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The branch set of a quasiregular mapping between metric manifolds

L'ensemble de branchement d'une application quasi régulière entre variétés métriques

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A R T I C L E I N F O

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A B S T R A C T

In this note, we announce some new results on quantitative countable porosity of the branch set of a quasiregular mapping in very general metric spaces. As applications, we solve a recent conjecture of Fässler et al., an open problem of Heinonen–Rickman, and an open question of Heinonen–Semmes.

R É S U M É

Dans cette note, nous annonçons de nouveaux résultats quant à la porosité dénombrable quantitative de l'ensemble des branchements d'une application quasi régulière dans un cadre très général d'espaces métriques. Comme applications de nos résultats, nous répondons à une conjecture récente de Fässler et al., à un problème ouvert de Heinonen–Rickman et à une question ouverte de Heinonen–Semmes.

1. Introduction and main results

A continuous mapping $f : X \to Y$ between topological spaces is said to be a *branched covering* if f is *discrete* and *open*, i.e., f is an open mapping and if, for each $y \in Y$, the preimage $f^{-1}(y)$ is a discrete subset of X. The *branch set* $\mathcal{B}_f$ of f is the closed set of points in X where f does not define a local homeomorphism. In the case where X and Y are *generalized n-manifolds*, $\mathcal{B}_f$ can be interpreted alternatively as the set of points at which the *local index* $i(x, f) = 1$.

For a branched covering $f : X \to Y$ between two metric spaces, $x \in X$ and $r > 0$, set

$$H_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)},$$

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where

$$L_f(x,r) := \sup\{d(f(x), f(y)) : d(x,y) = r\},$$

and

$$l_f(x,r) := \inf\{d(f(x), f(y)) : d(x,y) = r\}.$$

Then the *linear dilatation function* of f at x is defined pointwise by

$$H_f(x) = \limsup_{r\to 0} H_f(x,r).$$

For each $x \in X$, denote by $U(x,r)$ the component of x in $f^{-1}(B(f(x),r))$. Set

$$H_f^*(x,s) = \frac{L_f^*(x,s)}{l_f^*(x,s)},$$

where

$$L_f^*(x,s) = \sup_{z\in\partial U(x,s)} d(x,z) \quad \textit{and} \quad l_f^*(x,s) = \inf_{z\in\partial U(x,s)} d(x,z).$$

The *inverse linear dilatation function* of f at x is defined pointwise by

$$H_f^*(x) = \limsup_{s\to 0} H_f^*(x,s).$$

A mapping $f : X \to Y$ between two metric measure spaces is termed *H-quasiregular* if the linear dilatation function H_f is finite everywhere and essentially bounded from above by H. We call f a *quasiregular mapping* if it is H-quasiregular for some $H \in [1,\infty)$.

The branch set of a quasiregular mapping can be very wild, for instance, it may contain many wild Cantor sets, such as the Antoine's necklace [10], of classical geometric topology. In his 2002 ICM address [8, Section 3], Heinonen asked about the following question: *Can we describe the geometry and topology of allowable branch sets of quasiregular mappings between metric n-manifolds?*

In [6], we explore the (geometric) porosity of $\mathcal{B}_f \cap A$ and $f(\mathcal{B}_f \cap A)$ when the linear dilatation of f is finite on A. Our main result states that if X satisfies a quantitative local connectivity assumption, the aforementioned sets are quantitatively porous.

In the remainder of this introduction, we take as standing assumptions that X and Y are *compact* and *doubling metric spaces*, which are also *generalized n-manifolds*, that X is *linearly locally n-connected*, and that Y has *bounded turning*. Recall that X is *λ-linearly locally n-connected* (abbreviated λ-LLC^n) if for each $x \in X$ and $r < 2d(x,\partial X)/\lambda$, the ball $B(x,r)$ is n-connected in $B(x,\lambda r/2)$. (The other definitions are standard and can be found for instance in [11,6].)

The following result is a special case of [6, Theorem 1.1] and it says that the branch set of a quasiregular mapping as well as its image are quantitatively porous.

