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Partial differential equations

Optimal transport of closed differential forms for convex costs

Transport optimal des formes fermées pour des coûts convexes

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ABSTRACT

Let $c : \Lambda^{k-1} \to \mathbb{R}_+$ be convex and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let f_0 and f_1 be two closed k-forms on Ω satisfying appropriate boundary conditions. We discuss the minimization of $\int_\Omega c(A)\,dx$ over a subset of $(k-1)$-forms A on Ω such that $dA + f_1 - f_0 = 0$, and its connection with a transport of symplectic forms. Section 3 mainly serves as a step toward Section 4, which is richer, as it connects to variational problems with multiple minimizers.

RÉSUMÉ

Soient $c : \Lambda^{k-1} \to \mathbb{R}_+$ une fonction convexe et $\Omega \subset \mathbb{R}^n$ un domaine borné. Soient f_0 et f_1 des k-formes fermées sur Ω satisfaisant des conditions de bord appropriées. Nous nous intéressons à la minimisation de $\int_\Omega c(A)\,dx$ sur l'ensemble des $(k-1)$-formes A telles que $dA + f_1 - f_0 = 0$, ainsi que sa relation à un problème de transport des formes symplectiques. La Section 3 sert d'étape intermédiaire vers la Section 4, qui est plus riche, car reliée à des problèmes variationnels avec une multitude de minimiseurs.

Version française abrégée

Soit n un entier positif pair, soit $\Omega \subset \mathbb{R}^n$ un ouvert borné contractile de bord régulier et de normale unitaire extérieure ν. Supposons que $f_0, f_1 \in C^1(\overline{\Omega}; \Lambda^2)$ soient des formes symplectiques telles que $\nu \wedge (f_0 - f_1)$ s'annule sur le bord $\partial\Omega$. Faisons l'hypothèse supplémentaire que $f_t := tf_1 + (1-t)\,f_0$ reste symplectique pour tout $t \in [0,1]$. Nous identifierons les éléments u de Λ^1 avec des champs vectoriels de $u : \Omega \to \mathbb{R}^n$. Rappelons que la définition de l'ensemble $\mathcal{C}(f_1 - f_0)$ apparaît dans la Definition 2. Montrons comment le problème variationnel

$$(P_2) \quad \inf_A \left\{ I_2(A) = \frac{1}{2} \int_\Omega |A|^2 : A \in \mathcal{C}(f_1 - f_0) \right\}$$

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peut être exploité pour produire des bijections qui soient des applications optimales transportant f_0 sur f_t. Notre affirmation repose aussi sur la Section 1 affirmant que le chemin $t \to (f_t, A_2)$ est optimal pour la fonction coût $\bar{c}(f, A) = |A|^2$ dans le problème (1).

Theorem 1. *Soit A_2 l'unique minimiseur de (P_2) (voir Theorem 4). Comme f_t est non dégénérée, soit $u_t \in \Lambda^1$ l'unique solution de $u_t \lrcorner f_t = A_2$. Soit enfin $\varphi : [0,1] \times \bar{\Omega} \to \bar{\Omega}$ le flot associé à u, défini par*

$$\partial_t \varphi_t = u_t \circ \varphi_t \quad \text{sur} \quad t \in [0,1] \times \Omega, \qquad \varphi_0 = \text{id} \quad \text{sur} \quad \Omega.$$

Alors, pour tout $t \in [0,1]$, $\varphi_t \in \text{Diff}^1\left(\overline{\Omega}; \overline{\Omega}\right)$ (en particulier $\varphi_t(\Omega) = \Omega$) et $\varphi_t^*(f_t) = f_0$ dans Ω.

Proof. Le résultat de régularité (12) nous donne que $A_2 \in C^{1,\alpha}$ pour tout $\alpha < 1$, et donc que $(t, x) \to u_t(x)$ est de classe $C^1\left([0,1] \times \overline{\Omega}; \mathbb{R}^n\right)$. Comme $\nu \wedge A_2 = 0$ sur $\partial\Omega$, nous en déduisons que $\langle \nu; u_t \rangle = 0$ sur $\partial\Omega$, d'où $\varphi_t \in \text{Diff}^1\left(\overline{\Omega}; \overline{\Omega}\right)$. Nous utilisons un résultat standard (voir par exemple le Theoreme 12.5 dans [3]) pour conclure que

$$\partial_t\left(\varphi_t^*(f_t)\right) = \varphi_t^*\left(\partial_t f_t + \mathrm{d}(u_t \lrcorner f_t) + u_t \lrcorner \mathrm{d} f_t\right).$$

