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On compact Ricci solitons in Finsler geometry

*Sur les solitons de Ricci compacts en géométrie finslérienne*

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ABSTRACT

Ricci solitons on Finsler spaces, previously developed by the present authors, are a generalization of Einstein spaces, which can be considered as a solution to the Ricci flow on compact Finsler manifolds. In the present work, it is shown that on a Finslerian space, a forward complete shrinking Ricci soliton is compact if and only if it is bounded. Moreover, it is proved that a compact shrinking Finslerian Ricci soliton has finite fundamental group, and hence the first de Rham cohomology group vanishes.

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R É S U M É

Les solitons de Ricci sur les espaces de Finsler, précédemment définis et étudiés par les auteurs de la présente note, sont une généralisation des espaces d'Einstein, et peuvent être considérés comme des solutions du flot de Ricci sur les variétés finslériennes compactes. Dans ce travail, on démontre qu'un soliton de Ricci complet contractant en temps croissant sur un espace de Finsler est compact si et seulement s'il est borné. En outre, il est démontré qu'un soliton de Ricci contractant compact donne lieu à un groupe fondamental de type fini et donc que le premier groupe de cohomologie s'annule.

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1. Introduction

The Ricci flow in Riemannian geometry was introduced by R.S. Hamilton in 1982, cf. [9], and since then has been extensively studied thanks to its applications in geometry, physics and different branches of real-world problems. Quasi-Einstein metrics or Ricci solitons are considered as solutions to the Ricci flow equation and are subject of great interest in geometry and physics, specially in relation with string theory, cf. [10].

Let (M, g) be a Riemannian manifold, a triple (M, g, X) is said to be a *quasi-Einstein metric* or *Ricci soliton* if g satisfies the equation

$$2 \operatorname{Ric} + \mathcal{L}_X g = 2\lambda g, \quad (1)$$

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where Ric is the Ricci tensor, X a smooth vector field on M , \mathcal{L}_X the Lie derivative along X , and λ a real constant. A Ricci soliton is said to be *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If the vector field X is the gradient of a function f , then (M, g, X) is said to be *gradient* and (1) takes the familiar form:

$$Ric + \nabla \nabla f = \lambda g.$$

On a compact Riemannian manifold, a quasi-Einstein metric is a special solution to the Ricci flow equation defined by

$$\frac{\partial}{\partial t} g(t) = -2 Ric, \quad g(t=0) := g_0.$$

J. Lott has shown that the fundamental group of a closed Riemannian manifold is finite for any gradient shrinking Ricci soliton, cf. [11]. M.F. López and E.G. Río have proved that a Riemannian compact shrinking Ricci soliton has finite fundamental group, cf. [10]. W. Wylie has shown that a Riemannian complete shrinking Ricci soliton has finite fundamental group, cf. [12].

The concept of Ricci flow on Finsler manifolds is defined first by D. Bao, cf. [5], using the Ricci tensor defined by H. Akbar-Zadeh, [2]. Recently the present authors have developed the concept of Ricci solitons as a generalization of Einstein spaces and convergence of Ricci flow on Finsler spaces, cf. [7,8]. It is proved that if there is a Ricci soliton on a compact Finsler manifold then there exists a solution to the Ricci flow equation and vice-versa. Since Finslerian Ricci solitons generalize Einstein manifolds, it is natural to ask whether classical results like the Bonnet–Myers theorem for Finsler–Einstein manifolds of positive Ricci scalar remain valid for Finslerian Ricci solitons. In the present work, in analogy with Riemannian space, the shrinking Finslerian Ricci soliton is defined and it is shown that a forward complete shrinking Finslerian Ricci soliton (M, F, V) is compact if and only if $\|V\|$ is bounded. Moreover, it is proved that in this case the fundamental group is finite and, as a consequence, the first de Rham cohomology group of M vanishes.

2. Preliminaries and notations

Let M be a real n -dimensional differentiable manifold. We denote by TM its tangent bundle and by $\pi : TM_0 \rightarrow M$, the fiber bundle of non-zero tangent vectors. A *Finsler structure* on M is a function $F : TM \rightarrow [0, \infty)$, with the following properties:

I. Regularity: F is C^∞ on the entire slit tangent bundle $TM_0 = TM \setminus 0$.

II. Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.

