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Partial differential equations

A simple criterion for transverse linear instability of nonlinear waves

Un critère simple d'instabilité transverse d'ondes non linéaires

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ABSTRACT

We prove a simple criterion for transverse linear instability of nonlinear waves for partial differential equations in a spatial domain $\Omega \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$. For stationary solutions depending upon $x \in \Omega$ only, the question of transverse (in)stability is concerned with their (in)stability with respect to perturbations depending upon $(x, y) \in \Omega \times \mathbb{R}$. Starting with a formulation of the PDE as a dynamical system in the transverse direction y, we give sufficient conditions for transverse linear instability. We apply the general result to the Davey–Stewartson equations, which arise as modulation equations for three-dimensional water waves.

RÉSUMÉ

Nous montrons un critère simple d'instabilité transverse linéaire d'ondes non linéaires d'équations aux dérivées partielles posées dans un domaine spatial $\Omega \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$. Pour des solutions stationnaires dépendant de $x \in \Omega$, la question de l'(in)stabilité transverse concerne leur (in)stabilité par rapport à des perturbations dépendantes de $(x, y) \in \Omega \times \mathbb{R}$. En utilisant une formulation de l'équation comme système dynamique par rapport à la direction transverse y, nous donnons des conditions suffisantes d'instabilité transverse linéaire. Nous appliquons ce résultat aux équations de Davey–Stewartson, qui apparaissent comme équations de modulation dans le problème des vagues en trois dimensions.

1. Introduction

We consider partial differential equations of the form

$$u_y = Du_t + F(u), \tag{1}$$

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where the unknown u depends upon the time variable $t \in \mathbb{R}$ and a space variable $y \in \mathbb{R}$, with values in a Banach space X, D is a linear operator acting in X and F a nonlinear map. Typically X represents a space of functions defined on a domain $\Omega \subset \mathbb{R}^n$. In this work we present a simple criterion for the linear instability of y-independent steady solutions of (1) with respect to y-dependent perturbations. We refer to such perturbations as transverse perturbations and the corresponding (in)stability is called transverse (in)stability. For a y-independent steady solution u_* to (1), hence satisfying $F(u_*) = 0$, the question of transverse linear instability concerns the existence of growing-in-time solutions to the linearized equation

$$u_y = Du_t + \mathcal{L}u, \tag{2}$$

in which $\mathcal{L} := \mathrm{d}F(u_*)$ is the differential of F at u_*. We give sufficient conditions for the existence of solutions to (2) that are exponentially growing in time. Our main assumptions are the reversibility of (2) and a condition on the purely imaginary spectrum of $\mathcal{L}$.

This criterion allows us to recover several existing results on transverse instability for both periodic and solitary waves, as, for instance, the transverse linear instability of solitary and periodic waves for the nonlinear Schrödinger and the Kadomtsev–Petviashvili-I equations [7,9,10,12], or for the classical water-wave problem [4,6,8] (see Section 4). Recently, Rousset and Tzvetkov [10] obtained a criterion of transverse instability for solitary waves in Hamiltonian systems. Besides the Hamiltonian structure, their main assumptions are also spectral conditions for the linearization about the steady wave. A key difference with the result here is the starting formulation of the stability problem. Our starting point is a formulation of the equations as an evolutionary problem in the transverse direction y, and the reversibility hypothesis concerns this formulation, whereas in [10] the system is an evolutionary problem in time t, and the Hamiltonian structure is required for this formulation. While the criterion in [10] is more suitable for a further study of the nonlinear transverse instability of solitary waves [11], the present result is more convenient for a further study of the bifurcations induced by this transverse instability [5,6,8]. The paper is organized as follows. We give the main assumptions and prove the instability result in Section 2. Then we use it in Section 3 for the study of the transverse linear instability of a family of periodic solutions to the Davey–Stewartson equations. We conclude with a short discussion.