Comme $\mathrm{d} f_t = 0$ et que $\mathrm{d}\,(u_t \lrcorner f_t) = \mathrm{d}A_2 = f_0 - f_1 = -\partial_t f_t$ nous en déduisons que $\varphi_t^*(f_t)$ est indépendante de t, ce qui termine la preuve, car $\varphi_0 = \text{id}$. $\square$

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded contractible smooth set and denote by ν the outward unit normal to $\partial\Omega$. Let $1 < p < \infty$ and let $f_0, f_1 \in L^p(\bar{\Omega}; \Lambda^k)$ be two closed forms (in the weak sense) such that $\nu \wedge (f_1 - f_0) = 0$ on $\partial\Omega$ (cf. Definition 2). When $k = 2$, $n = 2m$ and f_0 and f_1 are smooth and of maximal rank these forms are called symplectic.

Our original motivation is to find a map $\varphi : \bar{\Omega} \to \bar{\Omega}$, so that $\varphi^*(f_1) = f_0$. This is a very classical problem that goes back to the famous Darboux theorem. We want here to propose an "optimal" way of selecting such a φ. In our articles [5] and [6], we discuss other approaches to the problem.

Let us informally start with a description [5], to arrive at the content of the current manuscript. Denote by $\mathcal{F}$ the set of closed forms $h \in L^p(\Omega, \Lambda^k)$ such that $\nu \wedge (f_1 - h) = 0$ on $\partial\Omega$ in the weak sense. Denote by $P(f_0, f_1)$ the set of pairs $(\bar{f}, \bar{A})$ such that $\bar{f}$ is continuous in t, $\bar{f}$ starts at f_0, ends at f_1,

$$\begin{aligned} &\bar{A} \in L^1\left((0,1) \times \Omega; \Lambda^k\right), \quad \bar{f} \in C([0,1]; \mathcal{F}), \\ &\int_0^1 \left(\int_\Omega (\langle f; \partial_t h \rangle + \langle A; \delta h \rangle) \mathrm{d}x \right) \mathrm{d}t = \int_\Omega \langle f_1, h_1 \rangle - \langle f_0, h_0 \rangle, \quad \forall\, h \in C^1\left([0,1]; C^1(\bar{\Omega}, \Lambda^k)\right). \end{aligned} \tag{1}$$

Let $\bar{c} : \Lambda^k \times \Lambda^{k-1} \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function, bounded below. We are interested in proving the existence of minimizers and in characterizing the Euler–Lagrange equations of

$$\inf_{(\bar{f}, \bar{A})} \left\{ \int_0^1 \int_\Omega \bar{c}(\bar{f}_t(x), \bar{A}_t(x))\, \mathrm{d}x\, \mathrm{d}t \;\middle|\; (\bar{f}, \bar{A}) \in P(f_0, f_1) \right\}. \tag{2}$$

Let $\mathcal{C}(f_1 - f_0)$ be the set of $A \in L^1(\Omega; \Lambda^{k-1})$ that satisfy in the weak sense (cf. Definition 2):

$$\mathrm{d}A + f_1 - f_0 = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \nu \wedge A = 0 \quad \text{on} \quad \partial\Omega. \tag{3}$$

One of the simplest versions of the variational problem (2) is obtained by assuming the existence of a strictly convex function $c : \Lambda^{k-1} \to \mathbb{R}$ such that $\bar{c}(\bar{f}, \bar{A}) = c(\bar{A})$. Setting

$$A(x) = \int_0^1 \bar{A}_t(x)\, \mathrm{d}t, \quad \tilde{f}_t = (1 - t) f_0 + t f_1,$$

we have $(\tilde{f}, A) \in P(f_0, f_1)$, $A \in \mathcal{C}(f_1 - f_0)$ and by Jensen's inequality (which is strict unless $\bar{A}_t \equiv A$)

$$\int_0^1 \left(\int_\Omega \bar{c}(\bar{f}_t(x), \bar{A}_t(x)) \mathrm{d}x \right) \mathrm{d}t = \int_\Omega \left(\int_0^1 c(\bar{A}_t(x)) \mathrm{d}t \right) \mathrm{d}x \geq \int_\Omega c(A)\, \mathrm{d}x = \int_0^1 \left(\int_\Omega \bar{c}(\tilde{f}_t(x), A(x))\, \mathrm{d}x \right) \mathrm{d}t.$$

