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Mathematical analysis

Some simple conditions for univalence

Quelques conditions simples pour l'univalence

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ABSTRACT

We apply Ozaki–Umezawa's lemma on functions that are convex in one direction to find some sufficient conditions for univalence and closeness to convexity.

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RÉSUMÉ

Nous appliquons le lemme de Ozaki et Umezawa sur les fonctions convexes dans une direction, afin de trouver des conditions suffisantes pour l'univalence et la presque convexité.

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1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and denote by $\mathcal{A}$ the class of analytic functions in $\mathbb{D}$ and normalized, i.e. $\mathcal{A} = \{f \in \mathcal{H} : f(0) = 0, f'(0) = 1\}$. We say that $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in the unit disk $\mathbb{D}$, written $f \prec g$ if and only if there exists an analytic function $w \in \mathcal{H}$ such that $|w(z)| \le |z|$ and $f(z) = g[w(z)]$ for $z \in \mathbb{D}$. Therefore $f \prec g$ in $\mathbb{D}$ implies $f(\mathbb{D}) \subset g(\mathbb{D})$. In particular if g is univalent in $\mathbb{D}$, then the Subordination Principle says that $f \prec g$ if and only if $f(0) = g(0)$ and $f(|z| < r) \subset g(|z| < r)$, for all $r \in (0, 1]$.

Let us recall the Ozaki–Umezawa's lemma [6,8].

Lemma 1.1. *Let $f(z) = z + a_2 z^2 + \cdots$ be analytic for $|z| \le 1$ and $f'(z) \neq 0$ on $|z| = 1$. If there holds the relation*

$$\int_0^{2\pi} \left| 1 + \Re\mathfrak{e} \left\{ \frac{zf''(z)}{f'(z)} \right\} \right| d\theta < 4\pi, \quad |z| = 1, \tag{1.1}$$

then $f(z)$ is convex in one direction and hence $f(z)$ is univalent in $|z| \le 1$.

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2. Main result

Theorem 2.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$. Assume that*

$$\left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| \le |1 + \mathfrak{Re}\{\alpha_0 z\}| \quad (z \in \mathbb{D}), \tag{2.1}$$

where $\alpha_0 = 1/\cos t_0$ and t_0 is the positive root of the equation

$$\tan t = t + \pi/2, \quad 0 < t < \pi/2. \tag{2.2}$$

Then $f(z)$ is univalent in $\mathbb{D}$. Note that and $2.909 < \alpha_0 < 2.992$.

Proof. Applying [1] and [4], we have from (2.1)

$$\begin{aligned}\int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| \mathrm{d}\theta &= \int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right| \mathrm{d}\theta \\ &\le \int_0^{2\pi} \left|1 + \mathfrak{Re}\left\{\alpha_0 re^{i\theta}\right\}\right| \mathrm{d}\theta,\end{aligned}$$

where $0 < r < 1$. Letting $r \to 1$, we have

$$\begin{aligned}&\int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| \mathrm{d}\theta \\ &\le \int_0^{2\pi} |1 + \alpha_0 \cos\theta| \, \mathrm{d}\theta.\end{aligned}$$

We have

$$\cos^{-1}(-1/\alpha_0) = \pi - \cos^{-1}(1/\alpha_0). \tag{2.3}$$

Thus we obtain

$$\begin{aligned}&\int_0^{2\pi} |1 + \alpha_0 \cos\theta| \, \mathrm{d}\theta \\ &= 2\int_0^{\cos^{-1}(-1/\alpha_0)} (1 + \alpha_0 \cos\theta) \, \mathrm{d}\theta - \int_{\cos^{-1}(-1/\alpha_0)}^{2\pi - \cos^{-1}(-1/\alpha_0)} (1 + \alpha_0 \cos\theta) \, \mathrm{d}\theta \\ &= 2\left[\theta + \alpha_0 \sin\theta\right]_0^{\cos^{-1}(-1/\alpha_0)} - \left[\theta + \alpha_0 \sin\theta\right]_{\cos^{-1}(-1/\alpha_0)}^{2\pi - \cos^{-1}(-1/\alpha_0)} \\ &= 2\left[\theta + \alpha_0 \sin\theta\right]_0^{\pi - \cos^{-1}(1/\alpha_0)} - \left[\theta + \alpha_0 \sin\theta\right]_{\pi - \cos^{-1}(1/\alpha_0)}^{\pi + \cos^{-1}(1/\alpha_0)} \\ &= 3\left[\pi - \cos^{-1}(1/\alpha_0) + \alpha_0 \sin\left\{\cos^{-1}(1/\alpha_0)\right\}\right] \\ &\quad - \left[\pi + \cos^{-1}(1/\alpha_0) - \alpha_0 \sin\left\{\cos^{-1}(1/\alpha_0)\right\}\right] \\ &= 4\pi + \left[-2\pi - 4\cos^{-1}(1/\alpha_0) + 4\alpha_0 \sin\left\{\cos^{-1}(1/\alpha_0)\right\}\right].\end{aligned} \tag{2.4}$$