2. The main result

Consider the partial differential equation (1), in which F is a smooth map defined on a subspace $Y \subset X$, and D is a linear operator with domain $\mathrm{Dom}(D)$ such that $Y \subset \mathrm{Dom}(D)$. Suppose that $u_* \in Y$ is an equilibrium of the system (1), and consider the linearized equation (2) at u_*. We say that u_* is **transversely linearly unstable** if the equation (2) has a solution u of the form $u : (y,t) \mapsto \mathrm{e}^{\lambda t} v(y)$, where λ is a complex number with positive real part, $v(y) \in Y$ for any $y \in \mathbb{R}$, and the map $y \mapsto v(y)$ is bounded on $\mathbb{R}$. We make the following assumptions on (2).

(i) $\mathcal{L}$ and D are closed real operators in X with domains $\mathrm{Dom}(\mathcal{L}) = Y \subset \mathrm{Dom}(D)$.
(ii) The spectrum $\sigma(\mathcal{L})$ of the linear operator $\mathcal{L}$ contains a pair of isolated purely imaginary eigenvalues $\pm \mathrm{i}k_*, k_* \in \mathbb{R}^*$ with odd algebraic multiplicity.
(iii) The system (2) is reversible, *i.e.*, there exists a linear symmetry $R \in \mathcal{L}(Y) \cap \mathcal{L}(X)$ such that $R^2 = \mathrm{id}$ and $RDu = -DRu$, $R\mathcal{L}u = -\mathcal{L}Ru$, for all $u \in Y$.

Our main result is the following instability criterion.

Theorem 2.1 *(Transverse linear instability criterion). Under the assumptions (i), (ii), (iii), for any sufficiently small positive real number λ, the system (2) has a solution of the form $u : (y,t) \mapsto \mathrm{e}^{\lambda t} v(y)$, where $v(y) \in Y$, for any $y \in \mathbb{R}$ and the map $y \mapsto v(y)$ is smooth and periodic. Consequently, the equilibrium u_* is transversely linearly unstable.*

Proof. We claim that it is enough to prove that for any sufficiently small $\lambda > 0$ the spectrum $\sigma(\lambda D + \mathcal{L})$ of the real operator $\lambda D + \mathcal{L}$ contains a pair of purely imaginary eigenvalues $\pm \mathrm{i}k, k \in \mathbb{R}^*$. Indeed, if w denotes an eigenvector associated with $\mathrm{i}k$, its complex conjugate $\overline{w}$ is an eigenvector associated with $-\mathrm{i}k$, and the function v defined through $v(y) = \mathrm{e}^{\mathrm{i}ky} w + \mathrm{e}^{-\mathrm{i}ky}\overline{w}$, for all $y \in \mathbb{R}$, is a real, smooth, periodic solution to the equation

$$v_y = (\lambda D + \mathcal{L})v.$$

Consequently, the function u defined through $u(y,t) = \mathrm{e}^{\lambda t} v(y)$ satisfies (2), which proves the claim.

We show now that the operator $\lambda D + \mathcal{L}$ possesses a pair of isolated purely imaginary eigenvalues, for any sufficiently small $\lambda > 0$. Since the spectrum of $\mathcal{L}$ contains a pair of isolated purely imaginary eigenvalues of odd algebraic multiplicity, and since D is a relatively bounded perturbation of $\mathcal{L}$, a standard perturbation argument implies that for any $\lambda > 0$ sufficiently small, there exist two neighborhoods of $\pm \mathrm{i}k_*$, each containing an odd number of eigenvalues of $\lambda D + \mathcal{L}$, counted with their algebraic multiplicity. Then the reversibility of the linearized system (2) implies that R anticommutes with $\lambda D + \mathcal{L}$. Consequently, the spectrum $\sigma(\lambda D + \mathcal{L})$ is symmetric with respect to the origin. Moreover, the operator $\lambda D + \mathcal{L}$ is real, so that $\sigma(\lambda D + \mathcal{L})$ is symmetric with respect to the real axis. We deduce that $\sigma(\lambda D + \mathcal{L})$ is symmetric with respect to the imaginary axis. Then at least one pair of the eigenvalues close to the $\pm \mathrm{i}k_*$ above will be purely imaginary, since their number is odd. This completes the proof of the theorem. □

Remark 1. This result can be easily extended to systems of the form

$$u_y = Pu + F(u), \quad P = \sum_{k=1}^{n} D_k \partial_t^{(k)},$$

in which $D_1, \dots, D_n$ are closed linear operators whose domains contain Y and $\partial_t^{(k)}$ represents the differential operator of order k with respect to t (see also [6]).