Thus, the study of (1) reduces to that of the variational problem

$$(P) \quad \inf_A \left\{ I(A) = \int_\Omega c(A)\, \mathrm{d}x : A \in \mathcal{C}(f_1 - f_0) \right\}.$$

In the particular case where $c(A)=|A|^2/2$, $n=2m$ and $k=2$, (P) has a unique minimizer A that satisfies $A\in C^{l+1,\alpha}(\bar\Omega,\Lambda^1)$ if for instance $f_1, f_0\in C^{l,\alpha}(\bar\Omega,\Lambda^2)$ (cf. Theorem 4). If in addition $\tilde f_t=(1-t)f_0+tf_1$ remains a symplectic form for any $t\in[0,1]$ then we can define (cf. Theorem 1) $u\in C^1([0,1];C^{l,\alpha}(\bar\Omega,\Lambda^1))$, which we identify with a vector field and $\varphi:[0,1]\times\bar\Omega\to\bar\Omega$ so that

$$u_t\lrcorner f_t=A,\quad \text{and}\quad \begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t=u_t\circ\varphi_t & t\in[0,1]\\ \varphi_0=\mathrm{id}\,.\end{cases}$$

Consequently, for any $t\in[0,1]$, φ_t is a diffeomorphism from Ω onto Ω and $\varphi_t^*(f_t)=f_0$ in Ω.

Returning to a general strictly convex smooth c that satisfies growth conditions such as (7), the existence of a minimizer A is obtained by the standard method of the calculus of variation (cf. Theorem 4). Optimal regularity properties of A is a harder task to establish in general. Setting $q=p/(p-1)$, one identifies the dual problem of (P), obtained by maximizing over the set of $h\in W^{1,q}(\Omega;\Lambda^k)$,

$$\mathcal{D}(h):=\int_\Omega\Big(\langle f_1-f_0;h\rangle-c^*(\delta h)\Big)\,\mathrm{d}x.$$

A maximum is readily obtained (cf. Theorem 6) in this problem that we denote by (D). We discuss also the case where $c(A)=|A|$, the linear growth case. We obtain a duality result in weaker spaces (cf. Theorem 12).

2. Notation and definition

For simplicity, throughout the manuscript, $\Omega\subset\mathbb{R}^n$ is assumed to be an open contractible smooth set and ν denote the outward unit normal to $\partial\Omega$. Let $1\le k\le n$ be an integer. We assume that $p,q\in(1,\infty)$ are conjugate of each other in the sense that $p+q=pq$. We refer to [3] for this section and adopt the following notations. First, if $u\in\Lambda^1(\mathbb{R}^n)$ and $f\in\Lambda^k(\mathbb{R}^n)$, then $u\lrcorner f$ is the interior product of f with u. If $\varphi\in C^1(\overline{\Omega};\mathbb{R}^n)$, then $\varphi^*(f)$ is the pullback of f by φ. Recall that for $u\in\Lambda^1(\mathbb{R}^n)$, $f\in\Lambda^k(\mathbb{R}^n)$ and $h\in\Lambda^{k+1}(\mathbb{R}^n)$, we have $\langle u\wedge f;h\rangle=\langle f;u\lrcorner h\rangle$.

We now give a weak formulation to the notion of closedness as well as its dual counterpart. Let $1\le k\le n-1$ be an integer, $f\in L^1(\Omega;\Lambda^k)$.

