images with  $\kappa_0$ -adjacency and  $\kappa_1$ -adjacency, respectively. A function  $f : X \rightarrow Y$  is a  $(\kappa_0, \kappa_1)$ -isomorphism, if  $f$  is  $(\kappa_0, \kappa_1)$ -continuous and bijective and further  $f^{-1}$  is  $(\kappa_1, \kappa_0)$ -continuous. It is denoted by  $X \cong_{(\kappa_0, \kappa_1)} Y$ .

A  $(2, \kappa)$ -continuous function  $f : [0, m]_{\mathbb{Z}} \rightarrow X$  such that  $f(0) = x$  and  $f(m) = y$  is called a digital  $\kappa$ -path [2] from  $x$  to  $y$  in a digital image  $X$ . In a digital image  $(X, \kappa)$ , for every two points, if there is a  $\kappa$ -path, then  $X$  is called  $\kappa$ -path connected.

**Definition 2.6.** (See [2].) Let  $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$  and  $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$  be digital images. Two  $(\kappa_0, \kappa_1)$ -continuous functions  $f, g : X \rightarrow Y$  are digitally  $(\kappa_0, \kappa_1)$ -homotopic in  $Y$  if there is a positive integer  $m$  and a function  $H : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$  such that

- for all  $x \in X$ ,  $H(x, 0) = f(x)$  and  $H(x, m) = g(x)$ ;
- for all  $x \in X$ , the induced function  $H_x : [0, m]_{\mathbb{Z}} \rightarrow Y$  defined by  $H_x(t) = H(x, t)$  for all  $t \in [0, m]_{\mathbb{Z}}$ , is  $(2, \kappa_1)$ -continuous; and
- for all  $t \in [0, m]_{\mathbb{Z}}$ , the induced function  $H_t : X \rightarrow Y$  defined by  $H_t(x) = H(x, t)$  for all  $x \in X$ , is  $(\kappa_0, \kappa_1)$ -continuous.

The function  $H$  is called a digital  $(\kappa_0, \kappa_1)$ -homotopy between  $f$  and  $g$ . If these functions are digitally  $(\kappa_0, \kappa_1)$ -homotopic, it is denoted  $f \simeq_{(\kappa_0, \kappa_1)} g$ .

**Definition 2.7.** (See [2].) • A digital image  $(X, \kappa)$  is said to be  $\kappa$ -contractible if its identity map is  $(\kappa, \kappa)$ -homotopic to a constant function  $\bar{c}$  for some  $c \in X$  where the constant function  $\bar{c} : X \rightarrow X$  is defined by  $\bar{c}(x) = c$  for all  $x \in X$ .

• Let  $\emptyset \neq A \subset X$ . Assume  $\kappa$ -adjacency for  $A$  and  $X$ . We say that  $A$  is a  $\kappa$ -retract of  $X$  if and only if there is a  $\kappa$ -continuous function  $r : X \rightarrow A$  such that  $r(a) = a$  for all  $a \in A$ . The function  $r$  is called a  $\kappa$ -retraction of  $X$  onto  $A$ .

Let  $i : A \rightarrow X$  be the inclusion function.  $A$  is called a  $\kappa$ -deformation retract of  $X$  [3] if there exists a  $\kappa$ -homotopy  $H : X \times [0, m]_{\mathbb{Z}} \rightarrow X$  between the identity map  $id_X$  and  $i \circ r$ , for some  $\kappa$ -retraction  $r$  of  $X$  onto  $A$ .

A fixed point of a function  $f : X \rightarrow X$  is a point  $x \in X$  such that  $f(x) = x$ .

### 3. Digital homotopy fixed point theory

We firstly recall the fixed point property of a digital image. Let  $(X, \kappa)$  be a digital image and  $f : (X, \kappa) \rightarrow (X, \kappa)$  be any  $(\kappa, \kappa)$ -continuous function. We say the digital image  $(X, \kappa)$  has the fixed point property [8] if for every  $(\kappa, \kappa)$ -continuous map  $f : X \rightarrow X$  there exists  $x \in X$  such that  $f(x) = x$ . The notion of homotopy fixed point property is defined by Szymik [15]. We give new definitions and theorems using his idea.