Therefore, we will get the univalence of f in the unit disk by Ozaki–Umezawa's Lemma 1.1, whenever

$$-2\pi - 4\cos^{-1}(1/\alpha_0) + 4\alpha_0 \sin\left\{\cos^{-1}(1/\alpha_0)\right\} = 0. \tag{2.5}$$

If $t_0 = \cos^{-1}(1/\alpha_0)$, then (2.5) becomes

$$-2\pi - 4t_0 + 4\frac{1}{\cos t_0}\sin t_0 = 0,$$

which is assumed in (2.2). Note that $1.22 < t_0 < 1.23$. □

In the same way as Theorem 2.1 we can prove the following sufficient condition for convexity.

Theorem 2.2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$. Assume that*

$$\left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| \leq |1 + \mathfrak{Re}\{z\}| \quad (z \in \mathbb{D}), \tag{2.6}$$

then $f(z)$ is convex univalent in $\mathbb{D}$.

Proof. Applying the proof of Theorem 2.1, we have from (2.4)

$$\int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| d\theta \leq 4\pi + \left[-2\pi - 4\cos^{-1}(1/\alpha) + 4\alpha \sin\left\{\cos^{-1}(1/\alpha)\right\}\right],$$

for all $\alpha \in (-\infty - 1] \cup [1, +\infty)$. Putting $\alpha = 1$, we have

$$\int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| d\theta \leq 2\pi. \tag{2.7}$$

On the other hand, we have for $z = re^{i\theta}$

$$\begin{aligned} 2\pi &= \frac{1}{i} \int_{|z|=r} \left\{\frac{1}{z} + \frac{f''(z)}{f'(z)}\right\} dz \\ &= \int_0^{2\pi} \left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta. \end{aligned}$$

Hence

$$\int_0^{2\pi} \left\{1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right\} d\theta = 2\pi. \tag{2.8}$$

By (2.7) and (2.8) we have

$$\int_0^{2\pi} \left|1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right| d\theta \leq \int_0^{2\pi} \left\{1 + \mathfrak{Re}\frac{zf''(z)}{f'(z)}\right\} d\theta. \tag{2.9}$$

Therefore,

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq 0 \quad (z \in \mathbb{D}).$$

Hence $f(z)$ is convex univalent in $\mathbb{D}$ if we prove $1 + \mathfrak{Re}\left\{1 + zf''(z)/f'(z)\right\} \neq 0$ in $|z| < 1$. By the mathematical method of absurdity, if there exists a point $z_0 = r_0 \exp(i\theta_0)$, $0 < r_0 < 1$, $0 \leq \theta_0 < 2\pi$ for which $1 + \mathfrak{Re}\left\{1 + z_0 f''(z_0)/f'(z_0)\right\} = 0$, this shows that the image point, $1 + z_0 f''(z_0)/f'(z_0)$ is located on the imaginary axis of the w-plane. Let us consider the mapping of a very small domain $D : |z - z_0| < \delta$, where δ is sufficiently small and positive. Then the image domain of D by the mapping $w = 1 + zf''(z)/f'(z)$ must take negative real value, because the function $w = 1 + zf''(z)/f'(z)$ is a continuous function. This is a contradiction and it completes the proof. □

Umezawa in [8] proved that

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \sqrt{6} \quad (|z| \leq 1), \tag{2.10}$$

implies the univalence of $f(z)$ in $|z| \leq 1$. Notice also here that in [6] Ozaki proved that if $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is analytic in $\mathbb{D}$, with $f(z)f'(z)/z \neq 0$ there, and if either

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq -\frac{1}{2}$$

or

$$\mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\le\frac{3}{2} \tag{2.11}$$

holds throughout $\mathbb{D}$, then f is univalent and convex in at least one direction in $\mathbb{D}$. It has been generalized in [5,7]. The number $\sqrt{6}$ in (2.10), was improved to $3.05\ldots$ in [2]. Notice that the condition

$$1+\frac{zf''(z)}{f'(z)}\prec 1+\alpha z \quad (z\in\mathbb{D}),$$

$0\le\alpha<2.832\ldots$ is sufficient for starlikeness, [3, p. 273]. If $f\in\mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{e^{i\alpha}g(z)}\right\}>0,\ z\in\mathbb{D} \tag{2.12}$$

for some $g(z)\in\mathcal{S}^*$ and some $\alpha\in(-\pi/2,\pi/2)$, then $f(z)$ is said to be close to convex (with respect to $g(z)$) in $\mathbb{D}$ and denoted by $f(z)\in\mathcal{C}$. An univalent function $f\in\mathcal{S}$ belongs to $\mathcal{C}$ if and only if the complement E of the image-region $F=\{f(z):|z|<1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

