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Differential geometry

Moser-type results in Riemannian product spaces

Résultats à la Moser dans les espaces produit de Riemann

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A B S T R A C T

In this short paper, as applications of the well-known generalized maximum principle of Omori–Yau, we obtain new extensions of a classical theorem due to Moser [8]. More precisely, under suitable constraints on the norm of the gradient of the smooth function u that defines an entire CMC graph $\Sigma(u)$ constructed over a fiber M^n of a Riemannian product space of the type $\mathbb{R} \times M^n$, we show that u must actually be constant.

R É S U M É

Dans cette courte Note, nous obtenons de nouvelles extensions d'un théorème classique de Moser [8] comme application du principe bien connu du maximum généralisé de Omori–Yau. Plus précisément, soit u une fonction lisse définissant un graphe $\Sigma(u)$ entier, CMC, construit sur une fibre M^n d'un espace produit de Riemann du type $\mathbb{R} \times M^n$. Nous montrons alors que, sous des contraintes convenables sur la norme du gradient de u, cette fonction doit en fait être constante.

1. Introduction

The study of the rigidity of entire minimal or, more generally, constant mean curvature (CMC) graphs in a Riemannian space is a classical and fruitful theme into the theory of geometric analysis and it was started with Bernstein's theorem [2] (amended by Hopf in [7]), which asserts that the only entire minimal graphs in $\mathbb{R}^3$ are the planes. Later on, Simons [14] proved a result that, together with some theorems of de Giorgi [5] and Fleming [6], yield a proof of the extension of the Bernstein's theorem to $\mathbb{R}^n$, for $n \leq 7$. However, Bombieri, de Giorgi and Giusti [3] astonishingly showed that Bernstein's theorem does not hold for $n \geq 8$.

Consequently, it turns an interesting research topic in geometric analysis has been the possible extension of Bernstein's result to either higher dimension or another ambient space. A very notable contribution in this direction was made by Moser [8], who showed that the hyperplanes are the only entire minimal graphs of $\mathbb{R}^n$ whose gradient of the corresponding function has bounded norm. In the context of Riemannian product spaces, Rosenberg [11] showed that, when M^2 is a complete surface with nonnegative Gaussian curvature, an entire minimal graph in $\mathbb{R} \times M^2$ is totally geodesic. Hence, in

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this case, the graph is a horizontal slice or M^2 is a flat $\mathbb{R}^2$ and the graph is a tilted plane. In [1], Alías, Dajczer and Ripoll generalized this result to constant mean curvature entire graphs immersed in a Riemannian ambient endowed with a Killing vector field. More recently, Rosenberg, Schulze and Spruck [12] showed that an entire minimal graph with nonnegative height function in a product space $\mathbb{R} \times M^n$, whose fiber M^n is complete with nonnegative Ricci curvature and sectional curvature bounded from below, must be a slice.

Motivated by these works, our aim in this note is to present extensions of Moser's theorem concerning entire CMC graphs constructed over the fiber M^n of a Riemannian product space of the type $\mathbb{R} \times M^n$. For this, we recall that a graph over a domain Ω of a Riemannian manifold $\left(M^n, \langle\,,\,\rangle_M\right)$ is determined by a smooth function $u \in C^\infty(\Omega)$ and it is given by

$$\Sigma^n(u) = \{(u(p), p) : p \in \Omega\} \subset \mathbb{R} \times M^n.$$

The metric induced on Ω from the product metric on the ambient space via $\Sigma(u)$ is

$$\langle\,,\,\rangle = \mathrm{d}u^2 + \langle\,,\,\rangle_M.$$

The graph $\Sigma(u)$ is said to be *entire* if $\Omega = M^n$. Moreover, according to the current literature, since the mean curvature function $H(u)$ of $\Sigma(u)$ will be supposed constant, it will be called an entire H-graph.

In this setting, as a suitable application of the generalized maximum principle of Omori [9] and Yau [15] jointly with the previous mentioned Rosenberg–Schulze–Spruck result [12], we obtain the following theorem.

Theorem 1. *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature and sectional curvature bounded from below, and let $\Sigma(u) \subset \mathbb{R} \times M^n$ be an entire H-graph of a smooth function $u \in C^\infty(M)$ whose second fundamental form has bounded norm. If $|Du|_M \le C$, for some positive constant C, then $\Sigma(u)$ is minimal. In addition, if u is bounded from below on M^n, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

Here, Du stands for the gradient of the smooth function u on the fiber M^n and $|Du|_M$ is the norm of Du with respect to the metric $\langle\,,\,\rangle_M$. Proceeding, we also get the following theorem.

