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Partial differential equations

A local regularity condition involving two velocity components of Serrin-type for the Navier–Stokes equations

Une condition de la régularité locale impliquant deux composantes de la vitesse de type Serrin pour les équations de Navier–Stokes

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ABSTRACT

The present paper deals with the problem of local regularity of weak solutions to the Navier–Stokes equation in $\Omega \times (0,T)$ with forcing term $\boldsymbol{f}$ in L^2. We prove that $\boldsymbol{u}$ is strong in a sub-cylinder $Q_r \subset \Omega \times (0,T)$ if two velocity components u^1, u^2 satisfy a Serrin-type condition.

RÉSUMÉ

Le présent papier traite le problème de la régularité locale de solutions faibles à l'équation de Navier–Stokes en $\Omega \times (0,T)$ de terme de force $\boldsymbol{f}$ en L^2. Nous prouvons que $\boldsymbol{u}$ est forte dans un sous-cylindre $Q_r \subset \Omega \times (0,T)$ si deux composantes de la vitesse u^1, u^2 satisfont une condition de type Serrin.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open set and $0 < T < +\infty$. We consider the Navier–Stokes equations in the cylindrical domain $\Omega_T := \Omega \times (0,T)$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.1}$$

$$\partial_t u^i + (\boldsymbol{u} \cdot \nabla) u^i - \Delta u^i + \partial_i p = f^i, \quad i = 1,2,3, \tag{1.2}$$

where $\boldsymbol{u} = (u^1, u^2, u^3)$ and p are unknown velocity and pressure, respectively, and $\boldsymbol{f} = (f^1, f^2, f^3)$ is a known exterior force.

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The aim of the present paper is to show that if two components of $\boldsymbol{u}$, say u^1 and u^2, satisfy a Serrin condition in a sub cylinder $Q_r \subset \Omega_T$, i.e.

$$u^i \in L^s(t_0 - r^2, t_0; L^q(B_r)), \quad \frac{2}{s} + \frac{3}{q} = 1, \quad i = 1, 2, \quad (3 < q \le +\infty), \tag{1.3}$$

then $\boldsymbol{u}$ is regular in $Q_r := (t_0 - r^2, t_0) \times B_r$, where $B_r \subset \Omega$ is a ball of radius r.

The Serrin-type regularity criterion for the Navier–Stokes equations is studied a lot, especially in [6,12,13]. In the sense of componentwize Serrin criteria, the two-component regularity is studied for the vorticity in [4], and for the velocity in [1], which is published in [2]. There are many results on this problem for the two-component conditions, for example [3]. The one-component regularity condition is studied in [9,10,16].

The local version of the Serrin-type condition is studied in [5] for the vorticity, and in [11] for the axially symmetric case.

In this article, we study a local Serrin-type regularity criterion with two components, which completes the result in [1,2]. We remark that it is shown in [1,2] that if u^1, u^2 satisfy (1.3) with $q = 6$, $s = \infty$, then the weak solution is regular.

We begin our discussion by providing the notations used throughout the paper. For points $x, y \in \mathbb{R}^3$, by $x \cdot y$ we denote the usual scalar product. Vector functions as well as tensor-valued functions are denoted by bold-face letters. For two matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{3\times 3}$, we denote by $\boldsymbol{A}{:}\boldsymbol{B}$ the scalar product $\boldsymbol{A}{:}\boldsymbol{B} := \sum_{i,j=1}^{3} A_{ij} B_{ij}$.

The notations $W^{k,q}(\Omega)$, $W^{k,q}_0(\Omega)$ $(1 \le q \le +\infty; k \in \mathbb{N})$ stand for the usual Sobolev spaces. As it will be always clear, throughout this Note we will not distinguish between spaces of scalar valued functions and spaces of vector- or tensor-valued functions. For a given Banach space X, we denote by $L^s(a, b; X)$ the space of all Bochner measurable functions $f : (a, b) \to X$ such that $\|f(\cdot)\|_X \in L^s(a, b)$. Its norm is given by

$$\|f\|_{L^s(a,b;X)} := \begin{cases} \left(\int_a^b \|f(t)\|_X^s \, \mathrm{d}t \right)^{1/s} & \text{if} \quad 1 \le s < +\infty, \\ \operatorname{ess\,sup}_{t \in (a,b)} \|f(t)\|_X & \text{if} \quad s = +\infty. \end{cases}$$

