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Geometry

The ε-positive center set and its applications ☆

L'ensemble des centres ε-positifs et ses applications

Shengliang Pan [a], Yunlong Yang [a], Pingliang Huang [b]

[a] *Mathematics Department, Tongji University, Shanghai, 200092, PR China*
[b] *Mathematics Department, Shanghai University, Shanghai, 200444, PR China*

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ABSTRACT

In this paper we will first give a positive answer to Kaiser's conjecture on ε-positive centers for convex curves and then present its two applications.

RÉSUMÉ

Dans cette Note, nous apportons une réponse positive à la conjecture de Kaiser sur les centres ε-positifs des courbes convexes, puis nous en présentons deux applications.

1. Introduction

For a convex plane curve γ with length L and area A, Bonnesen [1] had proved the famous inequality that is now known as the *Bonnesen inequality*:

$$Lr - A - \pi r^2 \geq 0, \quad r_{in} \leq r \leq r_{out}, \tag{1.1}$$

where r_{in} and r_{out} are the inradius and circumradius of γ. The equality in (1.1) holds when $r = r_{in}$ if and only if γ is either a circle or a sausage curve and when $r = r_{out}$ if and only if γ is a circle. The proof of (1.1) can be found in [1–3,13,14], etc.

To understand the curve shortening problem (cf. [4,5,7]), Gage [6] introduced, for the first time, the positive center for a convex curve γ with length L and area A as a point for which its support function $h(\theta)$ satisfies

$$Lh(\theta) - A - \pi h(\theta)^2 \geq 0, \tag{1.2}$$

for all $\theta \in [0, 2\pi]$. Gage [6] has shown that the center of the minimal annulus must be a positive center and that many other natural "centers" of γ are not positive centers in general, such as the center of mass, the centroid and the Steiner point. Following Gage's idea, the authors of the present paper have proven in [10] that the positive center set of a convex curve is convex and shown that circles and sausage curves are the only examples of positive center sets of zero area. In

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E-mail addresses: slpan@tongji.edu.cn (S. Pan), 88ylyang@tongji.edu.cn (Y. Yang), huangpingliang@shu.edu.cn (P. Huang).

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1996, Kaiser [12] had defined the ε-positive center for a curve as Gage and put forward the following conjecture by some computer graphics:

Conjecture *(Kaiser). Let γ be a simple closed curve.*

(i) If γ has more than one positive center, then it has an ε-positive center for some $\varepsilon > 0$.
(ii) The ε-positive center set of γ is convex for any $\varepsilon \geq 0$.

Let K be the domain enclosed by γ and D the unit disk. For a point $c \in K$, let

$$r_{\text{in}}(c) = \max\{r \geq 0 \mid c + rD \subseteq K\}, \quad r_{\text{out}}(c) = \min\{r > 0 \mid c + rD \supseteq K\}.$$

Through the *Bonnesen function*

$$B(r) = Lr - A - \pi r^2, \tag{1.3}$$

one can get the equivalent definitions of positive centers and ε-positive centers. A point $c \in \operatorname{int} K$ is a *positive center* of γ if it satisfies

$$B(r_{\text{in}}(c)) \geq 0 \quad \text{and} \quad B(r_{\text{out}}(c)) \geq 0. \tag{1.4}$$

A point $c \in \operatorname{int} K$ is an *ε-positive center* of γ if there exists an $\varepsilon \geq 0$ such that

$$B(r_{\text{in}}(c)) \geq \varepsilon \quad \text{and} \quad B(r_{\text{out}}(c)) \geq \varepsilon. \tag{1.5}$$

It is obvious that $0 \leq \varepsilon \leq \min\{Lr_{\text{in}} - A - \pi r_{\text{in}}^2, Lr_{\text{out}} - A - \pi r_{\text{out}}^2\}$ and an ε-positive center must be a positive center.

The purpose of this paper is to describe the ε-positive center set and give a positive answer to Kaiser's conjecture for convex curves. As applications of ε-positive centers, we investigate the ε-positive center sets of constant width curves and give a shorter proof of a geometric inequality that is appeared in [8].