3. Application: periodic waves of the Davey–Stewartson equations

Consider the Davey–Stewartson equations

$$\mathrm{i}A_t + A_{xx} + \alpha A_{yy} + \lambda A + \delta B_x A + \gamma |A|^2 A = 0, \tag{3}$$

$$B_{xx} + \nu B_{yy} + \mu \left(|A|^2\right)_x = 0, \tag{4}$$

where the unknowns A and B are complex- and real-valued functions, respectively, depending on the space variables x, y and the time t, and where the coefficients α, λ, δ, γ, μ and ν are real. The system (3)–(4) arises as a model in the water-wave problem, approximately describing three-dimensional gravity-capillary waves (see [1]). The dynamics of the solutions to (3)–(4) strongly depends on the signs of the coefficients α, δ, μ and ν. Following [1], we focus on the three following cases, which are relevant for the water-wave problem:

- **Case 1:** $\alpha > 0$, $\delta < 0$, $\mu > 0$ and $\nu > 0$.
- **Case 2:** $\alpha > 0$, $\delta < 0$, $\mu < 0$ and $\nu < 0$.
- **Case 3:** $\alpha < 0$, $\delta > 0$, $\mu > 0$ and $\nu > 0$.

Our goal is to study the transverse linear instability of a family of y-independent periodic steady waves of (3)–(4). We check that the assumptions (i), (ii) and (iii) in Theorem 2.1 hold.

The system (3)–(4) has a family of one-dimensional periodic solutions (A^*, B^*), given by

$$A^*(x) = a_0 \mathrm{e}^{\mathrm{i}kx}, \qquad B_x^* = \chi - \mu |a_0|^2,$$

with $k^2 = (\gamma - \delta\mu)|a_0|^2 + \lambda + \delta\chi$, where χ is an arbitrary constant (see [3]). Since the system (3)–(4) is invariant under the change $B_x \longmapsto B_x + \chi$, $\lambda \longmapsto \lambda + \delta\chi$, without loss of generality, we may assume that $\chi = 0$. Notice that the system (3)–(4) is invariant under multiplication of the complex variable A by $\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$, so that we can restrict to the case $a_0 \in \mathbb{R}$.

In order to study the transverse stability of $\left(A^*, B^*\right)$, we set

$$A(x,y,t) = A^*(x)\left[1 + a(x,y,t)\right], \quad B_x(x,y,t) = B_x^* + b(x,y,t), \tag{5}$$

where the functions a and b are assumed to be periodic with respect to x, with wavenumber K, and b has zero mean. Notice that the perturbations A^*a and b belong to a rather general class. In particular, these perturbations are periodic in x with the same period as (A^*, B^*) if $K = nk$, for some $n \in \mathbb{Z}$. Then a and b satisfy the nonlinear system:

$$a_{yy} = -\frac{1}{\alpha}\left(\mathrm{i}a_t + a_{xx} + 2\mathrm{i}ka_x + \gamma a_0^2\left(a + \overline{a}\right) + \delta b + \delta ab + \gamma a_0^2\left(a^2 + 2a\overline{a} + a^2\overline{a}\right)\right), \tag{6}$$

$$b_{yy} = -\frac{1}{\nu}\left(b_{xx} + \mu a_0^2\left(a + \overline{a}\right)_{xx} + \mu a_0^2\left(a\overline{a}\right)_{xx}\right). \tag{7}$$

With this formulation, the steady solution (A^*, B^*) of the system (3)–(4) corresponds to the trivial equilibrium $(0,0)$ of (6)–(7).