The space of smooth solenoidal vector fields with compact support in Ω will be denoted by $C^\infty_{\mathrm{c,div}}(\Omega)$. Then, we define:

$$\begin{aligned} L^q_{\mathrm{div}}(\Omega) &:= \text{closure of } C^\infty_{\mathrm{c,div}}(\Omega) \text{ with respect to the } L^q \text{ norm}, \\ W^{1,q}_{0,\mathrm{div}}(\Omega) &:= \text{closure of } C^\infty_{\mathrm{c,div}}(\Omega) \text{ with respect to the } W^{1,q}_0 \text{ norm}, \\ V^2(\Omega_T) &:= L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)), \\ V^2_{\mathrm{div}}(\Omega_T) &:= \{\boldsymbol{u} \in V^2(\Omega_T) : \nabla \cdot \boldsymbol{u} = 0 \text{ a.e. in } \Omega_T\}. \end{aligned}$$

Next, we are going to introduce the notion of a local weak solution to (1.1), (1.2) with finite energy.

Definition 1.1. Let $\boldsymbol{f} \in L^2(\Omega_T)$. A vector function $\boldsymbol{u}$ is called a *weak solution to* (1.1), (1.2) *with finite energy* if $\boldsymbol{u} \in V^2_{\mathrm{div}}(\Omega_T)$ and the following integral identity holds for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{c}}(\Omega_T)$ with $\nabla \cdot \boldsymbol{\varphi} = 0$

$$\int_{\Omega_T} -\boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} - \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\varphi} + \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t.$$

Remark 1.2. By means of Sobolev's embedding theorem, we get the embedding

$$V^2_{\mathrm{div}}(\Omega_T) \hookrightarrow L^\alpha(0, T; L^\beta(\Omega)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = \frac{3}{2}, \quad \alpha, \beta \in [2, +\infty].$$

Our main result is the following.

Theorem 1.3. *Let $\boldsymbol{f} \in L^2(\Omega_T)$. Let $\boldsymbol{u} \in V^2(\Omega_T))$ be a local weak solution with finite energy to* (1.1), (1.2)*. Suppose that u^1, u^2 satisfy* (1.3) *in a sub-cylinder $Q_r = Q_r(x_0, t_0) \subset \Omega_T$. Then $\boldsymbol{u}$ is a strong solution in Q_r, i.e.*

$$\nabla^2 \boldsymbol{u} \in L^2(Q_\rho), \nabla \boldsymbol{u} \in L^\infty(t_0 - \rho^2, t_0; L^2(B_\rho)) \quad \forall 0 < \rho < r.$$

We remark that initial and boundary conditions are not important since we consider the local regularity.

We will prove the theorem in the next sections. For that, we consider a decomposition of the pressure in Section 2, and prove the regularity in the whole space under Serrin conditions in Section 3. Finally, in Section 4, we prove our theorem.

2. Local pressure and local suitable weak solutions

We briefly recall the definition of the local pressure (for details, cf. [15]). Let $U \subset \mathbb{R}^3$ be a bounded C^1 domain. By $W^{-1,q}(U)$ we denote the dual of $W_0^{1,q'}(U)$. Here, q' stands for the dual exponent of q, that is, $\frac{q}{q-1}$ if $1 < q < +\infty$, 1 if $q = +\infty$, and $+\infty$ if $q = 1$. Furthermore, we define the following subspaces of $W^{-1,q}(U)$

$$G^{-1,q}(U) := \Big\{ \nabla p \in W^{-1,q}(U) \,\Big|\, p \in L_0^q(U) \Big\},$$

$$W_{\mathrm{div}}^{-1,q}(U) := \Big\{ -\Delta \boldsymbol{v} \in W^{-1,q}(U) \,\Big|\, \boldsymbol{v} \in W_{0,\mathrm{div}}^{1,q}(U) \Big\}.$$

Here, $L_0^q(U)$ denotes the space of all $p \in L^q(U)$ such that $\int_U f \,\mathrm{d}x = 0$.