2. Preliminaries

Let E and F be two compact sets in $\mathbb{R}^2$, D the unit disk. The *Minkowski sum* of E and F is defined by

$$E + F = \{x + y \mid x \in E,\ y \in F\}.$$

The Minkowski sum of a disk and a line segment is called a *sausage body* (cf. [9]), its boundary is called a *sausage curve*. Let K be a convex domain with perimeter L and area A. The area of the *outer parallel body* of K at distance t, $K + tD$ $(t \geq 0)$, can be given by

$$A_K(t) \triangleq A(K + tD) = A + Lt + \pi t^2, \tag{2.1}$$

which is called the *Steiner polynomial* of K. If the boundary of K, ∂K, is a strictly convex and C^2 curve, then the area of $K + tD$ can be expressed in terms of the support function $h(\theta)$ of ∂K as

$$A_K(t) = \frac{1}{2}\int_0^{2\pi} \left((h(\theta) + t)^2 - h'(\theta)^2\right) \mathrm{d}\theta. \tag{2.2}$$

The *Minkowski difference* of E and F is defined by

$$E \sim F = \{x \in \mathbb{R}^2 \mid x + F \subseteq E\}.$$

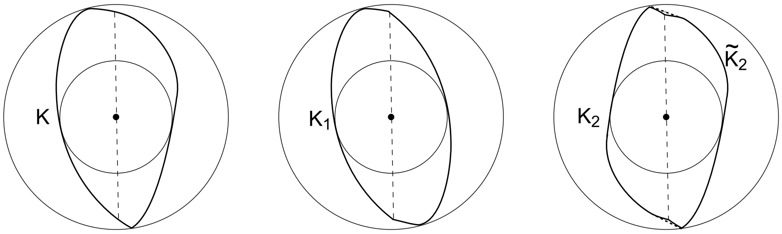
If E and F are both convex domains, then so is $E \sim F$. For convex domains E and F we say that F is a *summand* of E if there is a convex domain M such that $E = F + M$. It is clear that $(E + F) \sim F = E$ holds for any convex domains E and F, while $(E \sim F) + F = E$ holds if and only if F is a summand of E. Denote by r_{in} the inradius of a convex domain E. The set

$$E_{-\lambda} \triangleq E \sim \lambda D, \quad 0 \leq \lambda \leq r_{\text{in}},$$

is called an *inner parallel body of E at distance λ*.

If there exists an ε-positive center, then it is clear that the equation $B(r) = \varepsilon$ has two non-negative real roots. We denote them by $r_1(\varepsilon)$ and $r_2(\varepsilon)$ with $r_1(\varepsilon) \leq r_2(\varepsilon)$.

In the following, "convex curve" means "closed convex plane curve", the set of all positive centers of a convex curve γ is denoted by $\mathfrak{P}(\gamma)$ and that of all ε-positive centers is denoted by $\mathfrak{P}_\varepsilon(\gamma)$, and $C(x, r)$ represents the circle with radius r and centered at x.



(a) K (b) K_1 (c) K_2

Fig. 1. Symmetry.

3. The ε-positive center and Kaiser's conjecture

In this section, we will show that the ε-positive center set of a convex curve is a non-empty convex set. Firstly, we introduce a lemma about the positive center set for centrally symmetric convex curves.

Lemma 3.1. *(See [10].) If γ is a convex curve centrally symmetric with respect to point o, then o is the center of the minimal annulus of γ and $\mathfrak{P}(\gamma)$ is a centrally symmetric domain with the same symmetry center o.*

Proposition 3.2. *If a convex curve γ is neither a circle nor a sausage curve, then $o \in \operatorname{int}\mathfrak{P}(\gamma)$, where o is the center of the minimal annulus of γ.*

To prove the above proposition, we need the following lemma, which is a direct consequence of Proposition 1.6 and Theorem 1.8 of Gage [6].

Lemma 3.3. *(See [6].) Let γ be a convex plane curve, o the center of its minimal annulus. If $s, t \in \gamma \cap C(o, r_{\text{in}}(o))$ and $S, T \in \gamma \cap C(o, r_{\text{out}}(o))$ and the line segments $\overline{st}$ and $\overline{ST}$ satisfy $\overline{st} \cap \overline{ST} \neq \emptyset$, then there is a line l with the following properties:*

(i) $l \cap K$ is a line segment with o as its midpoint, where K is the domain enclosed by γ;
(ii) the points s and t lie on different sides of l, and so do S and T.