III. Strong convexity: the $n \times n$ Hessian matrix $g_{ij} = ([\frac{1}{2}F^2]_{y^i y^j})$ is positive definite at every point of TM_0 . A *Finsler manifold* (M, F) is a pair consisting of a differentiable manifold M and a Finsler structure F . The formal Christoffel symbols of second kind and spray coefficients are denoted respectively by

$$\gamma_{jk}^i := g^{is} \frac{1}{2} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right),$$

where $g_{ij}(x, y) = [\frac{1}{2}F^2]_{y^i y^j}$, and $G^i := \frac{1}{2} \gamma_{jk}^i y^j y^k$. We consider also the *reduced curvature tensor* R_k^i , which is expressed entirely in terms of the x and y derivatives of spray coefficients G^i .

$$R_k^i := \frac{1}{F^2} \left(2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k} \right). \quad (2)$$

In the general Finslerian setting, one of the Ricci tensors is introduced by H. Akbar-Zadeh [1] as follows:

$$Ric_{jk} := [\frac{1}{2}F^2 Ric]_{y^j y^k}, \quad (3)$$

where $Ric = R_i^i$ and R_k^i is defined by (2). Akbar-Zadeh's definition of Einstein–Finsler space related to this Ricci tensor is obtained as a critical point of an Einstein–Hilbert functional and, (see [2] chapter IV). One of the advantages of the Ricci quantity defined here is its independence of the choice of Cartan, Berwald or Chern (Rund) connections. Based on the Akbar-Zadeh's Ricci tensor, in analogy with Eq. (3), D. Bao has considered the following natural extension of *Ricci flow* in Finsler geometry, cf. [5]:

$$\frac{\partial}{\partial t} g_{jk} = -2 Ric_{jk}, \quad g(t=0) := g_0.$$

This equation leads to the following differential equation

$$\frac{\partial}{\partial t} (\log F(t)) = -Ric, \quad F(t=0) := F_0,$$

where F_0 is the initial Finsler structure. Let $V = V^i(x) \frac{\partial}{\partial x^i}$ be a vector field on M .

The Lie derivative of a Finsler metric tensor g_{jk} is given in the following tensorial form by

$$\mathcal{L}_{\hat{V}} g_{jk} = \nabla_j V_k + \nabla_k V_j + 2(\nabla_0 V^l) C_{ljk}, \quad (4)$$

where \hat{V} is the complete lift of a vector field V on M , ∇ is the Cartan connection, $\nabla_0 = y^p \nabla_p$ and $\nabla_p = \nabla_{\frac{\delta}{\delta x^p}}$, (see [13], p. 180, and see [6]).

Let M be a connected smooth manifold, then there exists a simply connected smooth manifold \tilde{M} , called the universal covering manifold of M , and a smooth covering map $p : \tilde{M} \rightarrow M$ such that it is unique up to a diffeomorphism. The complete lift of p is a map $\tilde{p} : T\tilde{M} \rightarrow TM$ is given by

$$\tilde{p}(\tilde{x}, \tilde{y}) = (p(\tilde{x}), \tilde{y}^i \frac{\partial p}{\partial \tilde{x}^i}) = (p(\tilde{x}), \tilde{y}^i \frac{\partial p^j}{\partial \tilde{x}^i} \frac{\partial}{\partial x^j}),$$

where $\tilde{y} \in T_{\tilde{x}}\tilde{M}$.

3. Shrinking Finslerian Ricci soliton

Let (M, F_0) be a Finsler manifold and $V = V^i(x) \frac{\partial}{\partial x^i}$ a vector field on M . We call the triple (M, F_0, V) a Finslerian quasi-Einstein or a Finslerian Ricci soliton if g_{jk} , the Hessian related to the Finsler structure F_0 , satisfies

$$2Ric_{jk} + \mathcal{L}_{\hat{V}} g_{jk} = 2\lambda g_{jk}, \tag{5}$$

where \hat{V} is the complete lift of V and $\lambda \in \mathbb{R}$. A Finslerian Ricci soliton is said to be *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. The Finslerian Ricci soliton is said to be forward complete (resp. compact) if (M, F_0) is forward complete (resp. compact). Note that according to the Hopf–Rinow’s theorem, the both notions forward complete and forward geodesically complete are equivalent. Denote by $S_x M$ the sphere bundle, defined by $S_x M := \bigcup_{x \in M} S_x M$ where

$S_x M := \{y \in T_x M | F(x, y) = 1\}$. For a vector field $X = X^i(x) \frac{\partial}{\partial x^i}$ on M , define

$$\|X\|_x = \max_{y \in S_x M} \sqrt{g_{ij}(x, y) X^i X^j}, \tag{6}$$

where $x \in M$ (see [4] at p. 321). Since $S_x M$ is compact, $\|X\|_x$ is well defined.

