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Optimal geometric estimates for fractional Sobolev capacities

*Estimées géométriques optimales pour les capacités fractionnelles de Sobolev*Jie Xiao¹

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ABSTRACT

This note discovers new sharp inequalities relating the fractional Sobolev capacity of a set to its standard volume and fractional perimeter, respectively, and consequently proves that the sharp fractional Sobolev inequality is equivalent to either the sharp fractional isocapacity inequality or the sharp fractional isoperimetric inequality.

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R É S U M É

Cette note révèle de nouvelles inégalités exactes mettant la capacité fractionnelle de Sobolev d'un ensemble en relation avec son volume standard et son périmètre fractionnel, respectivement, et démontre, par conséquence, que l'inégalité fractionnelle exacte de Sobolev est équivalente, soit à l'inégalité fractionnelle isocapacitaire exacte, soit à l'inégalité fractionnelle isopérimétrique exacte.

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Continuing from [19,20], this short article establishes not only two new optimal estimates linking the fractional Sobolev capacity of a set to its standard volume and fractional perimeter, but also shows that the sharp fractional Sobolev inequality, the sharp fractional isocapacity inequality and the sharp fractional isoperimetric inequality are mutually equivalent.

1. Fractional Sobolev capacities and their basic properties

Let $0 < \alpha < 1$ and C_0^∞ denote the class of all smooth functions with compact support in \mathbb{R}^n . Define the fractional Sobolev space (or the homogeneous $(\alpha, 1, 1)$ -Besov space) $\dot{\Lambda}_\alpha^{1,1}$ as the completion of all functions $f \in C_0^\infty$ with

$$\|f\|_{\dot{\Lambda}_\alpha^{1,1}} = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x+h) - f(x)| dx \right) \frac{dh}{|h|^{n+\alpha}}.$$

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Attached to $\dot{\Lambda}_\alpha^{1,1}$ is the following set-function:

$$\text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) = \inf \{ \|f\|_{\dot{\Lambda}_\alpha^{1,1}} : f \in C_0^\infty \ \& \ f \geq 1_K \} \quad \forall \text{ compact } K \subset \mathbb{R}^n.$$

Here and henceforth, 1_E stands for the indicator of a set $E \subset \mathbb{R}^n$. This definition is extended to any set $E \subset \mathbb{R}^n$ via

$$\text{cap}(E; \dot{\Lambda}_\alpha^{1,1}) = \inf_{\text{open } O \supset E} \text{cap}(O; \dot{\Lambda}_\alpha^{1,1}) = \inf_{\text{open } O \supset E} \left(\sup_{\text{compact } K \subset O} \text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) \right).$$

The number $\text{cap}(E; \dot{\Lambda}_\alpha^{1,1})$ is called the fractional Sobolev capacity (or the homogeneous end-point Besov capacity) of E ; see also [1–3,19,17]. Note that (cf. [15,16,4,5,11])

$$\lim_{\alpha \rightarrow 0} \alpha \|f\|_{\dot{\Lambda}_\alpha^{1,1}} = 2n\omega_n \|f\|_{L^1} \quad \& \quad \lim_{\alpha \rightarrow 1} (1 - \alpha) \|f\|_{\dot{\Lambda}_\alpha^{1,1}} = \tau_n \|\nabla f\|_{L^1}$$

where ω_n is the volume of the unit ball \mathbb{B}^n of \mathbb{R}^n and $\tau_n = \int_{\mathbb{S}^{n-1}} |\cos \theta| d\sigma$ with: \mathbb{S}^{n-1} being the unit sphere of \mathbb{R}^n ; θ being the angle deviation from the vertical direction; and $d\sigma$ being the standard area measure on \mathbb{S}^{n-1} . So, we have that for any compact $K \subset \mathbb{R}^n$,

$$\lim_{\alpha \rightarrow 0} \alpha \text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) = 2n\omega_n V(K) \quad \& \quad \lim_{\alpha \rightarrow 1} (1 - \alpha) \text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) = \tau_n \text{cap}(K; \dot{W}^{1,1})$$

where

$$V(K) = \int_K dx \quad \& \quad \text{cap}(K; \dot{W}^{1,1}) = \inf \{ \|\nabla f\|_{L^1} : f \in C_0^\infty \ \& \ f \geq 1_K \}$$

and

$$\text{cap}(E; \dot{W}^{1,1}) = \inf_{\text{open } O \supset E} \text{cap}(O; \dot{W}^{1,1}) = \inf_{\text{open } O \supset E} \left(\sup_{\text{compact } K \subset O} \text{cap}(K; \dot{W}^{1,1}) \right) \quad \forall \ E \subset \mathbb{R}^n.$$