Based on the result [8, Theorem 2.1] we see that

$$W^{-1,q}(U) = G^{-1,q}(U) \oplus W_{\mathrm{div}}^{-1,q}(U),$$

and there exists a unique projection $\boldsymbol{E}_U : W^{-1,q}(U) \to G^{-1,q}(U)$, such that

$$\boldsymbol{v}^* - \boldsymbol{E}_U \boldsymbol{v}^* \in W_{\mathrm{div}}^{-1,q}(U),$$

i.e. there exists a unique pair $(\boldsymbol{v}, p) \in W_{0,\mathrm{div}}^{1,q}(U) \times L_0^q(U)$, such that $\nabla p = \boldsymbol{E}_U \boldsymbol{v}^*$, which is a weak solution to the Stokes system

$$\begin{cases} \nabla \cdot \boldsymbol{v} = 0 \quad \text{a.e. in} \quad U, \\ -\Delta \boldsymbol{v} + \nabla p = \boldsymbol{v}^* \quad \text{in} \quad W^{-1,q}(U), \\ \boldsymbol{v} = \boldsymbol{0} \quad \text{a.e. in} \quad \partial U. \end{cases}$$

In particular, we have the estimate

$$\|p\|_{L^q} \le c \|\boldsymbol{v}^*\|_{W^{-1,q}}$$

with a constant $c > 0$ depending only on q and U. In case U coincides with a ball B_r, the constant c depends only on q.

Furthermore, by virtue of [7, Theorem IV.5.1], we have the following regularity result.

Lemma 2.1. *Let $U \subset \mathbb{R}^3$ be a bounded C^{k+1} domain. Then the restriction of $\boldsymbol{E}_U$ to $W^{k-1,q}(U)$ defines a projection in $W^{k-1,q}(U)$. In addition, there holds*

$$\|\nabla p\|_{W^{k-1,q}} \le c \|\boldsymbol{f}\|_{W^{k-1,q}},$$

where we have identified $W^{0,q}(U)$ with $L^q(U)$ and have used the canonical embedding $L^q(U) \hookrightarrow W^{-1,q}(U)$ given as

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int\limits_U \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x, \quad \boldsymbol{f} \in L^q(U), \quad \boldsymbol{v} \in W_0^{1,q}(U).$$

Here, $c = \mathrm{const} > 0$ and depends on q and on the geometric property of U only. In particular, if U equals a ball B_r, this constant is independent of $r > 0$.

By using the local projection $\boldsymbol{E}_U$, we have the following lemma.

Lemma 2.2. *Let $\boldsymbol{u} \in V_{\mathrm{div}}^2(\Omega_T)$. Then for every bounded C^2 domain $U \subset \Omega$ the following identity holds true that for every $\boldsymbol{\varphi} \in C_c^\infty(U \times (0,T))$,*

$$\int\limits_{\Omega_T} -(\boldsymbol{u} + \nabla \pi_{\mathrm{hm},U}) \cdot \partial_t \boldsymbol{\varphi} - \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\varphi} + \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int\limits_{\Omega_T} \boldsymbol{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T} (\pi_{1,U} + \pi_{2,U} + \pi_{3,U}) \nabla \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\nabla \pi_{\mathrm{hm},U} = -\boldsymbol{E}_U(\boldsymbol{u}), \qquad \nabla \pi_{1,U} = -\boldsymbol{E}_U((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}),$$
$$\nabla \pi_{2,U} = \boldsymbol{E}_U(\Delta \boldsymbol{u}), \qquad \nabla \pi_{3,U} = \boldsymbol{E}_U(\boldsymbol{f}).$$

For a detailed proof of Lemma 2.2, see [14, Lemma 2.4].

We proceed by providing the definition of a local suitable weak solution.

Definition 2.3. Let $\boldsymbol{u} \in V^2_{\mathrm{div}}(\Omega_T)$ be a local weak solution to (1.1), (1.2). Let $U \subset \Omega$ be a bounded C^2 domain and $0 \le t_1 < t_2 \le T$. Then $\boldsymbol{u}$ is called a *local suitable weak solution* to (1.1), (1.2) in $U \times (t_1, t_2)$ if