**Definition 3.1.** A digital image  $(X, \kappa)$  has the **digital fixed point property** with respect to a digital interval  $[0, m]_{\mathbb{Z}}$  if for all  $(\kappa_*, \kappa)$ -continuous maps  $f : [0, m]_{\mathbb{Z}} \times X \rightarrow X$  where  $\kappa_*$  is the adjacency for  $[0, m]_{\mathbb{Z}} \times X$ , there exists at least one  $(2, \kappa)$ -continuous map  $p : [0, m]_{\mathbb{Z}} \rightarrow X$  of fixed points.

**Definition 3.2.** A digital image  $(X, \kappa)$  has the **digital homotopy fixed point property** if it has the digital fixed point property with respect to  $[0, m]_{\mathbb{Z}}$  for all integers  $m \geq 0$ .

**Definition 3.2** can be stated as follows: a digital image  $(X, \kappa)$  has the digital homotopy fixed point property if for all digital homotopies  $f : [0, m]_{\mathbb{Z}} \times X \rightarrow X$  in  $X$ , there is a  $(2, \kappa)$ -continuous  $\kappa$ -path  $p : [0, m]_{\mathbb{Z}} \rightarrow X$  such that  $p(t)$  is a fixed point of  $f_t$  for all  $t$  in  $[0, m]_{\mathbb{Z}}$ .

Now we prove two propositions about digital retraction and the fixed point property of a digital image  $(X, \kappa)$  with respect to a digital interval  $[0, m]_{\mathbb{Z}}$ .

**Proposition 3.1.** *If a digital image  $(X, \kappa)$  has the digital fixed point property with respect to a digital interval  $[0, m]_{\mathbb{Z}}$ , then  $(X, \kappa)$  has the digital fixed point property with respect to  $[0, n]_{\mathbb{Z}}$  where  $n < m$ .*

**Proof.** We take two  $(2, 2)$ -continuous maps  $i : [0, n]_{\mathbb{Z}} \rightarrow [0, m]_{\mathbb{Z}}$  and  $r : [0, m]_{\mathbb{Z}} \rightarrow [0, n]_{\mathbb{Z}}$  such that  $r \circ i = id_{[0, n]_{\mathbb{Z}}}$ . From the hypothesis, we have a  $(2, \kappa)$ -continuous map  $q : [0, m]_{\mathbb{Z}} \rightarrow X$  of fixed points. If there exists a digital continuous map  $f : [0, n]_{\mathbb{Z}} \times X \rightarrow X$ , we get the digital continuous map  $g = f \circ (r \times id_X)$ , i.e.

$$[0, m]_{\mathbb{Z}} \times X \xrightarrow{r \times id_X} [0, n]_{\mathbb{Z}} \times X \xrightarrow{f} X.$$

It is clear that  $p = q \circ i$  is a  $(2, \kappa)$ -continuous map of fixed points for  $f$ .  $\square$

**Corollary 3.3.** *The digital homotopy fixed point property requires the digital fixed point property.*

**Example 1.** Let  $X = [a, a + 1]_{\mathbb{Z}}$  be a digital interval where  $a \in \mathbb{Z}$ . It is easy to see that  $(X, 2)$  has both the digital fixed point property and digital homotopy fixed point property.