(i) When we write $\mathrm{d}f=0$ (resp. $\delta f=0$) in the weak sense, we mean that

$$\int_\Omega\langle f;\delta h\rangle=0\quad\forall\quad h\in C_c^\infty\left(\Omega;\Lambda^{k+1}\right)\qquad\left(\text{resp.}\quad\int_\Omega\langle f;\mathrm{d}h\rangle=0\,\forall\,h\in C_c^\infty\left(\Omega;\Lambda^{k-1}\right)\right).$$

(ii) Similarly if we want to express in the weak sense

$$\text{(i)}\begin{cases}\mathrm{d}f=0 & \text{in }\Omega\\ \nu\wedge f=0 & \text{on }\partial\Omega\end{cases}\quad\left(\text{resp.}\quad\text{(ii)}\begin{cases}\delta f=0 & \text{in }\Omega\\ \nu\lrcorner f=0 & \text{on }\partial\Omega\end{cases}\right),\tag{4}$$

we write $\int_\Omega\langle f;\delta h\rangle=0\quad\forall\quad h\in C^\infty(\overline{\Omega};\Lambda^{k+1})$ (resp. $\int_\Omega\langle f;\mathrm{d}h\rangle=0\quad\forall\quad h\in C^\infty(\overline{\Omega};\Lambda^{k-1})$).

We will often use the following results in [3]: Theorem 6.5, the regularity result in Theorem 7.2, the classical integration by parts in Theorem 3.28, the particular version of Gaffney inequality in Theorem 5.21, and the remark following it.

Definition 2. Let $1\le k\le n-1$, and $f\in L^p(\Omega;\Lambda^k)$ be such that (4) (i) holds. We say that $A\in L^1(\Omega;\Lambda^{k-1})$ satisfies in the weak sense (3), and we write $A\in\mathcal{C}(f)$, if

$$\int_\Omega\langle A;\delta h\rangle=\int_\Omega\langle f;h\rangle\quad\text{for every }h\in C^\infty\left(\overline{\Omega};\Lambda^{k-1}\right).\tag{5}$$

Remark 3. (i) Note that $\mathcal{C}(f_1-f_0)$ is not empty. Indeed, combining (4) and Theorem 7.2 in [3], there exists $F\in W^{1,p}(\Omega;\Lambda^{k-1})$ such that $F\in\mathcal{C}(f_1-f_0)$ and $\delta F=0$.

(ii) Note that, when $k=1$ the minimization problem (P) is trivial since, noticing that d is here the gradient operator, $\mathcal{C}(f_1-f_0)=\{F\}$.

(iii) When $k=n$ the condition (4) has to be replaced by

$$\int_\Omega(f_1-f_0)=0.\tag{6}$$

Indeed (6) insures that the set $\mathcal{C}(f_1-f_0)$ is not empty (see, e.g., Theorem 7.2 in [3]).

3. The superlinear case

Let $\gamma_1, \cdots, \gamma_4 > 0$ and let $c : \Lambda^{k-1}(\mathbb{R}^n) \to \mathbb{R}_+$ be a C^1, strictly convex satisfying

$$\gamma_1 |A|^p - \gamma_2 \le c(A) \le \gamma_3 |A|^p + \gamma_4, \tag{7}$$

The following properties are easily derived (cf., e.g., Chapter 2 in [4]): if c^* denotes the Legendre transform of c, then $c^* \in C^1$ and there exist constants $\beta > 0$, $\alpha_1, \cdots, \alpha_4 > 0$ such that

$$\alpha_1 \left|A^*\right|^q - \alpha_2 \le c^*\left(A^*\right) \le \alpha_3 \left|A^*\right|^q + \alpha_4 \tag{8}$$

and

$$|\nabla c(A)| \le \beta\left(|A|^{p-1} + 1\right) \quad \text{and} \quad \left|\nabla c\left(A^*\right)\right| \le \beta\left(\left|A^*\right|^{q-1} + 1\right). \tag{9}$$

Let $1 \le k \le n-1$, $f_0, f_1 \in L^p\left(\Omega; \Lambda^k\right)$ be two k-forms such that, in the weak sense

$$f := f_1 - f_0 \quad \text{satisfies (4) (i) and} \quad \mathrm{d}f_0 = \mathrm{d}f_1 = 0 \quad \text{in} \quad \Omega. \tag{10}$$

We are mostly interested in the *symplectic* case, which means that $k = 2$ (but most of this paper will work for any k), $n = 2m$ and f_0 and f_1 satisfy, in addition to the previous hypotheses, $\operatorname{rank}[f_0] = \operatorname{rank}[f_1] = 2m$. The other relevant, and by now classical, problem is the case of *volume* forms where $k = n$ and $f_0 \cdot f_1 > 0$ in $\overline{\Omega}$, where we have identified the n-forms with scalar functions. Note that in this case the conditions (10) are automatically fulfilled. They have to be replaced by (6).