Theorem 1. The nonnegative set-function $E \mapsto \text{cap}(E; \dot{\Lambda}_\alpha^{1,1})$ enjoys the four essential properties below:

- (i) *Homogeneity:* $\text{cap}(rE; \dot{\Lambda}_\alpha^{1,1}) = r^{n-\alpha} \text{cap}(E; \dot{\Lambda}_\alpha^{1,1}) \ \forall rE = \{rx : x \in E\} \subset \mathbb{R}^n \ \& \ r \in [0, \infty)$.
- (ii) *Monotonicity:* $E_1 \subset E_2 \subset \mathbb{R}^n \Rightarrow \text{cap}(E_1; \dot{\Lambda}_\alpha^{1,1}) \leq \text{cap}(E_2; \dot{\Lambda}_\alpha^{1,1})$.
- (iii) *Subadditivity:* $\text{cap}(K_1 \cup K_2; \dot{\Lambda}_\alpha^{1,1}) \leq \text{cap}(K_1; \dot{\Lambda}_\alpha^{1,1}) + \text{cap}(K_2; \dot{\Lambda}_\alpha^{1,1}) \ \forall \text{ compact sets } K_1, K_2 \subset \mathbb{R}^n$.
- (iv) *Downward monotone convergence:* $\lim_{j \rightarrow \infty} \text{cap}(K_j; \dot{\Lambda}_\alpha^{1,1}) = \text{cap}(\cap_{j=1}^\infty K_j; \dot{\Lambda}_\alpha^{1,1}) \ \forall \text{ sequence } \{K_j\}_{j=1}^\infty \text{ of compact subsets of } \mathbb{R}^n \text{ with } K_1 \supset K_2 \supset K_3 \supset \dots$.

Proof. (i) follows from $\|f(r \cdot)\|_{\dot{\Lambda}_\alpha^{1,1}} = r^{\alpha-n} \|f\|_{\dot{\Lambda}_\alpha^{1,1}} \ \forall r \in [0, \infty)$ and the definition of $\text{cap}(\cdot; \dot{\Lambda}_\alpha^{1,1})$.

(ii) follows from the definition of $\text{cap}(\cdot; \dot{\Lambda}_\alpha^{1,1})$.

(iii) follows from an elementary argument. Clearly, we may assume $\text{cap}(K_j; \dot{\Lambda}_\alpha^{1,1}) < \infty$ with $j = 1, 2$. For any $\epsilon > 0$ there are $f_1, f_2 \in C_0^\infty$ such that

$$f_j \geq 1_{K_j} \ \& \ \|f_j\|_{\dot{\Lambda}_\alpha^{1,1}} < \text{cap}(K_j; \dot{\Lambda}_\alpha^{1,1}) + \epsilon \quad \forall \ j = 1, 2.$$

Clearly, $f = \max\{f_1, f_2\} \in C_0^\infty$ enjoys

$$f \geq 1_{K_1 \cup K_2} \ \& \ |f(x) - f(y)| \leq \max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|\} \quad \forall \ x, y \in \mathbb{R}^n.$$

This in turn implies

$$\text{cap}(K_1 \cup K_2; \dot{\Lambda}_\alpha^{1,1}) \leq \|f\|_{\dot{\Lambda}_\alpha^{1,1}} \leq \|f_1\|_{\dot{\Lambda}_\alpha^{1,1}} + \|f_2\|_{\dot{\Lambda}_\alpha^{1,1}} \leq \text{cap}(K_1; \dot{\Lambda}_\alpha^{1,1}) + \text{cap}(K_2; \dot{\Lambda}_\alpha^{1,1}) + 2\epsilon,$$

whence giving the desired one.