Theorem 1.1. *If $H_f(x) \le H$ or $H_f^*(x) \le H$ for every $x \in X$, then $\mathcal{B}_f$ and $f(\mathcal{B}_f)$ are countably δ-porous, quantitatively. Moreover, the porosity constant can be explicitly calculated.*

Recall that a set $E \subset X$ is said to be *α-porous* if for each $x \in E$,

$$\liminf_{r\to 0} r^{-1} \sup\left\{\rho : B(z,\rho) \subset B(x,r)\backslash E\right\} \ge \alpha. \tag{1}$$

A subset E of X is called *countablely (σ-)porous* if it is a countable union of (σ-)porous subsets of X.

In the special case where X and Y are Euclidean spaces, Theorem 1.1 strengthens the earlier quantitative porosity results of Bonk–Heinonen [1] and Onninen–Rajala [17] on the branch set of a quasiregular mapping. Moreover, the quantitative porosity bounds on $f(\mathcal{B}_f)$ are new and can be regarded as a strengthened version of the dimensional estimate of Sarvas [20].

Particularly important to the general theory of quasiconformal and quasisymmetric maps are *Ahlfors Q-regular spaces*. It is well known that porous subsets of such spaces have Hausdorff dimension strictly smaller than Q, quantitatively; see, e.g., [17, Lemma 9.2]. Thus we have the following consequence.

Corollary 1.2. *If X and Y are Ahlfors Q-regular, and $H_f(x) < \infty$ or $H_f^*(x) < \infty$ for all $x \in X$, then $\mathcal{H}^Q(\mathcal{B}_f) = \mathcal{H}^Q(f(\mathcal{B}_f)) = 0$. Moreover, if either $H_f(x) \le H$ or $H_f^*(x) \le H$ for all $x \in X$, then*

$$\max\left\{\dim_{\mathcal{H}}(\mathcal{B}_f), \dim_{\mathcal{H}}(f(\mathcal{B}_f))\right\} \le Q - \eta < Q,$$

where η depends only on H and the data of X and Y, and it can be explicitly calculated.

Applying the first part of Corollary 1.2 to the special case where X and Y are equiregular sub-Riemannian manifolds, it gives an affirmative answer to a recent conjecture [2, Remark 1.2].

To prove that the branch set of a quasiregular mapping is null with respect to the right Hausdorff measure, the earlier proofs are based on two important assumptions: the first fact is that the mapping in question is differentiable almost everywhere in a suitable sense and the differential is linear with respect to the group structure of the tangent space; the second fact is that the determinant of the mapping is positive almost everywhere; see [14] for Euclidean case, [11,5] for the generalized manifolds of type A case, and [7] for the subRiemannian case for detailed information of this approach.

To obtain a dimensional estimate as in Corollary 1.2, the earlier proofs of Bonk–Heinonen [1] and Onninen–Rajala [17] go along the following lines: one first obtains a quantitative porosity estimate for the set of points with sufficiently large local index; then one proves other quantitative porosity results for the set of points with a precise local index bound. The quantitative porosity bound on the branch set then follows by combining the above two estimates. In both approaches, two important assumptions are necessary: the first assumption is that the domain has to be Euclidean, since the McAuley–Robinson theorem [16] is necessary and its proof relies crucially on the affine structure of Euclidean spaces; the second assumption is that certain abstract Poincaré inequalities in the sense of Heinonen–Koskela [9] are necessary, since we need to use the standard modulus (of a curve family) techniques.

To illustrate our idea for Theorem 1.1, we need the following terminology introduced in [6]. Let $f : X \to Y$ be a branched covering between two metric spaces. Fix $x_0 \in X$, $y_0 = f(x_0)$, $r > 0$. We say a map $g\colon B(y_0, r) \to X$ is a *local left homotopy inverse* for f at x_0 if $g \circ f|_{U(x_0,r)}$ is homotopic to the identity on $U(x_0, r)$, via a homotopy H_t for which $x_0 \notin H_t(\partial U(x_0, r))$ for all t. Similarly, g is a *local right homotopy inverse* for f if $f \circ g$ is homotopic to the identity on $B(y_0, r)$, via a homotopy H_t with $y_0 \notin H_t(\partial B(y_0, r))$ for all t. If g is a left and right local homotopy inverse, we simply call it a *local homotopy inverse*. We denote by $\mathcal{B}^*_f$ the *homotopy branch set* of f, i.e., the set of points in X for which f has no (two-sided) local homotopy inverse. We also let $\mathcal{B}^{*,l}_f$ denote the *left homotopy branch set*, i.e., the set of points in X at which f has no left homotopy inverse. It is clear that if X and Y are generalized n-manifolds, then $\mathcal{B}_f = \mathcal{B}^{*,l}_f$.