**Proposition 3.2.** *If a digital image  $(X, \kappa)$  is a digital  $\kappa$ -retract of a digital image  $(Y, \kappa)$  that has the digital fixed point property with respect to a digital interval  $[0, m]_{\mathbb{Z}}$ , then  $(X, \kappa)$  also has the digital fixed point property with respect to  $[0, m]_{\mathbb{Z}}$ .*

**Proof.** Let  $(X, \kappa)$  be a digital  $\kappa$ -retract of  $(Y, \kappa)$ . Then there exists a  $\kappa$ -continuous map  $r : Y \rightarrow X$  such that  $r \circ i = id_X$  where  $i : X \rightarrow Y$  is an inclusion map. Considering the digital continuous map  $f : [0, m]_{\mathbb{Z}} \times X \rightarrow X$ , we have a digital continuous map  $g = i \circ f \circ (id_{[0, m]_{\mathbb{Z}}} \times r)$ , i.e.

$$[0, m]_{\mathbb{Z}} \times Y \xrightarrow{id_{[0, m]_{\mathbb{Z}}} \times r} [0, m]_{\mathbb{Z}} \times X \xrightarrow{f} X \xrightarrow{i} Y.$$

By the hypothesis,  $g$  has a  $(2, \kappa)$ -continuous map  $\alpha : [0, m]_{\mathbb{Z}} \rightarrow Y$  of fixed points, i.e.

$$i \circ f(u, r \circ \alpha(u)) = \alpha(u)$$

for all  $u \in [0, m]_{\mathbb{Z}}$ . If we apply  $r$  to the last equality, we have

$$r \circ i \circ f(u, r \circ \alpha(u)) = r \circ \alpha(u)$$

and  $r \circ \alpha = p$  is a  $(2, \kappa)$ -continuous map of fixed points for  $f$ . As a result,  $(X, \kappa)$  has the digital fixed point property with respect to  $[0, m]_{\mathbb{Z}}$ .  $\square$

**Theorem 3.4.** *Let  $(X, \kappa)$  and  $(Y, \kappa')$  be digital images such that  $X \cong_{(\kappa, \kappa')} Y$ . If  $(X, \kappa)$  has the digital homotopy fixed point property, then  $(Y, \kappa')$  has the digital homotopy fixed point property.*

**Proof.** There exists a bijective function  $h : (X, \kappa) \rightarrow (Y, \kappa')$  such that  $h$  is  $(\kappa, \kappa')$ -continuous and that its inverse  $h^{-1}$  is  $(\kappa', \kappa)$ -continuous because  $X \cong_{(\kappa, \kappa')} Y$ . Also,  $X$  has the digital homotopy fixed point property; i.e., for all digital homotopies  $f : [0, m]_{\mathbb{Z}} \times X \rightarrow X$  in  $X$  there is a  $(2, \kappa)$ -continuous  $\kappa$ -path  $p : [0, m]_{\mathbb{Z}} \rightarrow X$  such that  $p(t)$  is a fixed point of  $f_t$  for all  $t$  in  $[0, m]_{\mathbb{Z}}$ . Consider the following diagram:

$$[0, n]_{\mathbb{Z}} \xrightarrow{i} [0, n]_{\mathbb{Z}} \times X \xrightarrow{\alpha \times id_X} [0, m]_{\mathbb{Z}} \times X \xrightarrow{f} X \xrightarrow{h} Y.$$

If we take the composition of all digital continuous maps, for all digital homotopies  $g : [0, n]_{\mathbb{Z}} \times Y \rightarrow Y$  in  $Y$ , we have a  $(2, \kappa)$ -continuous  $\kappa'$ -path  $q : [0, n]_{\mathbb{Z}} \rightarrow Y$  such that  $q(t)$  is a fixed point of  $g_t$  for all  $t$  in  $[0, n]_{\mathbb{Z}}$  where  $q = h \circ f \circ (\alpha \times id_X) \circ i$ . Thus we get the required result.  $\square$

**Corollary 3.5.** *The digital homotopy fixed point property is a topological invariant for digital images.*

#### 4. An application

In this section, we deal with an application on the digital homotopy fixed point theory. Using Proposition 3.2, we give an application. It is clear that the digital image [6]

$$MSC'_8 = \{c_0 = (1, 0), c_1 = (0, 1), c_2 = (-1, 0), c_3 = (0, -1)\} \subset \mathbb{Z}^2$$

