Complex analysis/Topology

The branch set of a quasiregular mapping between metric manifolds

L'ensemble de branchement d'une application quasi régulière entre variétés métriques

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\section*{A B S T R A C T}
In this note, we announce some new results on quantitative countable porosity of the branch set of a quasiregular mapping in very general metric spaces. As applications, we solve a recent conjecture of Fässler et al., an open problem of Heinonen–Rickman, and an open question of Heinonen–Semmes.
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\section*{RÉSUMÉ}
Dans cette note, nous annonçons de nouveaux résultats quant à la porosité dénombrable quantitative de l'ensemble des branchements d'une application quasi régulière dans un cadre très général d'espaces métriques. Comme applications de nos résultats, nous répondons à une conjecture récente de Fässler et al., à un problème ouvert de Heinonen–Rickman et à une question ouverte de Heinonen–Semmes.
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1. Introduction and main results
A continuous mapping \( f : X \to Y \) between topological spaces is said to be a branched covering if \( f \) is discrete and open, i.e., \( f \) is an open mapping and if, for each \( y \in Y \), the preimage \( f^{-1}(y) \) is a discrete subset of \( X \). The branch set \( B_f \) of \( f \) is the closed set of points in \( X \) where \( f \) does not define a local homeomorphism. In the case where \( X \) and \( Y \) are generalized \( n \)-manifolds, \( B_f \) can be interpreted alternatively as the set of points at which the local index \( i(x, f) = 1 \).

For a branched covering \( f : X \to Y \) between two metric spaces, \( x \in X \) and \( r > 0 \), set

\[ H_f(x, r) = \frac{L_f(x, r)}{L_f(x, r)}. \]
where
\[ L_f(x, r) := \sup \{ d(f(x), f(y)) : d(x, y) = r \}, \]
and
\[ I_f(x, r) := \inf \{ d(f(x), f(y)) : d(x, y) = r \}. \]
Then the linear dilatation function of \( f \) at \( x \) is defined pointwise by
\[ H_f(x) = \limsup_{r \to 0} H_f(x, r). \]
For each \( x \in X \), denote by \( U(x, r) \) the component of \( x \) in \( f^{-1}(B(f(x), r)) \). Set
\[ H_f^*(x, s) = \frac{L_f^*(x, s)}{I_f^*(x, s)}, \]
where
\[ L_f^*(x, s) := \sup_{z \in \partial U(x, s)} d(x, z) \quad \text{and} \quad I_f^*(x, s) := \inf_{z \in \partial U(x, s)} d(x, z). \]
The inverse linear dilatation function of \( f \) at \( x \) is defined pointwise by
\[ H_f^*(x) = \limsup_{s \to 0} H_f^*(x, s). \]
A mapping \( f : X \to Y \) between two metric measure spaces is termed \( H \)-quasiregular if the linear dilatation function \( H_f \) is finite everywhere and essentially bounded from above by \( H \). We call \( f \) a quasiregular mapping if it is \( H \)-quasiregular for some \( H \in [1, \infty) \).

The branch set of a quasiregular mapping can be very wild, for instance, it may contain many wild Cantor sets, such as the Antoine's necklace [10], of classical geometric topology. In his 2002 ICM address [8, Section 3], Heinonen asked about the following question: Can we describe the geometry and topology of allowable branch sets of quasiregular mappings between metric \( n \)-manifolds?

In [6], we explore the (geometric) porosity of \( B_f \cap A \) and \( f(B_f \cap A) \) when the linear dilatation of \( f \) is finite on \( A \). Our main result states that if \( X \) satisfies a quantitative local connectivity assumption, the aforementioned sets are quantitatively porous.

In the remainder of this introduction, we take as standing assumptions that \( X \) and \( Y \) are compact and doubling metric spaces, which are also generalized \( n \)-manifolds, that \( X \) is linearly locally \( n \)-connected, and that \( Y \) has bounded turning. Recall that \( X \) is \( \lambda \)-linearly locally \( n \)-connected (abbreviated \( \lambda \)-LLC\(^n\)) if for each \( x \in X \) and \( r < 2d(x, \partial X)/\lambda \), the ball \( B(x, r) \) is \( n \)-connected in \( B(x, \lambda r/2) \). (The other definitions are standard and can be found for instance in [11,6].)

The following result is a special case of [6, Theorem 1.1] and it says that the branch set of a quasiregular mapping as well as its image are quantitatively porous.

**Theorem 1.1.** If \( H_f(x) \leq H \) or \( H_f^*(x) \leq H \) for every \( x \in X \), then \( B_f \) and \( f(B_f) \) are countably \( \delta \)-porous, quantitatively. Moreover, the porosity constant can be explicitly calculated.

Recall that a set \( E \subset X \) is said to be \( \alpha \)-porous if for each \( x \in E \),
\[
\liminf_{r \to 0} r^{-1} \sup_{\rho : B(z, \rho) \subset B(x, r) \setminus E} \rho \geq \alpha.
\]
A subset \( E \) of \( X \) is called countably \( (\sigma-)porous \) if it is a countable union of \((\sigma-)porous \) subsets of \( X \).

In the special case where \( X \) and \( Y \) are Euclidean spaces, Theorem 1.1 strengthens the earlier quantitative porosity results of Bonk–Heinonen [1] and Onninen–Rajala [17] on the branch set of a quasiregular mapping. Moreover, the quantitative porosity bounds on \( f(B_f) \) are new and can be regarded as a strengthened version of the dimensional estimate of Sarvas [20].

Particularly important to the general theory of quasiconformal and quasisymmetric maps are Ahlfors \( Q \)-regular spaces. It is well known that porous subsets of such spaces have Hausdorff dimension strictly smaller than \( Q \), quantitatively; see, e.g., [17, Lemma 9.2]. Thus we have the following consequence.