(iv) follows from a careful treatment. Suppose $\{K_j\}_{j=1}^\infty$ is a decreasing sequence of compact subsets of \mathbb{R}^n . Then $K = \cap_{j=1}^\infty K_j$ is compact. Following the argument for [9, Theorem 2.2(iv)], for any $\epsilon \in (0, 1)$ there is a function $f \in C_0^\infty$ such that

$$f \geq 1_K \quad \& \quad \|f\|_{\dot{\Lambda}_\alpha^{1,1}} < \text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) + \epsilon.$$

Note that if j is sufficiently large then K_j is contained in the compact set $\{x \in \mathbb{R}^n : f(x) \geq 1 - \epsilon\}$. So, an application of (ii) and the definition of $\text{cap}(\cdot; \dot{\Lambda}_\alpha^{1,1})$ derives

$$\lim_{j \rightarrow \infty} \text{cap}(K_j; \dot{\Lambda}_\alpha^{1,1}) \leq \text{cap}(\{x \in \mathbb{R}^n : f(x) \geq 1 - \epsilon\}; \dot{\Lambda}_\alpha^{1,1}) \leq (1 - \epsilon)^{-1} \|f\|_{\dot{\Lambda}_\alpha^{1,1}} \leq \frac{\text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) + \epsilon}{1 - \epsilon}.$$

Upon letting $\epsilon \rightarrow 0$ and using (ii) again, we get

$$\text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) \leq \lim_{j \rightarrow \infty} \text{cap}(K_j; \dot{\Lambda}_\alpha^{1,1}) \leq \text{cap}(K; \dot{\Lambda}_\alpha^{1,1}),$$

as desired. \square

For any set $E \subset \mathbb{R}^n$, let $E^c = \mathbb{R}^n \setminus E$ and compute

$$\|1_E\|_{\dot{\Lambda}_\alpha^{1,1}} = 2 \int_E \int_{E^c} \frac{dx dy}{|x - y|^{n+\alpha}} \equiv 2P_\alpha(E)$$

whose half $P_\alpha(E)$ is called the fractional α -perimeter; see, e.g., [8,12]. Notice that

$$\lim_{\alpha \rightarrow 0} \alpha P_\alpha(E) = n\omega_n V(E) \quad \& \quad \lim_{\alpha \rightarrow 1} (1 - \alpha)P_\alpha(E) = 2^{-1}\tau_n P(E)$$

where $P(E)$ is the perimeter of E . So, we get an extension of [14, Lemma 2.2.5] from the limit $\alpha \rightarrow 1$ to the intermediate value $0 < \alpha < 1$ that connects the fractional Sobolev capacity and the fractional perimeter.

Theorem 2. *If K is a compact subset of \mathbb{R}^n , then*

$$\text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) = 2 \inf_{O \in \mathcal{O}^\infty(K)} P_\alpha(O)$$

where $\mathcal{O}^\infty(K)$ denotes the class of all open sets with C^∞ boundary that contain K .

Proof. Given a compact $K \subset \mathbb{R}^n$. On the one hand, if $f \in C_0^\infty$ and $f \geq 1_K$ then $K \subset \{x \in \mathbb{R}^n : f(x) > t\} \forall t \in (0, 1)$, and hence an application of the generalized co-area formula in [18] (cf. [19, Theorem 1.2] for another version of the co-area formula of dimension $n - \alpha$) gives

$$\|f\|_{\dot{\Lambda}_\alpha^{1,1}} = 2 \int_0^\infty P_\alpha(\{x \in \mathbb{R}^n : f(x) > t\}) dt \geq 2 \int_0^1 P_\alpha(\{x \in \mathbb{R}^n : f(x) > t\}) dt \geq 2 \inf_{O \in \mathcal{O}^\infty(K)} P_\alpha(O).$$

This, along with the definition of $\text{cap}(K; \dot{\Lambda}_\alpha^{1,1})$, implies $\text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) \geq 2 \inf_{O \in \mathcal{O}^\infty(K)} P_\alpha(O)$.