Theorem 2. *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature and sectional curvature bounded from below, and let $\Sigma^n(u) \subset \mathbb{R} \times M^n$ be an entire H-graph over M^n, whose second fundamental form A has bounded norm. If $|Du|_M \le \alpha|A|$, for some positive constant α, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

Considering the Gauss map of $\Sigma(u)$, which is described in equation (3.6), with aid of Proposition 7.35 of [10], we can verify that its second fundamental form A is given by

$$AX = \frac{1}{\sqrt{1+|Du|_M^2}} D_X Du + \frac{\langle D_X Du, Du\rangle_M}{(1+|Du|_M^2)^{3/2}} Du, \tag{1.1}$$

for any tangent vector X on Ω, where D denotes the Levi-Civita connection in M^n with respect to the metric $\langle\,,\,\rangle_M$.

Hence, related to Theorems 1 and 2, if we assume that $|u|_{C^2(M)} < +\infty$, where $|u|_{C^2(M)} := \max_{|\gamma|\le 2} |D^\gamma u|_{L^\infty(M)}$, from (1.1), we see that the boundedness of $|A|$ is automatically satisfied. Furthermore, from (1.1), we also get that the mean curvature function $H(u)$ of $\Sigma(u)$ is given by the following equation:

$$nH(u) = \mathrm{Div}\left(\frac{Du}{\sqrt{1+|Du|_M^2}}\right), \tag{1.2}$$

where Div stands for the divergence on M^n. Consequently, when M^n is assumed to be compact, since $\Sigma(u)$ is an entire graph, it is also compact. In this case, applying the divergence theorem in (1.2), we conclude that every entire H-graph must be minimal and, hence, a slice if M^n is not flat (see, for instance, the beginning of the proof of Theorem 4 in [1] for the reasoning in the two-dimensional case; see also [13] for the case that M^n is complete noncompact with zero Cheeger constant).

The proofs of Theorems 1 and 2 are given in Section 3.

2. Preliminaries

In what follows, let us consider an $(n+1)$-dimensional product space $\overline{M}^{n+1}$ of the form $\mathbb{R} \times M^n$, where M^n is an n-dimensional connected Riemannian manifold and $\overline{M}^{n+1}$ is endowed with the standard product metric

$$\langle\,,\,\rangle = \pi_{\mathbb{R}}^*(dt^2) + \pi_M^*(\langle\,,\,\rangle_M),$$

where $\pi_{\mathbb{R}}$ and π_M denote the canonical projections from $\mathbb{R} \times M^n$ onto each factor, and $\langle\,,\,\rangle_M$ is the Riemannian metric on M^n. For simplicity, we will just write $\overline{M}^{n+1} = \mathbb{R} \times M^n$. For a fixed $t_0 \in \mathbb{R}$, we say that $M^n_{t_0} = \{t_0\} \times M^n$ is a *slice* of $\overline{M}^{n+1}$. It is not difficult to verify that such a slice of $\overline{M}^{n+1}$ is a totally geodesic hypersurface (see, for instance, [10]).

In what follows we will deal with an orientable hypersurface $\psi : \Sigma^n \to \mathbb{R} \times M^n$, for which we will choose a unit normal vector field N, and let us denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections in $\mathbb{R} \times M^n$ and Σ^n, respectively. Then, the Gauss and Weingarten formulas for ψ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N \tag{2.1}$$

and

$$AX = -\overline{\nabla}_X N, \tag{2.2}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here, $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the Weingarten endomorphism (or shape operator) of Σ^n with respect to N.

In this context, we consider two particular functions naturally attached to such a hypersurface Σ^n, namely, the (vertical) height function $h = (\pi_{\mathbb{R}})|_\Sigma$ and the angle function $\eta = \langle N, \partial_t\rangle$, where ∂_t stands for the unit vector field that determines on $\overline{M}^{n+1}$ a codimension-one foliation by totally geodesic slices M^n_t.

A simple computation shows that the gradient of $\pi_{\mathbb{R}}$ on $\mathbb{R} \times M^n$ is given by

$$\overline{\nabla}\pi_{\mathbb{R}} = \langle \overline{\nabla}\pi_{\mathbb{R}}, \partial_t\rangle \partial_t = \partial_t. \tag{2.3}$$

Consequently, from (2.3), we have that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_{\mathbb{R}})^\top = \partial_t^\top = \partial_t - \eta N, \tag{2.4}$$

where $()^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n. Hence, from (2.4), we get the following relation

$$\eta^2 = 1 - |\nabla h|^2, \tag{2.5}$$

where $|\,|$ denotes the norm of a vector field on Σ^n.