Next we set $a = a_r + \mathrm{i}a_i$, $a_y = \widetilde{a}_r + \mathrm{i}\widetilde{a}_i$, $b_y = \widetilde{b}$ and $U = \left(a_r,\ a_i,\ b,\ \widetilde{a}_r,\ \widetilde{a}_i,\ \widetilde{b}\right)^{\mathrm{T}}$, so that the system (6)–(7) is written in the form (1), with D and $F(U)$ that can be easily computed from (6)–(7). Linearizing this system at the origin, we find a system of the form (2) with

$$\mathcal{L} = \mathrm{d}F(0) = \begin{pmatrix} 0_3 & I_3 \\ C & 0_3 \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{1}{\alpha}(\partial_{xx} + 2\gamma a_0^2) & \frac{2}{\alpha}k\partial_x & -\frac{\delta}{\alpha} \\ -\frac{2}{\alpha}k\partial_x & -\frac{1}{\alpha}\partial_{xx} & 0 \\ -\frac{2}{\nu}\mu a_0^2\partial_{xx} & 0 & -\frac{1}{\nu}\partial_{xx} \end{pmatrix},$$

where 0_3 is the zero matrix of order 3 and I_3 is the identity matrix of order 3. For the function space X we choose the set $X = \left(H^1_{\mathrm{per}}\right)^2 \times \widetilde{H^1_{\mathrm{per}}} \times \left(L^2_{\mathrm{per}}\right)^2 \times \widetilde{L^2_{\mathrm{per}}}$, where

$$L^2_{\rm per} = \left\{ v \in L^2_{\rm loc}(\mathbb{R}),\, v\left(x + \frac{2\pi}{K}\right) = v(x),\ \text{for a.e. } x \in \mathbb{R} \right\},$$
$$H^j_{\rm per} = \left\{ v \in H^j_{\rm loc}(\mathbb{R}),\, v\left(x + \frac{2\pi}{K}\right) = v(x), \forall\, x \in \mathbb{R} \right\},\ \text{for } j \in \mathbb{N}^*,$$

and $\widetilde{L^2_{\rm per}} \subset L^2_{\rm per}$, $\widetilde{H^j_{\rm per}} \subset H^j_{\rm per}$ are the subspaces consisting of functions with zero mean.

Theorem 3.1. *The periodic wave (A^*, B^*) is transversely linearly unstable with respect to perturbations of the form (5) in Case 1 if $K^2 - 2(3k^2 - \lambda) < 0$, or if $K^2 - 2(3k^2 - \lambda) > 0$ and $\gamma > 0$, and in Cases 2 and 3.*

Proof. We apply Theorem 2.1 and check that the assumptions (i), (ii) and (iii) are satisfied by $\mathcal{L}$ and D.

With the choice of X above, the operator $\mathcal{L}$ is closed with domain $Y = \left(H^2_{\rm per}\right)^2 \times \widetilde{H^2_{\rm per}} \times \left(H^1_{\rm per}\right)^2 \times \widetilde{H^1_{\rm per}}$. Furthermore D is a bounded operator in X, so that the assumption (i) is satisfied. Next, the linearized system is reversible, since $\mathcal{L}$ and D anti-commute with the linear symmetry $R = \operatorname{diag}(1, 1, 1, -1, -1, -1)$. This proves the assumption (iii). It remains to check the assumption (ii) and show that the spectrum of the operator $\mathcal{L}$ contains at least one pair of purely imaginary eigenvalues with odd algebraic multiplicity.