Proof of Proposition 3.2. From [10, Theorems 2.6 and 2.7], we have known that $\operatorname{int}\mathfrak{P}(\gamma) \neq \emptyset$ when γ is neither a circle nor a sausage curve. Since the center o of the minimal annulus of γ must be a point of $\mathfrak{P}(\gamma)$, $o \in \operatorname{int}\mathfrak{P}(\gamma)$ or $o \in \partial\mathfrak{P}(\gamma)$. If $o \in \partial\mathfrak{P}(\gamma)$, then γ is not symmetric with respect to o by Lemma 3.1. The domain K enclosed by γ can be cut into two parts by a chord through o as shown in Fig. 1a by Lemma 3.3. Denote by L_i and A_i $(i = 1, 2)$ the length and the area of the two parts, respectively. Through a symmetrization of the two parts with respect to o, we obtain two centrally symmetric domains K_1 and K_2 as shown in Figs. 1b and 1c. It is obvious that the $r_{\text{in}}(o)$s in these three figures are equal and so are $r_{\text{out}}(o)$s.

Since K_1 is convex, from Lemma 3.1, we have

$$2L_1 r_{\text{in}}(o) - 2A_1 - \pi r_{\text{in}}^2(o) \geq 0, \quad 2L_1 r_{\text{out}}(o) - 2A_1 - \pi r_{\text{out}}^2(o) \geq 0.$$

As for K_2, as it is unnecessarily convex, we consider its convex hull $\widetilde{K}_2$, denote its perimeter and area by $\widetilde{L}_2$ and $\widetilde{A}_2$, respectively. Again by Lemma 3.1 and the fact that $\widetilde{L}_2 \leq 2L_2$ and $\widetilde{A}_2 \geq 2A_2$, we get

$$2L_2 r_{\text{in}}(o) - 2A_2 - \pi r_{\text{in}}^2(o) \geq \widetilde{L}_2 r_{\text{in}}(o) - \widetilde{A}_2 - \pi r_{\text{in}}^2(o) \geq 0,$$
$$2L_2 r_{\text{out}}(o) - 2A_2 - \pi r_{\text{out}}^2(o) \geq \widetilde{L}_2 r_{\text{out}}(o) - \widetilde{A}_2 - \pi r_{\text{out}}^2(o) \geq 0.$$

Hence

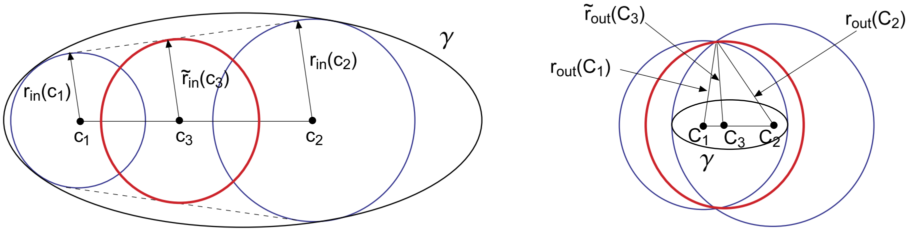
$$B(r_{\text{in}}(o)) = L r_{\text{in}}(o) - A - \pi r_{\text{in}}^2(o) \geq 0,$$
$$B(r_{\text{out}}(o)) = L r_{\text{out}}(o) - A - \pi r_{\text{out}}^2(o) \geq 0.$$

From [10, Theorem 2.1] and the fact that $o \in \partial\mathfrak{P}(\gamma)$, it follows that $B(r_{\text{in}}(o)) = 0$ or $B(r_{\text{out}}(o)) = 0$.

If $B(r_{\text{in}}(o)) = 0$, then

$$2L_1 r_{\text{in}}(o) - 2A_1 - \pi r_{\text{in}}^2(o) = 0,$$
$$2L_2 r_{\text{in}}(o) - 2A_2 - \pi r_{\text{in}}^2(o) = \widetilde{L}_2 r_{\text{in}}(o) - \widetilde{A}_2 - \pi r_{\text{in}}^2(o) = 0.$$

Therefore, $\widetilde{K}_2 = K_2$. Since K_1 and K_2 are centrally symmetric with respect to o, $r_{\text{in}} = r_{\text{in}}(o)$ and $r_{\text{out}} = r_{\text{out}}(o)$, which implies that ∂K_1 is a circle or a sausage curve, so is ∂K_2. If either ∂K_1 is a circle and ∂K_2 is a sausage curve or ∂K_1 is a sausage