Theorem 1. *Let (M, F_0) be a forward geodesically complete Finsler manifold satisfying*

$$2 Ric_{jk} + \mathcal{L}_{\hat{V}} g_{jk} \geq 2\lambda g_{jk}, \tag{7}$$

where $\lambda > 0$. Then, M is compact if and only if $\|V\|$ is bounded on M by a constant D and moreover, in such a case, $\text{diam}(M) \leq \frac{\pi}{\lambda} (D + \sqrt{D^2 + \lambda(n-1)})$.

Proof. Let M be a compact manifold, it is clear that $\|V\|$ is bounded on M . Conversely, let p, q be two points in M joined by a minimal geodesic γ parameterized by the arc length t , $\gamma : [0, \infty) \rightarrow M$. Using (4) we have along γ :

$$\gamma'^j \gamma'^k \mathcal{L}_{\hat{V}} g_{jk} = \gamma'^j \gamma'^k (\nabla_j V_k + \nabla_k V_j + 2(\nabla_0 V^l) C_{ljk}). \tag{8}$$

Along γ , we have $\gamma'^j \gamma'^k (\nabla_0 V^l) C_{ljk}(\gamma(t), \gamma'(t)) = 0$. Hence (8) reduces to

$$\gamma'^j \gamma'^k \mathcal{L}_{\hat{V}} g_{jk} = 2\gamma'^j \gamma'^k \nabla_j V_k. \tag{9}$$

On the other hand, by compatibility of metric with the Cartan connection, we have along the geodesic γ :

$$\gamma'^j \gamma'^k \nabla_j V_k = \nabla_{\gamma'^j \frac{\delta}{\delta x^j}} (\gamma'^k V_k) = \nabla_{\hat{\gamma}'} (\gamma'^k V_k) = \frac{d}{dt} (\gamma'^k V_k), \tag{10}$$

where $\hat{\gamma}' = \gamma'^j \frac{\delta}{\delta x^j}$. Replacing (10) in (9), we have:

$$\gamma'^j \gamma'^k \mathcal{L}_{\hat{V}} g_{jk} = 2 \frac{d}{dt} (\gamma'^k V_k). \tag{11}$$

By means of (7) and (11) we get:

$$2\gamma'^j \gamma'^k Ric_{jk} + 2 \frac{d}{dt} (\gamma'^k V_k) \geq 2\lambda \gamma'^j \gamma'^k g_{jk}.$$

By the last inequality, we conclude that

$$\gamma'^j \gamma'^k Ric_{jk} \geq \lambda \gamma'^j \gamma'^k g_{jk} - \frac{d}{dt} (\gamma'^k V_k) = \lambda + \frac{d}{dt} (-\gamma'^k V_k).$$

On the other hand, by means of the Cauchy–Schwarz inequality, we have along the geodesic γ :

$$\begin{aligned} |-\gamma'^k V_k| &= |\gamma'^k V_k| = |g_{kl}(\gamma(t), \gamma'(t))\gamma'^k V^l| \\ &\leq |g_{pq}(\gamma(t), \gamma'(t))\gamma'^p \gamma'^q|^{\frac{1}{2}} |g_{rs}(\gamma(t), \gamma'(t))V^r V^s|^{\frac{1}{2}} \\ &\leq \max_{y \in S_{\gamma(t)}M} |g_{rs}(\gamma(t), y)V^r V^s|^{\frac{1}{2}} = \|V\|_{\gamma(t)}. \end{aligned}$$

Since $\|V\|$ is assumed to be bounded on M , there exists a positive constant D such that $\|V\|_{\gamma(t)} \leq D$ and therefore along γ , $|-\gamma'^k V_k| \leq D$. Now, the result follows from generalization of Meyers Theorem, cf. [3]. That is, M is compact and moreover it is bounded from above by $\text{diam}(M) \leq \frac{\pi}{\lambda}(D + \sqrt{D^2 + \lambda(n-1)})$. This completes the proof. \square

Corollary 2. *Let (M, F, V) be a forward complete shrinking Finslerian Ricci soliton. Then, M is compact if and only if $\|V\|$ is bounded on M by a constant D and moreover, in this case, $\text{diam}(M) \leq \frac{\pi}{\lambda}(D + \sqrt{D^2 + \lambda(n-1)})$.*