The compact embedding $Y \subset X$ ensures that the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ is a discrete set of isolated eigenvalues with finite multiplicities. Using Fourier series, we obtain that $\sigma(\mathcal{L})$ is the set of the roots of the polynomials

$$P_n = s^6 - p_4 s^4 + p_2 s^2 - \frac{n^4 K^4 (n^2 K^2 - 2(3k^2 - \lambda))}{\nu \alpha^2},$$

for $n \in \mathbb{Z}$, where

$$p_4 = \frac{n^2 K^2 (\alpha + 2\nu) - 2\nu\gamma a_0^2}{\nu\alpha}, \quad p_2 = \frac{n^2 K^2 \left[2\alpha(n^2 K^2 - (\gamma - \delta\mu) a_0^2) + \nu(n^2 K^2 - 2(2k^2 + \gamma a_0^2))\right]}{\nu\alpha^2}.$$

Notice that $P_n = P_{-n}$, so that the eigenvalues of $\mathcal{L}$ have even algebraic multiplicity. In order to check the assumption (ii), which requires purely imaginary eigenvalues with odd algebraic multiplicity, we restrict the analysis to the invariant subspace

$$X_r = \left\{ (a_r,\, a_i,\, b,\, \widetilde{a_r},\, \widetilde{a_i},\, \widetilde{b}) \in X \,\middle|\, a_r,\, b,\, \widetilde{a_r},\, \widetilde{b} \text{ even functions}, a_i,\, \widetilde{a_i} \text{ odd functions} \right\}.$$

With this restriction, for $n \neq 0$, we can count the multiplicity of the eigenvalues of $\mathcal{L}$ from the roots of P_n, with $n > 0$. When $n = 0$ we are left with the two eigenvalues $\pm\sqrt{-2\gamma a_0^2/\alpha}$, which are purely imaginary if $\alpha\gamma > 0$.

Case 1. This case has been analyzed in [3]. The results in [3] imply that the spectrum $\sigma(\mathcal{L})$ contains a pair of simple purely imaginary eigenvalues and prove the result in this case.

Case 2. Suppose that $\alpha > 0, \delta < 0, \mu < 0$ and $\nu < 0$. If n is large enough, the cubic polynomial P_n possesses a simple negative root $s^2_{0,n}$ and two positive roots $s^2_{1,n}$ and $s^2_{2,n}$. The negative root $s^2_{0,n}$ has the asymptotic expansion

$$s^2_{0,n} = \frac{n^2 K^2}{\nu} + O(1),$$

as n tends to $+\infty$. In particular, the sequence $\left(s^2_{0,n}\right)_{n\geqslant 1}$ is strictly decreasing when n is large enough, so that the negative roots s^2_{0,n_1} and s^2_{0,n_2} are distinct for $n_1 \neq n_2$ sufficiently large. Consequently, for n large enough, the spectrum of $\mathcal{L}$ contains at least one pair of simple purely imaginary eigenvalues, and proves the result in Case 2.

Case 3. Suppose that $\alpha < 0, \delta > 0, \mu > 0$ and $\nu > 0$. For n sufficiently large, the cubic polynomial P_n possesses, in this case, two distinct negative roots $s^2_{1,n}$, $s^2_{2,n}$ and one positive root $s^2_{0,n}$. Using the Newton polygon method (*e.g.*, see [2]) we find the asymptotic expansion of the negative roots

$$s^2_{1,n} = \frac{n^2 K^2}{\alpha} - \frac{2k}{\alpha}|nK| + O(1), \quad s^2_{2,n} = \frac{n^2 K^2}{\alpha} + \frac{2k}{\alpha}|nK| + O(1),$$

as n tends to $+\infty$. Using the fact that the sequences $\left(s^2_{1,n}\right)_{n\geqslant 1}$ and $\left(s^2_{2,n}\right)_{n\geqslant 1}$ are strictly decreasing when n is large enough, we conclude that these roots are simple and distinct for $n_1 \neq n_2$. As in the previous case, this implies that $\mathcal{L}$ possesses at least one pair of simple purely imaginary eigenvalues, which completes the proof of the theorem. □