Theorem 2.3. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in $\mathbb{D}$ and suppose that there exists a starlike function $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ for which*

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right\}\le\mathfrak{Re}\{\alpha z\} \quad (z\in\mathbb{D}), \tag{2.13}$$

where $0<\alpha<\pi/4$. Then $f(z)$ is close to convex in $\mathbb{D}$ with respect to $g(z)$.

Proof. It follows that

$$\begin{aligned}&\arg\left\{\frac{zf'(z)}{g(z)}\right\}-\arg\left\{\frac{z_0f'(z_0)}{g(z_0)}\right\}\\&=\mathfrak{Re}\int_{z_0}^{z} i\left\{\frac{(zf'(z))'}{zf'(z)}-\frac{g'(z)}{g(z)}\right\}dz\\&=\mathfrak{Re}\int_{\theta_0}^{\theta}\left\{\frac{z(zf'(z))'}{zf'(z)}-\frac{zg'(z)}{g(z)}\right\}d\theta,\end{aligned}$$

where $z=re^{i\theta}$, $0<r\le1$, $0\le\theta\le2\pi$ and $z_0=re^{i\theta_0}$. Since

$$\left(\frac{zf'(z)}{g(z)}\right)_{z=0}=1$$

and the image domain of $|z|\le1$ under the mapping $w(z)=zf'(z)/g(z)$ contains the point $z=1$, there exist points z_1,z_2 for which

$$\arg\left(\frac{z_if'(z_i)}{g(z_i)}\right)=0,\quad i=1,2. \tag{2.14}$$

Therefore, for each $z=e^{i\theta}$ we can find $z_i=e^{i\theta_i}$ such that $|\theta-\theta_i|\le\pi$ and (2.14) holds.

Letting $r\to1$ and applying (2.13), we have

$$\begin{aligned}&\left|\arg\frac{zf'(z)}{g(z)}\right|\\&\le\int_{\theta_i}^{\theta}\left|\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right\}\right|d\theta\\&\le\int_{\theta_i}^{\theta}|\mathfrak{Re}\{\alpha z\}|\,d\theta\end{aligned}$$

$$
\begin{aligned}
&\le \alpha \int_{\theta_i}^{\theta} |\cos\theta|\, \mathrm{d}\theta \\
&\le \alpha \int_{\theta_i}^{\theta_i+\pi} |\cos\theta|\, \mathrm{d}\theta \\
&\le 2\alpha \\
&< \pi/2.
\end{aligned}
$$

The maximum principle of harmonic functions shows that

$$\left|\arg \frac{zf'(z)}{g(z)}\right| < \frac{\pi}{2} \quad (z \in \mathbb{D}).$$

Therefore, f is close to convex with respect to g. This completes the proof of Theorem 2.3 □

Applying the same method as used in the proof of Theorem 2.3, we have the following corollaries.

Corollary 2.4. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and suppose that there exists a convex function $g(z) = zh'(z) = z + \sum_{n=2}^{\infty} b_n z^n$ for which*

$$\mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{zh''(z)}{h'(z)}\right\} \le \mathfrak{Re}\{\alpha z\} \quad (z \in \mathbb{D}), \tag{2.15}$$

where $0 < \alpha < \pi/4$. Then $f(z)$ is close to convex in $\mathbb{D}$ with respect to $g(z)$.

If $f \in \mathcal{A}$ satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f^{1-\beta}(z)h^{\beta}(z)}\right\} > 0, \; z \in \mathbb{D}$$

for some $h(z) \in \mathcal{S}^*$ and some $\beta \in (0, \infty)$, then $f(z)$ is said to be a Bazilevič function of type β and is denoted by $f(z) \in \mathcal{B}(\beta)$.

Taking $g(z) = f^{1-\beta}(z)h^{\beta}(z)$ in Theorem 2.3 we obtain the following result.

Corollary 2.5. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and suppose that there exists a starlike function $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$ for which*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - (1-\beta)\frac{zf'(z)}{f(z)} - \beta\frac{zh'(z)}{h(z)}\right\} \le \mathfrak{Re}\{\alpha z\} \quad (z \in \mathbb{D}), \tag{2.16}$$

where $0 < \alpha < \pi/4$. Then $f(z)$ is a Bazilevič function of type β and it is univalent in $|z| \le 1$.

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