3.1. Existence of a minimizer

Theorem 4. *If $1 \le k \le n-1$ then there exists a unique minimizer $\overline{A} \in L^p\left(\Omega; \Lambda^{k-1}\right)$ of (P).*

(i) It satisfies in the weak sense

$$\delta(\nabla c(\overline{A})) = 0 \quad \text{in } \Omega. \tag{11}$$

(ii) If we further assume that $c(A) = \frac{1}{2}|A|^2$, then $\overline{A}$ has the optimal regularity; namely, let l be an integer, $0 < \alpha < 1$ and $1 < r < \infty$, then

$$\overline{A} \in \begin{cases} C^{l+1,\alpha}\left(\overline{\Omega}; \Lambda^k\right) & \text{if } f_1 - f_0 \in C^{l,\alpha}\left(\overline{\Omega}; \Lambda^k\right) \\ W^{l+1,r}\left(\Omega; \Lambda^k\right) & \text{if } f_1 - f_0 \in W^{l,r}\left(\Omega; \Lambda^k\right). \end{cases} \tag{12}$$

Proof. *Step 1.* Existence and uniqueness of a minimizer in (P) is given by standard methods of the calculus of variations (cf., e.g., [4]). Indeed, the growth condition (7) and the convexity of c ensures that $A \to \int_\Omega c(A)\,\mathrm{d}x$ is weakly lower semi-continuous on $L^p(\Omega; \Lambda^{k-1})$ and its sub-level subsets are weakly compact. By Remark 3, $L^p(\Omega; \Lambda^{k-1}) \cap \mathcal{C}(f_1 - f_0) \neq \emptyset$. Furthermore, the latter set is weakly closed. Hence, (P) has a minimizer $\bar{A}$ over $\mathcal{C}(f_1 - f_0)$, which turns out to be in $L^p(\Omega; \Lambda^{k-1}) \cap \mathcal{C}(f_1 - f_0)$. The strict convexity of c ensures uniqueness of the minimizer.

Step 2. Let $h \in C_0^\infty\left(\Omega; \Lambda^{k-2}\right)$. Then $\overline{A} + \epsilon\,\mathrm{d}h \in \mathcal{C}(f_1 - f_0)$. The growth condition on $|\nabla c|$ in (9) ensures that the real valued function $\epsilon \to \int_\Omega c\left(\overline{A} + \epsilon\,\mathrm{d}h\right)$ is differentiable at 0. Since it achieves its minimum there, its derivative must vanish, which is precisely (11).

Step 3. We assume now that $c(A) = \frac{1}{2}|A|^2$ and prove (ii) only for Hölder spaces, since the proof in the other case is similar. By Theorem 7.2 [3], there exists $\bar{F} \in C^{l+1,\alpha}\left(\overline{\Omega}; \Lambda^{k-1}\right)$ such that $\bar{F} \in \mathcal{C}(f_1 - f_0)$ and $\delta\bar{F} = 0$ in Ω. We use (i) to conclude that $\mathrm{d}(\bar{F} - \bar{A}) = 0$ in Ω, $\delta(\bar{F} - \bar{A}) = 0$ in Ω and $\nu \wedge (\bar{F} - \bar{A}) = 0$ on $\partial\Omega$. Hence, by Theorem 6.5 [3], $\bar{F} = \overline{A}$, which concludes the proof. □

Remark 5. (i) When $c(A) = \frac{1}{p}|A|^p$ with $1 < p < 2$, we conjecture that $\overline{A} \in C^{0,\alpha}$, for some $\alpha > 0$, is in general the best regularity that can be expected. Indeed, it is proven in [8] that when $q \neq 2$, the solution to

$$\mathrm{d}\left(\delta\bar{h}\left|\delta\bar{h}\right|^{q-2}\right) = 0$$

satisfies $\bar{h} \in C^{0,\alpha}$ locally for some $\alpha > 0$. One can anticipate that it should be possible to extend this result to the non-zero right-hand side $f_1 - f_0$. Note also that $C^{0,\alpha}$ is, in general, the optimal regularity for $\delta\bar{h}$ when the system of equations reduces to the so-called q-Laplacian scalar equation.