$$\frac{1}{2}\int_U \phi|\boldsymbol{v}(t)|^2\,\mathrm{d}x + \int_{t_1}^t\int_U \phi|\nabla\boldsymbol{v}|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$= \frac{1}{2}\int_{t_1}^t\int_U |\boldsymbol{v}|^2(\partial_t\phi + \Delta\phi)\,\mathrm{d}x\,\mathrm{d}s\Big(\frac{1}{2}|\boldsymbol{v}|^2 + \pi_{1,U} + \pi_{2,U} + \pi_{3,U}\Big)\boldsymbol{v}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}s + \int_{t_1}^t\int_U \phi\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t \tag{2.1}$$

for all nonnegative $\phi \in C^\infty_c(U \times (t_1, t_2))$ for a.e. $t \in (t_1, t_2)$, where

$$\boldsymbol{v} = \boldsymbol{u} + \nabla\pi_{\mathrm{hm},U}.$$

Remark 2.4. Let $\boldsymbol{u}$ be a local suitable weak solution to (1.1), (1.2) in $U \times (t_1, t_2)$. Using the method in [14], it can be checked easily that each point $z_0 \in U \times (t_1, t_2)$, with

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r(z_0)} |\nabla\boldsymbol{u}|^2\,\mathrm{d}x\,\mathrm{d}t = 0,$$

is a regular point, i.e. that there exists $\sigma > 0$ such that

$$\nabla\boldsymbol{u} \in V^2(Q_\sigma(z_0)).$$

Now, we are in a position to prove the following local energy equality.

Lemma 2.5. *Let $\boldsymbol{u} \in V^2_{\mathrm{div}}(\Omega_T)$ be a local weak solution to (1.1), (1.2). Assume that u^1, u^2 satisfy (1.3) in a subcylinder $Q_r(x_0, t_0)$. Then $\boldsymbol{u}$ is a local suitable weak solution to (1.1), (1.2) in Q_r. In fact, (2.1) holds with equal sign.*

Proof. In view of (1.1), we may write

$$\begin{aligned}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) &= \sum_{j=1}^3\sum_{i=1}^2 \partial_j(u^ju^i) + \sum_{j=1}^2 \partial_j(u^iu^3) + 2u^3\partial_3u^3\\
&= \sum_{j=1}^3\sum_{i=1}^2 \partial_j(u^ju^i) + \sum_{j=1}^2 \partial_j(u^ju^3) - 2\sum_{j=1}^2 u^3\partial_ju^j\\
&= \sum_{j=1}^3\sum_{i=1}^2 \partial_j(u^ju^i) - \sum_{j=1}^2 \partial_j(u^ju^3) + 2\sum_{j=1}^2 \partial_ju^3u^j = \nabla\cdot\boldsymbol{A} + \boldsymbol{b}.\end{aligned}$$

Owing to (1.3) and the embedding $V^2(\Omega_T) \hookrightarrow L^\alpha(0, T; L^\beta)$ with $\frac{2}{\alpha} + \frac{3}{\beta} = \frac{3}{2}$ for all $\alpha, \beta \in [2, +\infty]$, we find

$$\boldsymbol{A} \in L^2(Q_r), \quad \boldsymbol{b} \in L^\gamma(t_0 - r^2, t_0; L^\delta(B_r)), \quad \frac{2}{\gamma} + \frac{3}{\delta} = \frac{7}{2}, \quad \gamma, \delta \in (1, 2).$$

We define the local pressures $\pi_{\mathrm{hm}}, \pi_{1,1}, \pi_{1,2}, \pi_2$ and π_3 in the following ways:

$$\nabla\pi_{\mathrm{hm}} = -\boldsymbol{E}_{B_r}(\boldsymbol{u}), \qquad \nabla\pi_{1,1} = \boldsymbol{E}_{B_r}(\nabla\cdot\boldsymbol{A}),$$
$$\nabla\pi_{1,2} = \boldsymbol{E}_{B_r}(\boldsymbol{b}), \qquad \nabla\pi_2 = \boldsymbol{E}_{B_r}(\Delta\boldsymbol{u}), \qquad \nabla\pi_3 = \boldsymbol{E}_{B_r}(\boldsymbol{f}),$$

where $\boldsymbol{E}_{B_r} : W^{-1,q}(B_r) \to W^{-1,q}(B_r)$, $(1 < q < +\infty)$, defined above. Setting $\boldsymbol{v} := \boldsymbol{u} + \nabla\pi_{\mathrm{hm}}$ we see that