(a) $r_{in}(c_3)$ (b) $r_{out}(c_3)$

Fig. 2. $r_{\mathrm{in}}(c_3)$ and $r_{\mathrm{out}}(c_3)$.

curve and ∂K_2 is a circle, then it contradicts the fact that K_1 and K_2 have the same $r_{\mathrm{in}}(o)$ and $r_{\mathrm{out}}(o)$. If both ∂K_1 and ∂K_2 are circles or sausage curves, then γ must be a circle or a sausage curve, which is a contradiction of the fact that γ is not centrally symmetric.

If $B(r_{\mathrm{out}}(o)) = 0$, a similar argument implies that γ is a circle, which is impossible. Therefore, $o \in \operatorname{int}\mathfrak{P}(\gamma)$. □

Theorem 3.4. *If a convex curve γ is neither a circle nor a sausage curve, then there exists a positive number $\varepsilon > 0$ such that $\mathfrak{P}_\varepsilon(\gamma) \neq \emptyset$.*

Proof. By Proposition 3.2, one can see that

$$B(r_{\mathrm{in}}(o)) > 0 \quad \text{and} \quad B(r_{\mathrm{out}}(o)) > 0,$$

where o is the center of the minimal annulus of γ. It follows from the continuities of $r_{\mathrm{in}}(\cdot)$, $r_{\mathrm{out}}(\cdot)$, $B(r_{\mathrm{in}}(\cdot))$ and $B(r_{\mathrm{out}}(\cdot))$ that there exists an $\varepsilon > 0$ such that

$$B(r_{\mathrm{in}}(o)) \geq \varepsilon \quad \text{and} \quad B(r_{\mathrm{out}}(o)) \geq \varepsilon.$$

Hence, $o \in \mathfrak{P}_\varepsilon(\gamma)$, that is to say, $\mathfrak{P}_\varepsilon(\gamma) \neq \emptyset$. □

Remark 3.5. This theorem gives a positive answer to Conjecture (i) of Kaiser.

Corollary 3.6. *If γ is a strictly convex non-circular curve, then there exists an $\varepsilon > 0$ such that $\mathfrak{P}_\varepsilon(\gamma) \neq \emptyset$.*

To prove the convexity of the ε-positive center set of a convex curve, we need the following lemma.

Lemma 3.7. *Let γ be a convex curve. If c_1 and c_2 are two ε-positive centers of γ, then for any point c_3 on line segment $\overline{c_1c_2}$, one can get*

$$B(r_{\mathrm{in}}(c_3)) \geq \varepsilon \quad \textit{and} \quad B(r_{\mathrm{out}}(c_3)) \geq \varepsilon.$$

Proof. Let $C(c_3, \widetilde{r}_{\mathrm{in}}(c_3))$ be the largest inscribed circle of the convex hull of circles $C(c_1, r_{\mathrm{in}}(c_1))$ and $C(c_2, r_{\mathrm{in}}(c_2))$, $C(c_3, \widetilde{r}_{\mathrm{out}}(c_3))$ the circle that contains the two intersection points of the circles $C(c_1, r_{\mathrm{out}}(c_1))$ and $C(c_2, r_{\mathrm{out}}(c_2))$ (see Fig. 2). Since γ is convex, for the case $r_{\mathrm{in}}(\cdot)$, γ contains circles $C(c_1, r_{\mathrm{in}}(c_1))$, $C(c_2, r_{\mathrm{in}}(c_2))$ and $C(c_3, \widetilde{r}_{\mathrm{in}}(c_3))$; for the case $r_{\mathrm{out}}(\cdot)$, circles $C(c_1, r_{\mathrm{out}}(c_1))$, $C(c_2, r_{\mathrm{out}}(c_2))$ and $C(c_3, \widetilde{r}_{\mathrm{out}}(c_3))$ contain γ. From Fig. 2, it is clear that

$$\min\{r_{\mathrm{in}}(c_1), r_{\mathrm{in}}(c_2)\} \leq \widetilde{r}_{\mathrm{in}}(c_3) \leq r_{\mathrm{in}}(c_3), \tag{3.1}$$

$$r_{\mathrm{out}}(c_3) \leq \widetilde{r}_{\mathrm{out}}(c_3) < \max\{r_{\mathrm{out}}(c_1), r_{\mathrm{out}}(c_2)\}. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$r_1(\varepsilon) \leq \min\{r_{\mathrm{in}}(c_1), r_{\mathrm{in}}(c_2)\} \leq r_{\mathrm{in}}(c_3) \leq r_{\mathrm{out}}(c_3) \leq \max\{r_{\mathrm{out}}(c_1), r_{\mathrm{out}}(c_2)\} \leq r_2(\varepsilon).$$