Theorem 3. *Let (M, F) be a compact Finsler manifold satisfying (7). Then the fundamental group $\pi_1(M)$ of M is finite and its first cohomology group vanishes, i.e., $H^1_{dR}(M) = 0$.*

Proof. Let \tilde{M} be the universal covering manifold of M with the smooth covering map $p : \tilde{M} \rightarrow M$. Pull back of complete lift of the smooth covering map p , i.e., $\tilde{p}^*F := F \circ \tilde{p} : T\tilde{M} \rightarrow [0, \infty)$ is a Finsler structure on \tilde{M} . In fact, we check simply the three conditions of the Finsler structure. We have the regularity condition since F and p are C^∞ , and so is \tilde{p}^*F . Next,

$$\begin{aligned} \tilde{p}^*F(x, \lambda y) &= F \circ \tilde{p}(x, \lambda y) = F(p(x), \lambda y^i \frac{\partial p}{\partial x^i}) \\ &= \lambda F(p(x), y^i \frac{\partial p}{\partial x^i}) = \lambda \tilde{p}^*F(x, y). \end{aligned}$$

Thus the positive homogeneity is satisfied. Finally, assume that $\tilde{p}^*x^i = \tilde{x}^i$ and $\tilde{p}^*y^i = \tilde{y}^i$. For strong convexity we have:

$$\tilde{g}_{ij} := [\frac{1}{2}(\tilde{p}^*F)^2]_{\tilde{y}^i \tilde{y}^j} = \frac{1}{2} \frac{\partial^2((\tilde{p}^*F)^2)}{\partial \tilde{y}^i \partial \tilde{y}^j} = \frac{1}{2} \frac{\partial^2(\tilde{p}^*F^2)}{\partial \tilde{y}^i \partial \tilde{y}^j}.$$

One can easily check that

$$\frac{\partial(\tilde{p}^*F^2)}{\partial \tilde{y}^i} = \tilde{p}^* \frac{\partial F^2}{\partial y^i},$$

from which

$$\tilde{g}_{ij} = [\frac{1}{2}(\tilde{p}^*F^2)]_{\tilde{y}^i \tilde{y}^j} = \frac{1}{2} \frac{\partial^2(\tilde{p}^*F^2)}{\partial \tilde{y}^i \partial \tilde{y}^j} = \tilde{p}^*[\frac{1}{2}F^2]_{y^i y^j} = \tilde{p}^*g_{ij}. \tag{12}$$

Using the facts that $[\frac{1}{2}F^2]_{y^i y^j}$ is positive definite on TM_0 and \tilde{p}^* is a local diffeomorphism (note that p is the smooth covering map), $\tilde{p}^*[\frac{1}{2}F^2]_{y^i y^j}$ is also positive definite on $T\tilde{M}_0$ and hence $\tilde{F} := \tilde{p}^*F$ defines a Finsler structure on $T\tilde{M}_0$. Moreover, (\tilde{M}, \tilde{F}) is locally isometric to (M, F) . Let W denote the lift of V , that is, $W := p^*V = (p^{-1})_*V$. More precisely, since p is a local diffeomorphism, we can define $W := p^*V = (p^{-1})_*V$. By means of the local isometry $p : (\tilde{M}, \tilde{F}) \rightarrow (M, F)$ and the inequality (7), we have:

$$\tilde{p}^*(2\text{Ric}_{jk} + \mathcal{L}_{\hat{V}}g_{jk}) \geq 2\tilde{p}^*(\lambda g_{jk}).$$

By linearity of \tilde{p}^* we get:

$$2\tilde{p}^*\text{Ric}_{jk} + \tilde{p}^*\mathcal{L}_{\hat{V}}g_{jk} \geq 2\lambda\tilde{p}^*(g_{jk}). \tag{13}$$

By means of (12), $W = p^*V$ and commutativity of Lie derivative and the pull back \tilde{p}^* , we obtain:

$$\tilde{p}^*\mathcal{L}_{\hat{V}}g_{jk} = \mathcal{L}_{\tilde{W}}\tilde{g}_{jk}. \tag{14}$$

On the other hand, one can easily check that $\tilde{R}ic_{jk} = \tilde{p}^*\text{Ric}_{jk}$. In fact we have:

$$\begin{aligned} \tilde{p}^*\text{Ric}_{jk} &= \tilde{p}^*[\frac{1}{2}F^2\mathcal{R}ic]_{y^j y^k} = \frac{1}{2}\tilde{p}^* \frac{\partial^2(F^2\mathcal{R}ic)}{\partial y^i \partial y^j} \\ &= \frac{1}{2} \frac{\partial^2}{\partial \tilde{y}^i \partial \tilde{y}^j} (\tilde{p}^*(F^2\mathcal{R}ic)) = \frac{1}{2} \frac{\partial^2}{\partial \tilde{y}^i \partial \tilde{y}^j} (\tilde{p}^*(F^2)\tilde{p}^*(\mathcal{R}ic)). \end{aligned}$$