(ii) The same analysis is valid when $k = n$ under the natural hypothesis (6).

Theorem 6. *The maximum of $\mathcal{D}$ over $\{h \in W^{1,q}(\Omega, \Lambda^k) : |\delta h| \le 1\}$ is achieved at $\bar{h}$ such that $\nabla c(\bar{A}) = \delta\bar{h}$ and it can moreover be assumed to verify $\mathrm{d}\bar{h} = 0$ in Ω and $\nu \wedge \bar{h} = 0$ on $\partial\Omega$. Furthermore, (P) and (D) are dual of each other.*

Proof. Since $\bar{A} \in L^p(\Omega; \Lambda^{k-1})$, the growth condition on $|\nabla c|$ in (9) and that on c in (7) imply $\nabla c(\bar{A}) \in L^q(\Omega; \Lambda^{k-1})$. We use (11) and Theorem 7.2 [3] to find $\bar{h} \in W^{1,q}(\Omega, \Lambda^k)$ such that $\nabla c(\bar{A}) = \delta\bar{h}$ in Ω, $d\bar{h} = 0$ in Ω and $\nu \wedge \bar{h} = 0$ on $\partial\Omega$.

Let $h \in W^{1,q}(\Omega, \Lambda^k)$ and $A \in \mathcal{C}(f_1 - f_0)$. We first use that c and c^* are Legendre transform of each other, we then use the fact that $A \in \mathcal{C}(f_1 - f_0)$ to obtain

$$\int_\Omega \big(c(A) + c^*(\delta h)\big)\mathrm{d}x \geq \int_\Omega \langle A; \delta h\rangle \,\mathrm{d}x = \int_\Omega \langle f_1 - f_0; h\rangle \,\mathrm{d}x. \tag{13}$$

The inequality in (13) becomes an equality if $(A, \delta h) = (\bar{A}, \delta\bar{h})$. Rearranging, we have proven that $I(A) > \mathcal{D}(h)$ and equality holds if $\nabla c(\bar{A}) = \delta\bar{h}$. □

Definition 7. For $f \in \mathcal{C}(0)$ and f_0, f_1 as above, we define

$$|f|_p = \inf_{A \in \mathcal{C}(f)} \left(\int_\Omega |A|^p\right)^{1/p}, \quad M_p(f_0, f_1) = |f_1 - f_0|_p .$$

Recall that $\mathcal{C}(f_1 - f_0)$ is the set of $(k-1)$-forms $A \in L^1\left(\Omega; \Lambda^{k-1}\right)$ verifying, in the weak sense,

$$\mathrm{d}A + f_1 - f_0 = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge A = 0 \text{ on } \partial\Omega.$$

The first claim in Proposition 8 implies the second one. When $p = 1$, $\mathcal{C}(f)$ has to be replaced by the set of currents (cf. Section 4).

Proposition 8 *(Metrics for k-forms). Let $1 \leq p < \infty$. Then $|\cdot|_p$ is a norm and $M_p(\cdot,\cdot)$ is a distance.*

Remark 9. (i) When $1 < p < \infty$, then there exists a unique geodesic of M_p of minimal length connecting f_0 to f_1. It is independent of p and is given by $(1-t)f_0 + tf_1$.

(ii) When $k = n$, M_2 has been studied by Brenier [2] and M_1 is the Monge–Kantorovich metric [1,7].

4. The case of linear growth

Here, $f_0, f_1 \in L^p\left(\Omega; \Lambda^k\right)$ are still two k-forms such that (10) holds in the weak sense. In this section, we plan to replace the strictly convex smooth super linear cost $c(A)$ of the previous section by the "linear cost" $|A|$. In that case, we expect (1) to have multiple solutions. We postpone the study of the question, which is to characterize the optimal paths $(\bar{f}, \bar{A})$ such that $\bar{f} \not\equiv (1-t)f_0 + tf_1$, to [5].