with 8-adjacency is a digital 8-deformation retract of the digital image  $[-3, 3]_{\mathbb{Z}} \setminus \{(0, 0)\}$  (see Fig. 4). Let  $f : MSC'_8 \rightarrow MSC'_8$  be defined by  $f(c_i) = c_{(i+1) \bmod 4}$  where  $c_i \in X$ .  $f$  is an  $(8, 8)$ -continuous function, but  $f$  has no fixed point. As a result,  $MSC'_8$  has no digital fixed point property with respect to any digital interval  $[0, m]_{\mathbb{Z}}$ . By Proposition 3.2, we conclude that  $[-3, 3]_{\mathbb{Z}} \setminus \{(0, 0)\}$  has no digital fixed point property with respect to any digital interval  $[0, m]_{\mathbb{Z}}$ .

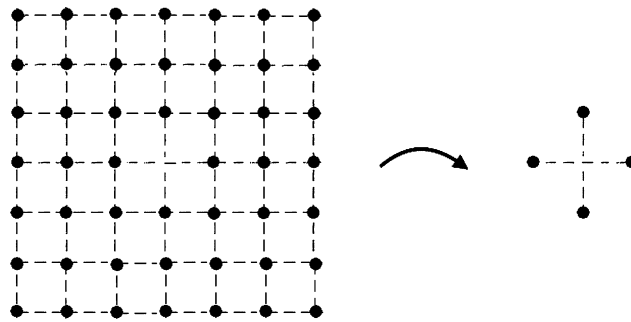


Fig. 4. 8-Deformation retract of  $[-3, 3]_{\mathbb{Z}}^2 \setminus \{(0, 0)\}$ .

## 5. Conclusion

The main purpose of paper is to deal with the digital homotopy fixed point theory. We hope that the fixed point theory with some applications will be useful in the digital topology and image processing. We finally show that digital homotopy fixed point theory could be used to solve some image processing problems.

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## References

- [1] L. Boxer, Digitally continuous functions, *Pattern Recognit. Lett.* 15 (1994) 833–839.
- [2] L. Boxer, A classical construction for the digital fundamental group, *J. Math. Imaging Vis.* 10 (1999) 51–62.
- [3] L. Boxer, Properties of digital homotopy, *J. Math. Imaging Vis.* 22 (2005) 19–26.
- [4] L. Boxer, Homotopy properties of sphere-like digital images, *J. Math. Imaging Vis.* 24 (2006) 167–175.
- [5] L. Boxer, Digital products, wedges and covering spaces, *J. Math. Imaging Vis.* 25 (2006) 159–171.
- [6] L. Boxer, Fundamental groups of unbounded digital images, *J. Math. Imaging Vis.* 27 (2007) 121–127.
- [7] O. Ege, I. Karaca, Fundamental properties of simplicial homology groups for digital images, *Amer. J. Comput. Technol. Appl.* 1 (2) (2013) 25–42.
- [8] O. Ege, I. Karaca, Lefschetz fixed point theorem for digital images, *Fixed Point Theory Appl.* 2013 (253) (2013).
- [9] O. Ege, I. Karaca, Applications of the Lefschetz number to digital images, *Bull. Belg. Math. Soc. Simon Stevin* 21 (5) (2014) 823–839.
- [10] O. Ege, I. Karaca, Banach fixed point theorem for digital images, *J. Nonlinear Sci. Appl.* 8 (3) (2015) 237–245.
- [11] S.E. Han, An extended digital  $(k_0, k_1)$ -continuity, *J. Appl. Math. Comput.* 16 (1–2) (2004) 445–452.
- [12] S.E. Han, Non-product property of the digital fundamental group, *Inf. Sci.* 171 (2005) 73–91.
- [13] G.T. Herman, Oriented surfaces in digital spaces, *CVGIP, Graph. Models Image Process.* 55 (1993) 381–396.
- [14] I. Karaca, O. Ege, Some results on simplicial homology groups of 2D digital images, *Int. J. Inf. Comput. Sci.* 1 (8) (2012) 198–203.
- [15] M. Szymik, Homotopies and the universal fixed point property, *Order* (2014), <http://dx.doi.org/10.1007/s11083-014-9332-x>.