Our starting point is to construct local left homotopy inverses away from a porous set. In the first step, we further developed a quantitative ENR theory, inspired by Groce, Petersen and Wu [3,4,18], and Semmes [21], for LLC^n spaces. As an immediate consequence, we obtain a generalized McAuley–Robinson theorem (cf. Theorem 4.1), which provides a criterion of being non-branching. The second step is to control the distortion of annuli, quantitatively, at points of finite dilatation, away from a porous set. The moral here is that if either of the sets

$$S_{H,R} = \{x \in X : H_f(x, r) \le H \text{ for all } r < R\}$$

or $f(S_{H,R})$ is “dense” at some point at a certain scale, then the annular distortion around that point will get controlled. Thus we may construct a local homotopy inverse around that point and use the generalized McAuley–Robinson theorem to conclude that the point is non-branching. This, together with a simple decomposition argument (cf. [6, Proof of Theorem 1.1]), will lead to Theorem 1.1.

Our standing assumptions for the underlying spaces X and Y, except the local linear n-connectivity on X, are quite mild. On the other hand, the local linear n-connectivity is necessary for the validity of all the previous results, as [24, Theorem 1.2] indicates.

Theorem 1.3. *For each $n \ge 3$, there exists an Ahlfors n-regular metric space X that is homeomorphic to $\mathbb{R}^n$ and supports a $(1,1)$-Poincaré inequality, and a 1-quasiregular mapping $f : X \to \mathbb{R}^n$, such that* $\min\{\mathcal{H}^n(\mathcal{B}_f), \mathcal{H}^n(f(\mathcal{B}_f))\} > 0$.

The example in Theorem 1.3 is indeed 1-BLD and it disproves the following well-known conjecture of Heinonen and Rickman [11, Remark 6.32 (b)]: *if X be a locally BLD n-Euclidean space that is locally bi-Lipschitz embeddable to some Euclidean space, then for all (Lipschitz) BLD-maps $f : X \to \mathbb{R}^n$, the branch set $\mathcal{B}_f$ has zero Hausdorff n-measure*. As a consequence, one cannot delete [13, Axiom II] from the a priori assumptions in [13, Theorem 2.1]; see [13, Section 5.1] and [24] for more detailed discussions.

2. Väisälä’s inequality

The Väisälä’s inequality was first proved by Väisälä [22] and it plays an important role in the theory of quasiregular mappings, in particular, many profound value-distributional type results; see [19].

Definition 2.1 *(Väisälä’s inequality).* We say that f satisfies *Väisälä’s inequality* with constant K_I if the following condition holds: suppose $m \in \mathbb{N}$, and Γ and Γ' are curve families in X and Y respectively, such that for each $\gamma' \in \Gamma'$, there are curves $\gamma_1, \dots, \gamma_m \in \Gamma$ such that $f(\gamma_k)$ is a subcurve of γ' for each k, and for each $t \in [0, l(\gamma)]$ and each $x \in X$, we have $\#\{k : \gamma_k(t) = x\} \le i(x, f)$. Then

$$\mathrm{Mod}_Q(\Gamma') \le K_I \,\mathrm{Mod}_Q(\Gamma)/m.$$

As an application of Corollary 1.2 and [23, Theorem 1.1], we obtain the following very general Väisälä’s inequality in [6].

Theorem 2.1 *(Väisälä's inequality). Let X and Y be Ahlfors Q-regular generalized n-manifolds, where X is LLC^n and Y is linearly locally connected, and suppose $f : X \to Y$ is discrete and open, with $H_f(x) < \infty$ for all $x \in X$, and $H_f(x) \le H$ for $\mathcal{H}^Q$-almost every $x \in X$.*

Then f satisfies Väisälä's inequality for some constant K_I depending only on H and the data of X and Y.