On the other hand, according to Theorem 1(ii) and [10, Theorem 3.1], we have

$$\text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) \leq \text{cap}(\bar{O}; \dot{\Lambda}_\alpha^{1,1}) \leq 2P_\alpha(O) \quad \forall O \in \mathcal{O}^\infty(K),$$

whence $\text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) \leq 2 \inf_{O \in \mathcal{O}^\infty(K)} P_\alpha(O)$. Therefore, the desired formula for $\text{cap}(K; \dot{\Lambda}_\alpha^{1,1})$ follows. \square

As an immediate consequence of Theorem 2, we have:

$$\lim_{\alpha \rightarrow 0} \alpha \text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) = 2n\omega_n V(K) \quad \& \quad \lim_{\alpha \rightarrow 1} (1 - \alpha) \text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) = \tau_n P(K) \quad \forall \text{ compact } K \subset \mathbb{R}^n.$$

2. Fractional Sobolev inequalities and their geometric forms

The next analytic–geometric assertion indicates that the fractional Sobolev capacity plays a decisive role in improving the fractional isoperimetric inequality [7, (4.2)].

Theorem 3. *Let $\kappa_{n,\alpha} = \omega_n^{\frac{n-\alpha}{n}} (2P_\alpha(\mathbb{B}^n))^{-1}$. Then:*

(i) *The analytic inequality*

$$\|f\|_{L^{\frac{n}{n-\alpha}}} \leq \kappa_{n,\alpha} \left(\int_0^\infty (\text{cap}(\{x \in \mathbb{R}^n : |f(x)| \geq t\}; \dot{\Lambda}_\alpha^{1,1}))^{\frac{n}{n-\alpha}} dt^{\frac{n}{n-\alpha}} \right)^{\frac{n-\alpha}{n}} \quad \forall f \in C_0^\infty \tag{1}$$

is equivalent to the geometric inequality

$$(V(O))^{\frac{n-\alpha}{n}} \leq \kappa_{n,\alpha} \text{cap}(\bar{O}; \dot{\Lambda}_\alpha^{1,1}) \quad \forall \text{ bounded domain } O \subset \mathbb{R}^n \text{ with } C^\infty \text{ boundary } \partial O. \tag{2}$$

Moreover, both (1) and (2) are true and sharp.

(ii) *The analytic inequality*

$$\left(\int_0^\infty (\text{cap}(\{x \in \mathbb{R}^n : |f(x)| \geq t\}; \dot{\Lambda}_\alpha^{1,1}))^{\frac{n}{n-\alpha}} dt^{\frac{n}{n-\alpha}} \right)^{\frac{n-\alpha}{n}} \leq \|f\|_{\dot{\Lambda}_\alpha^{1,1}} \quad \forall f \in C_0^\infty \tag{3}$$

is equivalent to the geometric inequality

$$\text{cap}(\bar{O}; \dot{\Lambda}_\alpha^{1,1}) \leq 2P_\alpha(O) \quad \forall \text{ bounded domain } O \subset \mathbb{R}^n \text{ with } C^\infty \text{ boundary } \partial O. \tag{4}$$

Moreover, both (3) and (4) are true and sharp.

Proof. (i) Suppose (2) is valid. For any C_0^∞ function f , set $O_t(f) = \{x \in \mathbb{R}^n : |f(x)| > t\} \forall t \geq 0$. Then an application of (2) to $O_t(f)$ yields

$$\|f\|_{\frac{n}{n-\alpha}} = \left(\int_0^\infty V(O_t(f)) dt^{\frac{n}{n-\alpha}} \right)^{\frac{n-\alpha}{n}} \leq \kappa_{n,\alpha} \left(\int_0^\infty (\text{cap}(\overline{O_t(f)}; \dot{\Lambda}_\alpha^{1,1}))^{\frac{n}{n-\alpha}} dt^{\frac{n}{n-\alpha}} \right)^{\frac{n-\alpha}{n}},$$

deriving (1). Conversely, suppose (1) is valid. For any bounded domain $O \subset \mathbb{R}^n$ with C^∞ boundary ∂O , the Euclidean distance $\text{dist}(x, E)$ of a point x to a set E , and $0 < \epsilon < 1$, let

$$f_\epsilon(x) = \begin{cases} 1 - \epsilon^{-1} \text{dist}(x, \bar{O}) & \text{as } \text{dist}(x, \bar{O}) < \epsilon \\ 0 & \text{as } \text{dist}(x, \bar{O}) \geq \epsilon. \end{cases}$$