$$\partial_t\boldsymbol{v} - \Delta\boldsymbol{v} = \nabla\cdot\boldsymbol{A} + \boldsymbol{b} - \nabla(\pi_{1,1} + \pi_{1,2} + \pi_2 + \pi_3) + \boldsymbol{f} \quad \text{in} \quad Q_r$$

in sense of distributions. Clearly, $\boldsymbol{v} \in V^2(Q_r)$. Furthermore,

$$\pi_{1,1} + \pi_2 + \pi_3 \in L^2(Q_r),$$
$$\nabla\pi_{1,2} \in L^\gamma(t_0 - r^2, t_0L^\delta(B_r)), \qquad \frac{2}{\gamma} + \frac{3}{\delta} = \frac{7}{2}, \quad (\gamma, \delta \in [1, 2]).$$

Accordingly, the following local energy equality holds:

$$\frac{1}{2}\int_{B_r} |\boldsymbol{v}(t)|^2\phi \,\mathrm{d}x + \int_{t_0-r^2}^{t}\int_{B_r} |\nabla \boldsymbol{v}|^2\phi \,\mathrm{d}x\,\mathrm{d}s$$

$$= \frac{1}{2}\int_{t_0-r^2}^{t}\int_{B_r} |\boldsymbol{v}|^2(\partial_t\phi + \Delta\phi)\,\mathrm{d}x\,\mathrm{d}s - \int_{t_0-r^2}^{t}\int_{B_r} \boldsymbol{A} : \nabla(\boldsymbol{v}\phi) + (\pi_{1,1}+\pi_2+\pi_3)\nabla\cdot(\boldsymbol{v}\phi)\,\mathrm{d}x\,\mathrm{d}s$$

$$+ \int_{t_0-r^2}^{t}\int_{B_r} \boldsymbol{b}\cdot\boldsymbol{v}\phi - \nabla\pi_{1,2}\boldsymbol{v}\phi \,\mathrm{d}x\,\mathrm{d}s + \int_{t_0-r^2}^{t}\int_{B_r} \phi \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}s$$

for all non-negative $\phi \in C^\infty_{\mathrm{c}}(Q_r)$. This shows that $\boldsymbol{u}$ is a local suitable weak solution to (1.1), (1.2) in Q_r. □

3. Global regularity

In the present section, we provide a Serrin-type condition on two velocity components in the whole space. Note that a similar result has been proved in [2]. We have the following theorem.

Theorem 3.1. *Let $\boldsymbol{f} \in L^2$ and let $\boldsymbol{u}_0 \in W^{1,2}_{\mathrm{div}}$. Let $\boldsymbol{u} \in V^2(\Omega_T)$ be a weak solution to (1.1)–(1.2) in $\mathbb{R}^3 \times (0,T)$. Suppose*

$$u^i \in L^s(0,T;L^q), \quad \frac{2}{s} + \frac{3}{q} = 1, \quad i = 1,2 \quad (3 < q \le +\infty). \tag{3.1}$$

Then $\boldsymbol{u}$ is a strong solution in $\mathbb{R}^3 \times [\tau, T]$ for every $0 < \tau < T$.

Proof. Let $0 < \tau < T$ such that $\boldsymbol{u}(\tau) \in W^{1,2}$. Assume that $\boldsymbol{u}$ is not strong on $\mathbb{R}^3 \times [\tau, T]$. Let $\tau < T_* \le T$ denote the first time of blow up of $\boldsymbol{u}$, the existence of which is guaranteed by the local-in-time existence of a strong solution. Let $\tau < T_0 < T_*$ be suitably fixed. Since $\boldsymbol{u}$ is strong in $[\tau, T_0]$, we may multiply both sides of (1.2) by $-\Delta\boldsymbol{u}$ and integrate the result over $\mathbb{R}^3 \times (T_0, t_0)$ $(T_0 < t_0 < T_*)$. By integration by parts, this leads to

$$\frac{1}{2}\|\nabla\boldsymbol{u}(t_0)\|^2_{L^2} + \int_{T_0}^{t_0}\int_{\mathbb{R}^3} |\Delta\boldsymbol{u}|^2\,\mathrm{d}x\,\mathrm{d}t = \int_{T_0}^{t_0}\int_{\mathbb{R}^3} u^j\partial_j u^i \partial_k\partial_k u^i\,\mathrm{d}x\,\mathrm{d}t - \int_{T_0}^{t_0}\int_{\mathbb{R}^3} \boldsymbol{f}\cdot\Delta\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\|\nabla\boldsymbol{u}(T_0)\|^2_{L^2}. \tag{3.2}$$