Moreover, as a particular case of the Proposition 3.1 of [4], we obtain the following formula for the Laplacian on Σ^n of the angle function η (see also Proposition 6 of [1])

$$\Delta\eta = -\left(\mathrm{Ric}_M(N^*, N^*) + |A|^2\right)\eta, \tag{2.6}$$

where Ric_M denotes the Ricci curvature of the fiber M^n, $N^* = N - \eta\partial_t$ is the projection of the unit normal vector field N onto the fiber M^n and $|A|$ is the Hilbert–Schmidt norm of the shape operator A.

3. Proofs of Theorems 1 and 2

In order to prove our Moser-type results, we will need two key lemmas. The first one gives a suitable lower estimate for the Ricci curvature of a hypersurface immersed in $\mathbb{R} \times M^n$.

Lemma 1. *Let Σ^n be an oriented hypersurface immersed in a Riemannian product space $\mathbb{R} \times M^n$, whose fiber M^n has sectional curvature bounded from below. If the second fundamental form A of Σ^n has bounded norm, then the Ricci curvature of Σ^n is bounded from below.*

Proof. We recall that, using the formulas (2.1) and (2.2), the curvature tensor R of the hypersurface Σ^n can be described in terms of the shape operator A and the curvature tensor $\overline{R}$ of $\mathbb{R} \times M^n$ by the so-called Gauss equation given by

$$R(X, Y)Z = (\overline{R}(X, Y)Z)^\top + \langle AX, Z\rangle AY - \langle AY, Z\rangle AX, \tag{3.1}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$. Here, as in [10], the curvature tensor R of a hypersurface $\psi : \Sigma^n \to \mathbb{R} \times M^n$ is given by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\,,\,]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

Let us consider $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, \ldots, E_n\} \subset \mathfrak{X}(\Sigma)$. Then, it follows from (3.1) that

$$\mathrm{Ric}_\Sigma(X, X) = \sum_i \langle \overline{R}(X, E_i)X, E_i\rangle + nH\langle AX, X\rangle - \langle AX, AX\rangle, \tag{3.2}$$

where Ric_Σ and $H = \frac{1}{n}\,\mathrm{tr}(A)$ are the Ricci curvature and the mean curvature of Σ^n, respectively.

On the other hand, we have that

$$\langle \overline{R}(X, E_i)X, E_i\rangle = K_M(X^*, E_i^*)(\langle X^*, X^*\rangle_M \langle E_i^*, E_i^*\rangle_M - \langle X^*, E_i^*\rangle_M^2), \tag{3.3}$$

where $X^* = X - \langle X, \partial_t \rangle \partial_t$ and $E_i^* = E_i - \langle E_i, \partial_t \rangle \partial_t$ are the projections of the tangent vector fields X and E_i onto M^n, respectively, and K_M stands for the sectional curvature of M^n.

Since our assumption on the sectional curvature of M^n guarantees the existence of a positive constant κ such that $K_M \geq -\kappa$, summing up relation (3.3), we get

$$\sum_i \langle \bar{R}(X, E_i)X, E_i \rangle \geq -(n-1)\kappa \left(1 - |\nabla h|^2\right) |X|^2. \tag{3.4}$$

Hence, from (3.2) and (3.4), we infer that the Ricci curvature of Σ^n satisfies the following estimate

$$\begin{aligned} \mathrm{Ric}_\Sigma(X, X) &\geq -\big((n-1)\kappa\eta^2 + |A|^2 + n|H||A|\big)|X|^2 \\ &\geq -\big((n-1)\kappa + |A|^2 + n|H||A|\big)|X|^2 \\ &\geq -\big((n-1)\kappa + (1+\sqrt{n})|A|^2\big)|X|^2, \end{aligned} \tag{3.5}$$

where it was used the fact that $nH^2 \leq |A|^2$ to obtain the last inequality. Therefore, since we are assuming that $|A|$ is bounded on Σ^n, from (3.5) we conclude that Ric_Σ is bounded from below. □

The second auxiliary lemma is the well-known generalized maximum principle due to Omori [9] and Yau [15], which is quoted below.