Then the inequality in (1) is true for f_ϵ . Consequently, via setting $O_\epsilon = \{x \in \mathbb{R}^n : \text{dist}(x, \bar{O}) < \epsilon\}$ & $\epsilon \rightarrow 0$ and using Theorem 1(iv), we gain:

$$(V(O))^{\frac{n-\alpha}{n}} \leftarrow \|f_\epsilon\|_{L^{\frac{n}{n-\alpha}}} \leq \kappa_{n,\alpha} \left(\int_0^\infty (\text{cap}(\overline{O_t(f_\epsilon)}; \dot{\Lambda}_\alpha^{1,1}))^{\frac{n}{n-\alpha}} dt^{\frac{n}{n-\alpha}} \right)^{\frac{n-\alpha}{n}} \leq \kappa_{n,\alpha} \text{cap}(\bar{O}_\epsilon; \dot{\Lambda}_\alpha^{1,1}) \rightarrow \kappa_{n,\alpha} \text{cap}(\bar{O}; \dot{\Lambda}_\alpha^{1,1}).$$

This proves (2). Moreover, the truth and the sharpness of (2) (and hence (1) via the just-checked equivalence) follow from the definition of $\text{cap}(O; \dot{\Lambda}_\alpha^{1,1})$ and the sharp fractional Sobolev inequality on [7, Theorem 4.1: $p = 1$]:

$$\|f\|_{L^{\frac{n}{n-\alpha}}} \leq \kappa_{n,\alpha} \|f\|_{\dot{\Lambda}_\alpha^{1,1}} \quad \forall f \in C_0^\infty.$$

(ii) Suppose (3) is valid. Given $\epsilon \in (0, 1)$ and a bounded domain $O \subset \mathbb{R}^n$ with C^∞ boundary ∂O , select again O_ϵ & f_ϵ as above. Then Theorem 1(ii) and certain elementary estimates with $\lim_{\epsilon \rightarrow 0} V(O_\epsilon \setminus O) = 0$ and $\alpha \in (0, 1)$ give

$$\text{cap}(\bar{O}; \dot{\Lambda}_\alpha^{1,1}) \leq \left(\int_0^1 (\text{cap}(\overline{O_t(f_\epsilon)}; \dot{\Lambda}_\alpha^{1,1}))^{\frac{n}{n-\alpha}} dt^{\frac{n}{n-\alpha}} \right)^{\frac{n-\alpha}{n}} \leq \|f_\epsilon\|_{\dot{\Lambda}_\alpha^{1,1}} \rightarrow \|1_O\|_{\dot{\Lambda}_\alpha^{1,1}} = 2P_\alpha(O).$$

In other words, (4) is true. Conversely, suppose (4) is valid. Upon noticing that for any C_0^∞ function f with $O_t(f)$ being as above, the function $t \mapsto \text{cap}(\overline{O_t(f)}; \dot{\Lambda}_\alpha^{1,1})$ decreases on $[0, \infty)$ (thanks to Theorem 1(ii)), we have

$$t^{\frac{\alpha}{n-\alpha}} (\text{cap}(\overline{O_t(f)}; \dot{\Lambda}_\alpha^{1,1}))^{\frac{n}{n-\alpha}} \leq \left(\frac{n-\alpha}{n} \right) \frac{d}{dt} \left(\int_0^t \text{cap}(\overline{O_s(f)}; \dot{\Lambda}_\alpha^{1,1}) ds \right)^{\frac{n}{n-\alpha}},$$

whence finding, along with Theorem 1(ii), (4), Theorem 2 and the previously-cited co-area formula,

$$\left(\int_0^\infty (\text{cap}(\overline{O_t(f)}; \dot{\Lambda}_\alpha^{1,1}))^{\frac{n}{n-\alpha}} dt^{\frac{n}{n-\alpha}} \right)^{\frac{n-\alpha}{n}} \leq \int_0^\infty \text{cap}(\overline{O_t(f)}; \dot{\Lambda}_\alpha^{1,1}) dt \leq 2 \int_0^\infty P_\alpha(O_t(f)) dt = \|f\|_{\dot{\Lambda}_\alpha^{1,1}}.$$