Thus

$$B(r_{\mathrm{in}}(c_3)) \geq \varepsilon \quad \text{and} \quad B(r_{\mathrm{out}}(c_3)) \geq \varepsilon. \qquad \square$$

Theorem 3.8. *If γ is a convex curve, then $\mathfrak{P}_\varepsilon(\gamma)$ is a closed convex set for any $\varepsilon \geq 0$. Moreover, if $\mathfrak{P}_\varepsilon(\gamma) \neq \emptyset$, then for any boundary point c of $\mathfrak{P}_\varepsilon(\gamma)$, at least one of $B(r_{\mathrm{in}}(c)) = \varepsilon$ and $B(r_{\mathrm{out}}(c)) = \varepsilon$ holds.*

Proof. From the definition of ε-positive centers and the continuity of $B(r)$, it follows that there exists a maximum of ε, denoted by $\varepsilon_{\max}$, such that $\mathfrak{P}_\varepsilon(\gamma)$ is not an empty set. If $\varepsilon > \varepsilon_{\max}$, then $\mathfrak{P}_\varepsilon(\gamma) = \emptyset$. If $0 \leq \varepsilon \leq \varepsilon_{\max}$, then it is clear that $\mathfrak{P}_\varepsilon(\gamma)$ is closed. Next, we deal with its convexity. If $\mathfrak{P}_\varepsilon(\gamma)$ has only one point, its convexity is obvious. If $\mathfrak{P}_\varepsilon(\gamma)$ has more than one point, then Lemma 3.7 can yield that $\mathfrak{P}_\varepsilon(\gamma)$ is a convex set. And therefore, for any boundary point c of $\mathfrak{P}_\varepsilon(\gamma)$, at least one of $B(r_{\mathrm{in}}(c)) = \varepsilon$ and $B(r_{\mathrm{out}}(c)) = \varepsilon$ holds when $0 \leq \varepsilon \leq \varepsilon_{\max}$. □

4. Applications

As an application of ε-positive centers, we describe the ε-positive center sets of constant width curves. We need the following lemma about constant width curves; its proof can be found in [10].

Lemma 4.1. *(See [10].) If γ is a curve of constant width w and K is the domain enclosed by γ, then*

$$r_{\text{in}}(c) + r_{\text{out}}(c) = w, \quad c \in K.$$

Proposition 4.2. *If γ is a curve of constant width w with area A, then for any $\varepsilon \in [0, \pi w r_{\text{in}} - A - \pi r_{\text{in}}^2]$, we have*

(i) $\mathfrak{P}_\varepsilon(\gamma)$ is its inner parallel body $K_{-r_1(\varepsilon)}$, where $r_1(\varepsilon)$ is the smaller root of $\pi w r - A - \pi r^2 = \varepsilon$. Moreover, if $\varepsilon = \pi w r_{\text{in}} - A - \pi r_{\text{in}}^2$, then $\mathfrak{P}_\varepsilon(\gamma)$ has only one point, which is just the center o of the minimal annulus of γ;
(ii) $B(r_{\text{in}}(c)) = B(r_{\text{out}}(c)) = \varepsilon$ holds for each boundary point c of $\mathfrak{P}_\varepsilon(\gamma)$.