Definition 10. A $(k-1)$-current A on $\overline{\Omega}$ is a linear form on $C_c\left(\mathbb{R}^n; \Lambda^{k-1}\right)$ whose support is contained in $\overline{\Omega}$ and whose total mass is finite. By the Riesz representation theorem, there exists a collection of $\binom{n}{k-1}$ signed Radon measures $A_{i_1\cdots i_{k-1}}$, $1 \leq i_1 < \cdots < i_{k-1} \leq n$, supported by $\overline{\Omega}$ with finite total mass that represents A in the following sense:

$$A(f) = \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq n} \int_{\overline{\Omega}} f_{i_1\cdots i_{k-1}} A_{i_1\cdots i_{k-1}}(\mathrm{d}x) =: \int_{\overline{\Omega}} \langle A(\mathrm{d}x); f\rangle,$$

when

$$f = \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq n} f_{i_1\cdots i_{k-1}} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_{k-1}} \in C_c\left(\mathbb{R}^n; \Lambda^{k-1}\right).$$

Define

$$||A|| := \sup_f \left\{|A(f)| \,:\, f \in C_c\left(\mathbb{R}^n\right) : \|f\|_{L^\infty} \leq 1\right\} = \int_{\bar{\Omega}} |A|. \tag{14}$$

Definition 11. The set $\mathcal{C}^*(f_1 - f_0)$ is the set of $(k-1)$-currents A on $\overline{\Omega}$ such that

$$\int_{\overline{\Omega}} \langle A(\mathrm{d}x); \delta h\rangle = \int_\Omega \langle f_1 - f_0; h\rangle \quad \text{for every } h \in C^1\left(\overline{\Omega}; \Lambda^k\right). \tag{15}$$

We have $\mathcal{C}(f_1 - f_0) \subset \mathcal{C}^*(f_1 - f_0)$ and so, by Remark 3 (i), theses sets are not empty. We define $\mathcal{F}_\infty$ to be the set of $h \in \cap_{s \geq 1} W^{1,s}\left(\Omega; \Lambda^k\right)$ such that $\|\delta h\|_{L^\infty} \leq 1$. We set

$$I_1^*(A) = ||A|| : A \in \mathcal{C}^*(f_1 - f_0), \quad \text{and} \quad D_\infty(h) = \int_\Omega \langle f_1 - f_0; h \rangle, \; h \in \mathcal{F}_\infty.$$

The problem at hand, which we denote by (P_1^*), consists in minimizing I_1^* over $\mathcal{C}^*(f_1 - f_0)$. We denote by (D_∞) the problem of maximizing D_∞ over $\mathcal{F}_\infty$.

Let $r \in (1, p)$ and $r' = r/(r-1)$ be its conjugate exponent. Since $f_0, f_1 \in L^r(\Omega; \Lambda^k)$ we can apply the results of Section 3 to $c(A) = |A|^r/r$ and denote by A_r the unique minimizer of (P) and by h_r the unique maximizer of (D).

Theorem 12. *(i) Up to a subsequence, $(A_r)_r$ converges weak ⋆ to some $A_1^* \in C^*(f_1 - f_0)$ and $(h_r)_r$ converges weakly to some h_∞ in $W^{1,s}$, for every $s \in (1, \infty)$, as r tends to 1. Moreover $\|\delta h_\infty\|_{L^\infty} \le 1$.*

(ii) A_1^* minimizes (P_1^*), h_∞ maximizes (D_∞) and duality holds, i.e.

$$I_1^*(A_1^*) = \inf(P_1^*) = \sup(D_\infty) = D_\infty(h_\infty).$$

Proof. *Step 1.* Let $F \in W^{1,p}(\Omega; \Lambda^{k-1})$ be given by Remark 3. For $r < p$, we first use Hölder inequality, then Theorem 6 to obtain $A_r|A_r|^{r-2} = \delta h_r$ and the minimality property of A_r to obtain

$$||F||_{L^r}^r \le ||F||_{L^p}^r |\Omega|^{1-\frac{r}{p}}, \quad ||\delta h_r||_{L^{r'}}^{r'} = ||A_r||_{L^r}^r, \quad ||A_r||_{L^r} \le ||F||_{L^r}. \tag{16}$$

The first and last inequalities in (16) prove that $\{||A_r||_{L^r} : r \in (1, p)\}$ and so, $\{||A_r||_{L^1} : r \in (1, p)\}$ are bounded by a constant C. Thus, up to a subsequence, $(A_r)_r$ converges narrowly to a $(k-1)$-current A_1^* on $\overline{\Omega}$. We conclude that $A_1^* \in C^*(f_1 - f_0)$ by using the fact that since $A_r \in \mathcal{C}(f_1 - f_0)$, we have for any $h \in C^1(\bar{\Omega}; \Lambda^k)$

$$\int_\Omega \langle f_1 - f_0; h \rangle = \lim_{r \to 1} \int_\Omega \langle A_r; \delta h \rangle = \int_{\overline{\Omega}} \langle A_1^*(\mathrm{d}x); \delta h \rangle.$$