So, (3) holds. Moreover, the truth of (4) (and hence (3) via the above equivalence) follows from Theorem 2. In fact, if (4) were not sharp, then an application of (2) would derive that the sharp fractional isoperimetric inequality (cf. [7, (4.2)])

$$(V(E))^{\frac{n-\alpha}{n}} \leq 2\kappa_{n,\alpha} P_\alpha(E) \quad \forall E \subset \mathbb{R}^n$$

is not sharp, thereby reaching a contradiction. Thus, (4) is sharp, and so is (3). \square

Theorem 3 comes actually from splitting both the sharp fractional Sobolev inequality and the fractional isoperimetric inequality whose equivalence (optimizing [10, Theorem 1.1] under $G = \mathbb{R}^n$) is described below.

Theorem 4. *The following three optimal statements are equivalent:*

- (i) *The fractional Sobolev inequality $\|f\|_{L^{\frac{n}{n-\alpha}}} \leq \kappa_{n,\alpha} \|f\|_{\dot{\Lambda}_\alpha^{1,1}}$ holds for any $f \in C_0^\infty$.*
- (ii) *The fractional isocapacitary inequality $(V(O))^{\frac{n-\alpha}{n}} \leq \kappa_{n,\alpha} \text{cap}(\bar{O}, \dot{\Lambda}_\alpha^{1,1})$ holds for any bounded domain $O \subset \mathbb{R}^n$ with C^∞ boundary ∂O .*
- (iii) *The fractional isoperimetric inequality $(V(O))^{\frac{n-\alpha}{n}} \leq 2\kappa_{n,\alpha} P_\alpha(O)$ holds for any bounded domain $O \subset \mathbb{R}^n$ with C^∞ boundary ∂O .*

Proof. (i) \Rightarrow (ii) follows from the definition of $\text{cap}(\bar{O}, \dot{\Lambda}_\alpha^{1,1})$. (ii) \Rightarrow (iii) follows from Theorem 2. (iii) \Rightarrow (i) follows from the idea verifying (4) \Rightarrow (3). As a matter of fact, assume (iii) is true. Given a function $f \in C_0^\infty$ with $O_t(f)$ being the same as in the proof of Theorem 3. Obviously, $t \mapsto V(O_t(f))$ is a decreasing function on $[0, \infty)$. This monotonicity, together with the layer-cake formula, the chain rule, (iii) for $O_t(f)$, and the above-used co-area formula, derives

$$\|f\|_{L^{\frac{n}{n-\alpha}}} = \int_0^\infty \left(\int_0^t V(O_s(f)) ds \right)^{-\frac{\alpha}{n}} V(O_t(f)) t^{\frac{\alpha}{n-\alpha}} dt \leq \int_0^\infty (V(O_t(f)))^{\frac{n-\alpha}{n}} dt \leq 2\kappa_{n,\alpha} \int_0^\infty P_\alpha(O_t(f)) dt,$$

whence reaching (i). \square

Note that for any compact set $K \subset \mathbb{R}^n$ and any bounded domain $O \subset \mathbb{R}^n$, one has

$$\begin{cases} \lim_{\alpha \rightarrow 1} (1-\alpha)^{-1} \kappa_{n,\alpha} = (n\omega_n^{\frac{1}{n}} \tau_n)^{-1} \\ \lim_{\alpha \rightarrow 1} (1-\alpha) \text{cap}(K; \dot{\Lambda}_\alpha^{1,1}) = \tau_n \text{cap}(K; \dot{W}^{1,1}) = \tau_n \inf_{\bar{O} \in \mathcal{O}^\infty(K)} P(\bar{O}) \\ \lim_{\alpha \rightarrow 1} (1-\alpha) P_\alpha(O) = 2^{-1} \tau_n P(O). \end{cases}$$

So, the limiting cases of Theorem 3 and Theorem 4 as $\alpha \rightarrow 1$ reduce to [20, Theorems 1.1–1.2] and [21, Proposition 3.1] plus the well-known Federer–Fleming–Maz’ya equivalence between the isoperimetric inequality and the Sobolev inequality (cf. [6,13]), respectively.

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