4. Discussion

Solitary waves. As mentioned in the introduction, our result can also be used to show transverse instability for solitary waves. A simple example is the Kadomtsev–Petviashvili-I equation

$$u_{yy} = (u_t + u_{xxx} + uu_x)_x,$$

which possesses a family of one-dimensional traveling solitary waves of the form $u(x, y, t) = u_*(x - ct)$. It turns out that the linearized equation at u_* can be written in the form (2), with $\mathcal{L}$ and D satisfying the assumptions (i)–(iii). In particular, the spectrum of $\mathcal{L}$ consists of the real line and a pair of simple, isolated purely imaginary eigenvalues. This result easily follows from the fact that the spectrum of the operator $\mathcal{L}_0 = \partial_{xxxx} - \partial_{xx} + \partial_{xx}(u_* \cdot)$ is the positive real axis $[0, +\infty)$ and a simple, isolated negative eigenvalue (see [10]). Our result allows us to recover the result in [10], showing that u_* is linearly transversely unstable.

Dimension breaking bifurcations. Our main result is a criterion for transverse linear instability of nonlinear waves (for instance periodic or solitary waves). A related issue is the one of bifurcations induced by this transverse linear instability (also called *dimension-breaking bifurcations*). In fact, the formulation of the equation (1) as a dynamical system is motivated by this issue. This bifurcation problem is concerned with solutions to the steady nonlinear equation

$$u_y = F(u), \tag{8}$$

which are close to the y-independent solution u_*. If u_* is transversely linearly unstable, in the sense of Theorem 2.1, we expect bifurcations of periodic solutions for the system (8), due to the presence of a pair of purely imaginary eigenvalues in the spectrum of the linear operator $\mathcal{L} = \mathrm{d}F(u_*)$. This spectral assumption, alone, is too weak to allow the study of such bifurcations, but under additional assumptions, they may be investigated with the help, for instance, of a center manifold reduction or the Lyapunov center theorem. We refer to [3,5,6,8] for results on dimension breaking bifurcations obtained in this way.

References

[1] M.J. Ablowitz, H. Segur, On the evolution of packets of water waves, J. Fluid Mech. 92 (04) (1979) 691–715.

[2] S.N. Chow, J.K. Hale, Methods of Bifurcation Theory, Springer, 1982.

[3] F. Dias, M. Hărăguş-Courcelle, On the transition from two-dimensional to three-dimensional water waves, Stud. Appl. Math. 104 (2) (2000) 91–127.

[4] M.D. Groves, M. Haragus, S.M. Sun, Transverse instability of gravity-capillary line solitary water waves, C. R. Acad. Sci. Paris, Ser. I 333 (5) (2001) 421–426.

[5] M.D. Groves, M. Haragus, S.M. Sun, A dimension-breaking phenomenon in the theory of steady gravity-capillary water waves, Philos. Trans. R. Soc. A, Math. Phys. Eng. Sci. 360 (1799) (2002) 2189–2243.

[6] M.D. Groves, S.M. Sun, E. Wahlén, A dimension-breaking phenomenon for water waves with weak surface tension, preprint Arch. Ration. Mech. Anal. (2015), http://dx.doi.org/10.1007/s00205-015-0941-3, in press.

[7] M. Haragus, Transverse spectral stability of small periodic traveling waves for the KP equation, Stud. Appl. Math. 126 (2) (2011) 157–185.

[8] M. Haragus, Transverse dynamics of two-dimensional gravity-capillary periodic water waves, J. Dyn. Differ. Equ. (2015), http://dx.doi.org/10.1007/s10884-013-9336-z, in press.

[9] M.A. Johnson, K. Zumbrun, Transverse instability of periodic traveling waves in the generalized Kadomtsev–Petviashvili equation, SIAM J. Math. Anal. 42 (6) (2010) 2681–2702.

[10] F. Rousset, N. Tzvetkov, A simple criterion of transverse linear instability for solitary waves, Math. Res. Lett. 17 (1) (2010) 157–170.

[11] F. Rousset, N. Tzvetkov, Transverse instability of the line solitary water-waves, Invent. Math. 184 (2) (2011) 257–388.

[12] A. Stefanov, M. Stanislavova, Z. Hakkaev, Transverse instability for periodic waves of KP-I and Schrödinger equations, Indiana Univ. Math. J. 61 (2) (2012) 461–492.