Step 2. If $s \le r'$ then by Hölder inequality $||\delta h_r||_{L^s} \le ||\delta h_r||_{L^{r'}} |\Omega|^{\frac{1}{s} - \frac{1}{r'}}$. This, together with (16) implies

$$||\delta h_r||_{L^s} \le ||F||_{L^p}^{\frac{r}{r'}} |\Omega|^{\frac{1}{s} - \frac{r-1}{p}}. \tag{17}$$

Hence, $\{||\delta h_r||_{L^s}\}_r$ is bounded by a constant C_s depending on s but independent of $r < s/(s-1)$. Since $\mathrm{d}h_r = 0$ in Ω and $\nu \wedge h_r = 0$ on $\partial\Omega$, Theorem 5.21 [3] yields that $\{h_r\}_r$ is weakly pre-compact in $W^{1,s}$. Hence, up to a subsequence, $\{h_r\}_r$ converges to some h_∞ weakly in $W^{1,s}$. By a diagonal sequence argument, we can choose a common subsequence for any $s \in \{n+1, n+2, \cdots\}$ to obtain that h_∞ is independent of s. The Sobolev embedding theorem yields that up to a subsequence $(h_r)_r$ converges uniformly to h_∞. Letting r tend to 1 and then s tend to ∞ in (17) we have $||\delta h_\infty||_{L^\infty} \le 1$. These show that (i) holds.

Step 3. The proof of the fact that the graph of I_1^* is above that of D_∞ can be given as in (13). We use first the duality $(P) = (D)$ for $c(A) = |A|^r/r$ and then the second identity in (16) to obtain that $\int_\Omega \langle f_1 - f_0; h_r \rangle \mathrm{d}x = ||A_r||_{L^r}^r$. Thus, by the weak lower semi-continuity of the total variations,

$$\int_{\overline{\Omega}} |A_1^*|(\mathrm{d}x) \le \underline{\lim}_{r \to 1^+} \int_\Omega |A_r| \le \underline{\lim}_{r \to 1^+} ||A_r||_r |\Omega|^{\frac{1}{r'}} = \underline{\lim}_{r \to 1^+} \left(\int_\Omega \langle f_1 - f_0; h_r \rangle \mathrm{d}x \right)^{\frac{1}{r}} |\Omega|^{\frac{1}{r'}} = \int_\Omega \langle f_1 - f_0; h_\infty \rangle \mathrm{d}x. \tag{18}$$

Thus, since the graph of I_1^* is above that of D_∞ and (18) reads off $D_\infty(h_\infty) \ge I_1^*(A_1)$, then (ii) holds. □

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References

[1] L. Ambrosio, N. Gigli, G. Savaré, Gradient Flows in Metric Spaces and in the Space of Probability Measures, Lectures in Mathematics, ETH Verlag/Birkhäuser, Zürich/Basel, Switzerland, 2005.
[2] Y. Brenier, Extended Monge–Kantorovich Mass Transfer Problem, Lecture Notes in Mathematics, vol. 1813, Springer, 2003, pp. 91–122.
[3] G. Csato, B. Dacorogna, O. Kneuss, The Pullback Equation for Differential Forms, Birkhaüser, 2012.
[4] B. Dacorogna, Direct Methods in the Calculus of Variations, second edition, Springer-Verlag, 2007.
[5] B. Dacorogna, W. Gangbo, O. Kneuss, Optimal transport of closed differential forms, preprint.
[6] B. Dacorogna, W. Gangbo, O. Kneuss, Symplectic decomposition, Darboux theorem and ellipticity, preprint.
[7] L.C. Evans, W. Gangbo, Differential equations methods for the Monge–Kantorovich mass transfer problem, Mem. Amr. Math. Soc. 137 (653) (1999), 66 p.
[8] K. Uhlenbeck, Regularity for a class of non-linear elliptic system, Acta Math. 138 (1977) 219–240.