Lemma 2. *Let Σ^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $\vartheta : \Sigma^n \to \mathbb{R}$ be a smooth function bounded from above on Σ^n. Then, there exists a sequence of points $(p_k)_{k \in \mathbb{N}} \subset \Sigma^n$ such that*

$$\lim_k \vartheta(p_k) = \sup_\Sigma \vartheta, \quad \lim_k |\nabla \vartheta(p_k)| = 0 \quad \text{and} \quad \limsup_k \Delta\vartheta(p_k) \leq 0.$$

Now, we are in position to proceed with the proofs of our theorems.

Proofs of Theorems 1 and 2. First, we observe that $\Sigma(u)$ is, in fact, complete. Indeed, an entire vertical graph is properly immersed into the Riemannian product space $\mathbb{R} \times M^n$, which is obviously complete when the fiber M^n is complete. Moreover, with a straightforward computation, we can verify that the unit vector field

$$N = \frac{1}{\sqrt{1 + |Du|_M^2}}(\partial_t - Du), \tag{3.6}$$

gives an orientation for $\Sigma(u)$ such that $0 < \eta \leq 1$ on it.

Now, considering $\Sigma(u)$ oriented by (3.6), we define a bounded smooth function $\vartheta : \Sigma(u) \to \mathbb{R}$ by

$$\vartheta = -e^\eta. \tag{3.7}$$

From (3.7) we have that

$$\nabla\vartheta = -e^\eta \nabla\eta \tag{3.8}$$

and, using formula (2.6),

$$\Delta\vartheta = e^\eta \left\{ -|\nabla\eta|^2 + (\mathrm{Ric}_M(N^*, N^*) + |A|^2)\eta \right\}. \tag{3.9}$$

On the other hand, from equation (2.4) it is not difficult to see that $N^{*\top} = \eta\nabla u$ and $|\nabla u|^2 = \langle N^*, N^* \rangle_M$. Here, we are taking into account that the height function h of $\Sigma(u)$ is nothing but the function u regarded as a function on $\Sigma(u)$. Thus, from (3.6), we obtain that

$$|\nabla u|^2 = \frac{|Du|_M^2}{1 + |Du|_M^2}. \tag{3.10}$$

Since we are supposing that there exists a positive constant C such that $|Du|_M \leq C$ (in the context of Theorem 2, as it was assumed that $|A|$ is bounded, we can take $C = \alpha \sup_{p \in \Sigma(u)} |A(p)|$), from (2.5) and (3.10) we have that

$$\eta \geq \frac{1}{\sqrt{1 + C^2}} > 0. \tag{3.11}$$

Since we are assuming that the fiber M^n has sectional curvature bounded from below and that $\sup_{p \in \Sigma(u)} |A(p)|^2 < +\infty$, Lemma 1 guarantees that the Ricci curvature of $\Sigma(u)$ is bounded from below. Hence, we can apply Lemma 2 to the function ϑ, obtaining a sequence of points $(p_k)_{k\in\mathbb{N}} \subset \Sigma^n(u)$ such that $\lim_k \vartheta(p_k) = \sup_{\Sigma(u)} \vartheta$, $\lim_k |\nabla\vartheta(p_k)| = 0$ and $\limsup_k \Delta\vartheta(p_k) \le 0$.

Consequently, taking into account that M^n has nonnegative Ricci curvature, from (3.7), (3.8) and (3.9) we have that

$$0 \ge \limsup_k \Delta\vartheta(p_k) = \limsup_k e^{\eta(p_k)}(\mathrm{Ric}_M(N^*, N^*) + |A|^2)\eta(p_k)$$
$$\ge e^{\inf_{p\in\Sigma(u)} \eta(p)} \limsup_k \Big(\mathrm{Ric}_M(N^*, N^*) + |A|^2\Big)(p_k) \inf_{p\in\Sigma(u)} \eta(p) \ge 0. \tag{3.12}$$

Thus, since (3.11) guarantees that $\inf_{p\in\Sigma(u)} \eta(p) > 0$, from (3.12) we get that $\lim_k |A(p_k)| = 0$. Hence, using once more the algebraic inequality $nH^2 \le |A|^2$, we obtain that $H = 0$, that is, $\Sigma(u)$ is minimal. In addition, assuming that $u \ge \beta$ for some constant β, we can apply the Rosenberg–Schulze–Spruck result [12] to the function $\widetilde{u} := u - \beta$ and conclude that $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.

Finally, assuming that $|Du|_M \le \alpha|A|$ for some positive constant α, from (3.10) we have that $\lim_k |\nabla u(p_k)|^2 = 0$. Hence, from (2.5) we get that $\inf_{p\in\Sigma(u)} \eta(p) = 1$. Therefore, also in this case, $u \equiv t_0$ for some $t_0 \in \mathbb{R}$. □

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