Elementary,

$$\int_{T_0}^{t_0}\int_{\mathbb{R}^3} u^j\partial_j u^i\partial_k\partial_k u^i\,\mathrm{d}x\,\mathrm{d}t$$

$$= \sum_{i=1}^{2}\int_{T_0}^{t_0}\int_{\mathbb{R}^3} u^j\partial_j u^i\partial_k\partial_k u^i\,\mathrm{d}x\,\mathrm{d}t + \sum_{j=1}^{2}\int_{T_0}^{t_0}\int_{\mathbb{R}^3} u^j\partial_j u^3\partial_k\partial_k u^3\,\mathrm{d}x\,\mathrm{d}t + \int_{T_0}^{t_0}\int_{\mathbb{R}^3} u^3\partial_3 u^3\partial_k\partial_k u^3\,\mathrm{d}x\,\mathrm{d}t$$

$$= -\sum_{i=1}^{2}\int_{T_0}^{t_0}\int_{\mathbb{R}^3} \partial_k u^j\partial_j u^i\partial_k u^i\,\mathrm{d}x\,\mathrm{d}t + \sum_{j=1}^{2}\int_{T_0}^{t_0}\int_{\mathbb{R}^3} u^j\partial_j u^3\partial_k\partial_k u^3\,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_{T_0}^{t_0}\int_{\mathbb{R}^3} \partial_k u^3\partial_3 u^3\partial_k u^3\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{T_0}^{t_0}\int_{\mathbb{R}^3} \partial_3 u^3|\nabla u^3|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$=: I_1 + I_2 + I_3 + I_4.$$

Applying integration by parts, we calculate

$$I_1 = \sum_{i=1}^{2}\int_{T_0}^{t_0}\int_{\mathbb{R}^3} \partial_k u^j u^i \partial_j\partial_k u^i\,\mathrm{d}x\,\mathrm{d}t.$$

By means of Hölder's inequality, we get

$$I_1 \le c\big(\|u^1\|_{L^s(T_0,T;L^q)} + \|u^2\|_{L^s(T_0,T;L^q)}\big)\|\nabla \boldsymbol{u}\|_{L^\gamma(T_0,T;L^\delta)}\|\Delta \boldsymbol{u}\|_{L^2}, \tag{3.3}$$

where

$$\frac{1}{s}+\frac{1}{\gamma}+\frac{1}{2}=1, \quad \frac{1}{q}+\frac{1}{\delta}+\frac{1}{2}=1.$$

Clearly in view of (3.1) we get

$$\frac{2}{\gamma}+\frac{3}{\delta}=\frac{3}{2}.$$

By means of Sobolev's embedding theorem, from (3.3) we infer

$$I_1 \le c_0\big(\|u^1\|_{L^s(T_0,T;L^q)} + \|u^2\|_{L^s(T_0,T;L^q)}\big)\big(\|\nabla \boldsymbol{u}\|^2_{L^\infty(T_0,t_0;L^2)} + \|\Delta \boldsymbol{u}\|^2_{L^2(T_0,t_0;L^2)}\big)$$

with an absolute constant $c_0 > 0$.

Similarly, owing to $\partial_3 u^3 = -\partial_1 u^1 - \partial_2 u^2$, we estimate

$$I_j \le c_0\big(\|u^1\|_{L^s(T_0,T;L^q)} + \|u^2\|_{L^s(T_0,T;L^q)}\big)\big(\|\nabla \boldsymbol{u}\|^2_{L^\infty(T_0,t_0;L^2)} + \|\Delta \boldsymbol{u}\|^2_{L^2(T_0,t_0;L^2)}\big)$$

$(j = 2, 3, 4)$.