Proof. (i) Let K be the domain bounded by γ. Since γ is a curve of constant width w, by Lemma 4.1, we have

$$r_{\text{in}}(c) + r_{\text{out}}(c) = w, \quad c \in K. \tag{4.1}$$

For any $\varepsilon \in [0, \pi w r_{\text{in}} - A - \pi r_{\text{in}}^2]$, the quadratic equation $B(r) = \varepsilon$ has two real roots $r_1(\varepsilon)$, $r_2(\varepsilon)$ and

$$r_1(\varepsilon) + r_2(\varepsilon) = w. \tag{4.2}$$

Eqs. (4.1) and (4.2) imply that $r_{\text{in}}(c)$ and $r_{\text{out}}(c)$ are symmetric with respect to $\frac{w}{2}$ and so are $r_1(\varepsilon)$ and $r_2(\varepsilon)$. Thus, if $r_{\text{in}}(c) \geq r_1(\varepsilon)$, then $r_{\text{out}}(c) \leq r_2(\varepsilon)$. It follows from the definitions of $\mathfrak{P}_\varepsilon(\gamma)$ and inner parallel body that $\mathfrak{P}_\varepsilon(\gamma)$ is the inner parallel body $K_{-r_1(\varepsilon)}$ of K.

If $\varepsilon = \pi w r_{\text{in}} - A - \pi r_{\text{in}}^2$, then it is clear that the center o of the minimal annulus of γ is the only point of $\mathfrak{P}_\varepsilon(\gamma)$.

(ii) Since $r_{\text{in}}(c)$ and $r_{\text{out}}(c)$ are symmetric with respect to $\frac{w}{2}$, $B(r_{\text{in}}(c)) = B(r_{\text{out}}(c))$, which together with Theorem 3.8 yields that $B(r_{\text{in}}(c)) = B(r_{\text{out}}(c)) = \varepsilon$ holds for any boundary point c of $\mathfrak{P}_\varepsilon(\gamma)$. $\square$

Motivated by Jetter's idea in [11], we give a different proof of Theorem 1.10 of [8] through ε-positive center and Blaschke's rolling theorem (cf. [15, Corollary 3.2.10]).

Proposition 4.3. *If γ is a strictly convex non-circular C^2 curve with length L and area A, then*

$$-\rho_{\max} < t_2 < -r_{\text{out}} < -\frac{L}{2\pi} < -r_{\text{in}} < t_1 < -\rho_{\min} < 0,$$

where $\rho_{\max}$ and $\rho_{\min}$ are the maximum and minimum curvature radii of γ, r_{in} and r_{out} are the inradius and circumradius of γ, t_1 and t_2 are the roots of the Steiner polynomial of domain K enclosed by γ.

Proof. Since $r_{\text{in}} D \subseteq K \subseteq r_{\text{out}} D$, $r_{\text{in}} \leq \frac{L}{2\pi} \leq r_{\text{out}}$ and the equalities hold if and only if K is a disk, that is, γ is a circle.

From Corollary 3.6, there exists an $\varepsilon > 0$ such that $\mathfrak{P}_\varepsilon(\gamma) \neq \emptyset$. For any point c of $\mathfrak{P}_\varepsilon(\gamma)$, we have

$$B(r_{\text{in}}(c)) > 0 \quad \text{and} \quad B(r_{\text{out}}(c)) > 0.$$

Thus, $-r_{\text{in}} \leq -r_{\text{in}}(c) < t_1$ and $t_2 < -r_{\text{out}}(c) \leq -r_{\text{out}}$.

Denote by $h(\theta)$ the support function of γ. Let $0 \leq m \leq \rho_{\min}$. It follows from the Blaschke rolling theorem (cf. [15, Corollary 3.2.10]) that $(K \sim mD) + mD = K$, hence $h_{K \sim mD} = h_K - m$. By (2.2), we obtain

$$A_{K \sim mD}(t) = \frac{1}{2} \int_0^{2\pi} \left((h(\theta) - m + t)^2 - h'(\theta)^2 \right) \mathrm{d}\theta = A_K(t - m).$$

From the fact that t_1, t_2 are the two roots of $A_K(t) = 0$, it follows that $t_1 + m$ and $t_2 + m$ are roots of $A_{K \sim mD}(t) = 0$. Since for any convex domain K, $A_K(t) = 0$ has two non-positive real roots, we have $t_1 + m \leq 0$ and the inequality is sharp when the area of K is positive. Hence, $t_1 \leq -m$, $\forall m \leq \rho_{\min}$. Set $m = \rho_{\min}$, we get $t_1 \leq -\rho_{\min}$. From the above discussions, $r_{\text{in}} > \rho_{\min}$, which implies that the area of $K \sim \rho_{\min} D$ is positive, and thus $t_1 < -\rho_{\min}$. Similarly, let $m \geq \rho_{\max}$, we can get $-\rho_{\max} < t_2$. $\square$

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