In the special case where Y is a generalized manifold with controlled geometry and topology and $X = \mathbb{R}^n$, Theorem 2.1 was first proved by Onninen and Rajala [17, Theorem 11.1].

3. Loewner spaces

There is a subtlety to the observation that Corollary 1.2 generalizes the Bonk–Heinonen theorem, which gave an index-free upper bound on $\dim_{\mathcal{H}} \mathcal{B}_f$. In general, the linear dilatation $H_f(x)$ of a quasiregular map in $\mathbb{R}^n$ does not need to be globally bounded – it is instead finite and essentially bounded, and at any point $x \in \mathbb{R}^n$, the dilatation depends quantitatively on not merely the essential supremum of H_f, but also on the index $i(x, f)$. That Corollary 1.2 is an actual generalization requires the fact that $H_f^*(x)$ is bounded everywhere by a constant H^* independent of $i(x, f)$. This latter fact was proved in the Euclidean case in [15], using the K_O- and Väisälä's inequalities, as well as the Loewner property of $\mathbb{R}^n$.

In the case where X and Y are Loewner, however, Väisälä's inequality allows us to generalize the corresponding result of [15], giving an index free upper bound on H_f^*.

Theorem 3.1. *Suppose (under the standing assumptions) that X and Y are locally Ahlfors Q-regular and Q-Loewner, $H_f(x) < \infty$ for all $x \in X$, and $H_f(x) \le H$ for $\mathcal{H}^Q$-almost every $x \in X$. Then $H_f^*(x) \le H^*$ for every $x \in X$, where H^* depends only on H and the data of X and Y, and the sets $\mathcal{B}_f$ and $f(\mathcal{B}_f)$ are δ-porous, for some δ depending only on H and the data.*

Combining Theorem 3.1 with Corollary 1.2, we obtain the following result, the first half of which is a true generalization of the Bonk–Heinonen theorem.

Corollary 3.2. *Under the assumptions of Theorem 3.1, we have*

$$\max\left\{\dim_{\mathcal{H}}(\mathcal{B}_f), \dim_{\mathcal{H}}(f(\mathcal{B}_f))\right\} \le Q - \eta < Q,$$

for some constant η depending only on H and the data of X and Y.

Corollary 3.2 answers affirmatively the open problem of Heinonen and Rickman [11, Remark 6.7 (b)] in a stronger form, namely, we obtain dimensional estimates for the class of quasiregular mappings, which is strictly large than the class of BLD mappings.

It was asked by Heinonen and Semmes [12, Question 27] that *if for a given branched covering $f : S^n \to S^n$, $n \ge 3$, there is a metric d on S^n so that (S^n, d) is an Ahlfors n-regular and locally linearly contractible metric space, and $f : (S^n, d) \to S^n$ is a BLD mapping.* By Corollary 3.2, the existence of such a metric d necessarily implies that $\mathcal{B}_f$ must be null with respect to the n-dimensional Hausdorff measure $\mathcal{H}^n$. On the other hand, there are plenty of branched coverings $f : S^n \to S^n$ such that $\mathcal{H}^n(\mathcal{B}_f) > 0$ and so we have the following negative answer to this question.

Corollary 3.3. *Not every branched covering $f : S^n \to S^n$, $n \ge 3$, can be made BLD by changing the metric in the domain but keeping the space Ahlfors n-regular and linearly locally contractible.*

4. Generalization of the McAuley–Robinson theorem

One of the crucial ingredient in the proof of Theorem 1.1 is the following generalization of the McAuley–Robinson theorem [16], which is of independent interest.

Theorem 4.1. *Let $A \subset X$, where X is a λ-LLC^n generalized n-manifold and $\dim_{top}(A) \le n$ and let Y be another generalized n-manifold. Let $f : X \to Y$ be a proper branched covering such that for some $x_0 \in A \setminus \partial A$, $f^{-1}(\{f(x_0)\}) = x_0$ and*

$$\sup_{x \in \partial A} \frac{\operatorname{diam} f^{-1}(\{f(x)\})}{d(x, x_0)} < \frac{1}{\lambda^{2n+1}}.$$

Then $x_0 \notin \mathcal{B}_f$.

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