Inserting the estimates of I_1, I_2, I_3 and I_4 into the right-hand side of (3.2) and applying Young's inequality, we are arrive at

$$\begin{aligned}&\Big(\|\nabla \boldsymbol{u}\|^2_{L^\infty(T_0,t_0;L^2)} + \|\Delta \boldsymbol{u}\|^2_{L^2(T_0,t_0;L^2)}\Big)\\ &\quad\le 12c_0\Big(\|u^1\|_{L^s(T_0,T;L^q)} + \|u^2\|_{L^s(T_0,T;L^q)}\Big)\Big(\|\nabla \boldsymbol{u}\|^2_{L^\infty(T_0,t_0;L^2)} + \|\Delta \boldsymbol{u}\|^2_{L^2(T_0,t_0;L^2)}\Big)\\ &\qquad + c\Big(\|\boldsymbol{f}\|^2_{L^2} + \|\nabla \boldsymbol{u}(T_0)\|^2_{L^2}\Big).\end{aligned}$$

In fact, we may choose T_0 such that

$$\|u^1\|_{L^s(T_0,T;L^q)} + \|u^2\|_{L^s(T_0,T;L^q)} \le \frac{1}{24c_0}.$$

Accordingly,

$$\|\nabla \boldsymbol{u}\|^2_{L^\infty(T_0,t_0;L^2)} + \|\Delta \boldsymbol{u}\|^2_{L^2(T_0,t_0;L^2)} \le 2c(|\boldsymbol{f}\|^2_{L^2} + \|\nabla \boldsymbol{u}(T_0)\|^2_{L^2}). \tag{3.4}$$

As the right-hand side of (3.4) is independent of t_0, we deduce from (3.4) that $\boldsymbol{u}$ is strong in $[\tau, T_*]$. However, this contradicts the definition of T_*. Whence, the statement of the theorem is true.

4. Proof of Theorem 1.3

Let $0 < r_0 < r$. By our assumption of Theorem 1.3 and Lemma 2.5, we see that $\boldsymbol{u}$ is a local suitable weak solution to (1.1), (1.2) in Q_r. As it has been proved in [14], such solutions are regular outside a singular set $\Sigma \subset Q_r$ whose one-dimensional Hausdorff measure is zero (cf. Remark 2.4). In particular, Σ does not contain a one-dimensional subset. Thus, there exists $0 < r_0 < \rho < r$ and a sufficiently small $\varepsilon > 0$, such that $\boldsymbol{u}$ is strong in the region

$$A_{\rho,\varepsilon} \times (t_0 - \rho^2, t_0) := \{\rho - \varepsilon < |x_0 - x| < \rho + \varepsilon\} \times (t_0 - \rho^2, t_0).$$

Let $\zeta \in C^\infty(\mathbb{R}^3)$ such that $\operatorname{supp}(\zeta) \subset B_{\rho+\varepsilon}$ and $\zeta \equiv 1$ on $\overline{B_\rho(x_0)}$.

Let π_{hm}, π_1, π_2 and π_3 denote the local pressure on Q_r, which has been defined in Lemma 2.2. As before, set $\boldsymbol{v} := \boldsymbol{u} + \nabla \pi_{\mathrm{hm}}$. Since $\pi_{\mathrm{hm}}(t)$ is harmonic in B_r, π_{hm} together with its derivatives $D^\alpha \pi_{\mathrm{hm}}$ for any multi-index α are continuous on $\overline{Q_{\rho+\varepsilon}}$.

Next, set

$$\boldsymbol{w} := \mathbf{P}(\zeta \boldsymbol{v}) = \zeta \boldsymbol{v} - \nabla Q,$$

where $\mathbf{P}$ denotes the usual Helmholtz projection and Q is defined by the Newton potential N as follows:

$$Q := -N * (\nabla\zeta \cdot \boldsymbol{v}) \quad \text{in} \quad \mathbb{R}^3 \times (t_0 - \rho^2, t_0).$$

Clearly, $Q = \Delta^{-1}(\nabla\zeta \cdot \boldsymbol{v})$. Furthermore, note that as $\operatorname{supp}(\zeta) \subset A_{\rho,\varepsilon}$ there holds $\nabla\zeta \cdot \boldsymbol{v} \in L^\infty(t_0-\rho^2, t_0; W^{1,2})$ it follows that

$$\nabla Q \in L^\infty(t_0-\rho^2, t_0; W^{2,2}), \quad \partial_t Q \in L^\infty(t_0-\rho^2, t_0; W^{1,2}). \tag{4.1}$$

Now, it remains to verify that $\boldsymbol{w}$ solves the Navier–Stokes equation in $\mathbb{R}^3 \times (t_0-\rho^2, t_0)$ with right-hand side $\overline{\boldsymbol{f}} \in L^2$, which is defined later.