Since $\tilde{p}^*(\mathcal{R}ic) = \tilde{\mathcal{R}}ic$, cf. [7], and $\tilde{p}^*(F^2) = \tilde{F}^2$, we get

$$\bar{p}^* Ric_{jk} = \frac{1}{2} \frac{\partial^2}{\partial \bar{y}^i \partial \bar{y}^j} (\bar{p}^*(F^2) \bar{p}^*(Ric)) = \frac{1}{2} \frac{\partial^2}{\partial \bar{y}^i \partial \bar{y}^j} (\bar{F}^2 \bar{Ric}) = \bar{Ric}_{jk}. \tag{15}$$

Replacing (12), (14) and (15) in (13), leads to

$$2\bar{Ric}_{jk} + \mathcal{L}_{\tilde{W}} \tilde{g}_{jk} \geq 2\lambda \tilde{g}_{jk}.$$

On the other hand, we have:

$$\begin{aligned} \|W\|_{\bar{x}} &= \max_{\bar{y} \in S_{\bar{x}} \tilde{M}} ((\bar{p}^* g_{ij})(\bar{x}, \bar{y}) W^i W^j)^{\frac{1}{2}} = \max_{\bar{y} \in S_{\bar{x}} \tilde{M}} (g_{ij}(p(\bar{x}), \bar{y}^i \frac{\partial p}{\partial \bar{x}^i}) \bar{p}_* W^i \bar{p}_* W^j)^{\frac{1}{2}} \\ &\leq \max_{y \in S_{\bar{x}} \tilde{M}} (g_{ij}(p(\bar{x}), y) \bar{p}_* W^i \bar{p}_* W^j)^{\frac{1}{2}} = \|\bar{p}_* W\|_{p(\bar{x})}. \end{aligned} \tag{16}$$

By compactness of M , the norm $\|\bar{p}_* W\|$ is bounded on M and therefore, by means of (16), the norm $\|W\|$ is bounded on \tilde{M} . It follows from Theorem 1 that (\tilde{M}, \tilde{F}) is compact. Thus the closed subset $p^{-1}(x)$ of \tilde{M} is compact and, being discrete, is finite. By assumption, M is connected, so all of its fundamental groups $\pi_1(M, x)$ are isomorphic, where x denotes the base point. Since \tilde{M} is a universal cover, $\pi_1(M, x)$ is bijective with $p^{-1}(x)$ and therefore $\pi_1(M)$ is finite. Thus, by a well-known result, the first cohomology group $H_{dR}^1(M) = 0$. This completes the proof. \square

Corollary 4. *Let (M, F, V) be a compact shrinking Finslerian Ricci soliton. Then the fundamental group $\pi_1(M)$ of M is finite and therefore $H_{dR}^1(M) = 0$.*

Corollary 5. *Let (M, F, V) be a compact shrinking Finslerian Ricci soliton. Then the fundamental group $\pi_1(SM)$ of SM is finite and therefore $H_{dR}^1(SM) = 0$.*

Proof. Let \tilde{M} be the universal covering manifold of M with the smooth covering map $p : \tilde{M} \rightarrow M$. It is well known that the homotopic sequence of the fiber bundle $(S\tilde{M}, \tilde{\pi}, \tilde{M}, S^{n-1})$ is exact. That is that

$$\dots \rightarrow \pi_1(S^{n-1}) \rightarrow \pi_1(S\tilde{M}) \rightarrow \pi_1(\tilde{M}) \rightarrow \dots, \tag{17}$$

is exact. Since \tilde{M} is simply connected, $\pi_1(\tilde{M}) = 0$. We know that $\pi_1(S^{n-1}) = 0$. Thus, by (17) we get $\pi_1(S\tilde{M}) = 0$. One can easily check that $\bar{p} : S\tilde{M} \rightarrow SM$ is a smooth covering map. Therefore, $S\tilde{M}$ is the universal covering manifold of SM . According to the proof of Theorem 3, \tilde{M} is compact and so is $S\tilde{M}$. Thus the fundamental group $\pi_1(SM)$ is finite and therefore $H_{dR}^1(SM) = 0$. \square

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