First, let us recall that $\boldsymbol{v}$ solves the equation

$$\partial_t \boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} = -\nabla(\pi_1+\pi_2+\pi_3) + \boldsymbol{f} \quad \text{in} \quad Q_r. \tag{4.2}$$

We evaluate the convective term as follows

$$\begin{aligned}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{v}) - \nabla\cdot(\boldsymbol{v}\otimes\nabla\pi_{\mathrm{hm}}) - \nabla\cdot(\nabla\pi_{\mathrm{hm}}\otimes\boldsymbol{v}) - \nabla\cdot(\nabla\pi_{\mathrm{hm}}\otimes\nabla\pi_{\mathrm{hm}})\\ &= (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\nabla\pi_{\mathrm{hm}} - (\nabla\pi_{\mathrm{hm}}\cdot\nabla)\boldsymbol{v} - \frac{1}{2}\nabla|\nabla\pi_{\mathrm{hm}}|^2\\ &=: (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \boldsymbol{f}_1.\end{aligned}$$

Clearly, $\boldsymbol{f} - \zeta\boldsymbol{f}_1 \in L^2$. From (4.2) multiplying both sides by ζ, we deduce that

$$\begin{aligned}&\partial_t\boldsymbol{w} + \zeta(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \Delta\boldsymbol{w}\\ &\quad = -2\nabla\zeta\cdot\nabla\boldsymbol{v} - \Delta\zeta\boldsymbol{v} - \zeta\nabla(\pi_1+\pi_2+\pi_3) - \nabla(\partial_t Q + \Delta Q) + \zeta\boldsymbol{f} - \zeta\boldsymbol{f}_1.\end{aligned} \tag{4.3}$$

On the other hand, we find:

$$\begin{aligned}(\boldsymbol{w}\cdot\nabla)\boldsymbol{w} &= (\zeta\boldsymbol{v}\cdot\nabla)(\zeta\boldsymbol{v}) - ((\zeta\boldsymbol{v})\cdot\nabla)\nabla Q - (\nabla Q\cdot\nabla)(\zeta\boldsymbol{v}) + \frac{1}{2}\nabla|\nabla Q|^2\\ &= \zeta(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + (\zeta\boldsymbol{v}\cdot\nabla)((\zeta-1)\boldsymbol{v}) - ((\zeta\boldsymbol{v})\cdot\nabla)\nabla Q - (\nabla Q\cdot\nabla)(\zeta\boldsymbol{v}) + \frac{1}{2}\nabla|\nabla Q|^2\\ &=: \zeta(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \boldsymbol{f}_2 + \frac{1}{2}\nabla|\nabla Q|^2.\end{aligned}$$

Observing (4.1) and recalling that $\nabla\boldsymbol{u} \in V^2(Q_{\rho+\varepsilon})$ we infer that $\boldsymbol{f}_2 \in L^2(t_0-\rho^2, t_0; L^2)$. Inserting this identity into (4.3), we arrive at

$$\partial_t\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} - \Delta\boldsymbol{w} = -\nabla P + \overline{\boldsymbol{f}} \quad \text{in} \quad \mathbb{R}^3 \times (t_0-\rho^2, t_0),$$

where

$$\begin{aligned}P &:= \zeta(\pi_1+\pi_2+\pi_3) + \partial_t Q - \Delta Q + \frac{1}{2}|\nabla Q|^2,\\ \overline{\boldsymbol{f}} &:= \nabla\zeta(\pi_1+\pi_2+\pi_3) + \zeta\boldsymbol{f} - \zeta\boldsymbol{f}_1 + \boldsymbol{f}_2 - 2\nabla\zeta\cdot\nabla\boldsymbol{v} - (\Delta\zeta)\boldsymbol{v}.\end{aligned}$$

As $\overline{\boldsymbol{f}} \in L^2(t_0-\rho^2, t_0; L^2)$, the claim follows thanks to Theorem 3.1.

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