**Corollary 1.2.** If \( X \) and \( Y \) are Ahlfors \( Q \)-regular, and \( H_f(x) < \infty \) or \( H_f^*(x) < \infty \) for all \( x \in X \), then \( \mathcal{H}^Q(B_f) = \mathcal{H}^Q(f(B_f)) = 0 \).

Moreover, if either \( H_f(x) \leq H \) or \( H_f^*(x) \leq H \) for all \( x \in X \), then
\[
\max \{ \dim_{\mathcal{H}}(B_f), \dim_{\mathcal{H}}(f(B_f)) \} \leq Q - \eta < Q.
\]
where \( \eta \) depends only on \( H \) and the data of \( X \) and \( Y \), and it can be explicitly calculated.
Applying the first part of Corollary 1.2 to the special case where $X$ and $Y$ are equiregular sub-Riemannian manifolds, it gives an affirmative answer to a recent conjecture [2, Remark 1.2].

To prove that the branch set of a quasiregular mapping is null with respect to the right Hausdorff measure, the earlier proofs are based on two important assumptions: the first fact is that the mapping in question is differentiable almost everywhere in a suitable sense and the differential is linear with respect to the group structure of the tangent space; the second fact is that the determinant of the mapping is positive almost everywhere; see [14] for Euclidean case, [11,5] for the generalized manifolds of type $A$ case, and [7] for the subRiemannian case for detailed information of this approach.

To obtain a dimensional estimate as in Corollary 1.2, the earlier proofs of Bonk–Heinonen [1] and Onninen–Rajala [17] go along the following lines: one first obtains a quantitative porosity estimate for the set of points with sufficiently large local index; then one proves other quantitative porosity results for the set of points with a precise local index bound. The quantitative porosity bound on the branch set then follows by combining the above two estimates. In both approaches, two important assumptions are necessary: the first assumption is that the domain has to be Euclidean, since the McAuley–Robinson theorem [16] is necessary and its proof relies crucially on the affine structure of Euclidean spaces; the second assumption is that certain abstract Poincaré inequalities in the sense of Heinonen–Koskela [9] are necessary, since we need to use the standard modulus (of a curve family) techniques.

To illustrate our idea for Theorem 1.1, we need the following terminology introduced in [6]. Let $f : X \to Y$ be a branched covering between two metric spaces. Fix $x_0 \in X$, $y_0 = f(x_0)$, $r > 0$. We say a map $g : B(y_0, r) \to X$ is a local left homotopy inverse for $f$ at $x_0$ if $g \circ f|_{U(x_0, r)}$ is homotopic to the identity on $U(x_0, r)$, via a homotopy $H_t$ for which $x_0 \notin H_t(\partial U(x_0, r))$ for all $t$. Similarly, $g$ is a local right homotopy inverse for $f$ if $f \circ g$ is homotopic to the identity on $B(y_0, r)$, via a homotopy $H_t$ with $y_0 \notin H_t(\partial B(y_0, r))$ for all $t$. If $g$ is a left and right local homotopy inverse, we simply call it a local homotopy inverse. We denote by $B_f^I$ the homotopy branch set of $f$, i.e., the set of points in $X$ for which $f$ has no (two-sided) local homotopy inverse. We also let $B_f^J$ denote the left homotopy branch set, i.e., the set of points in $X$ at which $f$ has no left homotopy inverse. It is clear that if $X$ and $Y$ are generalized $n$-manifolds, then $B_f = B_f^I$.

Our starting point is to construct local left homotopy inverses away from a porous set. In the first step, we further developed a quantitative ENR theory, inspired by Groce, Petersen and Wu [3,4,18], and Semmes [21], for LLC spaces. As an immediate consequence, we obtain a generalized McAuley–Robinson theorem (cf. Theorem 4.1), which provides a criterion of being non-branching. The second step is to control the distortion of annuli, quantitatively, at points of finite dilatation, away from a porous set. The moral here is that if either of the sets

$$S_{H,R} = \{x \in X : H_f(x,r) \leq H \text{ for all } r < R\}$$

or $f(S_{H,R})$ is “dense” at some point at a certain scale, then the annular distortion around that point will get controlled. Thus we may construct a local homotopy inverse around that point and use the generalized McAuley–Robinson theorem to conclude that the point is non-branching. This, together with a simple decomposition argument (cf. [6, Proof of Theorem 1.1]), will lead to Theorem 1.1.

Our standing assumptions for the underlying spaces $X$ and $Y$, except the local linear $n$-connectivity on $X$, are quite mild. On the other hand, the local linear $n$-connectivity is necessary for the validity of all the previous results, as [24, Theorem 1.2] indicates.

**Theorem 1.3.** For each $n \geq 3$, there exists an Ahlfors $n$-regular metric space $X$ that is homeomorphic to $\mathbb{R}^n$ and supports a $(1,1)$-Poincaré inequality, and a 1-quasiregular mapping $f : X \to \mathbb{R}^n$, such that $\min\{H^n(B_f), H^n(f(B_f))\} > 0$.

The example in Theorem 1.3 is indeed 1-BLD and it disproves the following well-known conjecture of Heinonen and Rickman [11, Remark 6.32 (b)]; if $X$ be a locally BLD $n$-Euclidean space that is locally bi-Lipschitz embeddable to some Euclidean space, then for all (Lipschitz) BLD-maps $f : X \to \mathbb{R}^n$, the branch set $B_f$ has zero Hausdorff $n$-measure. As a consequence, one cannot delete [13, Axiom II] from the a priori assumptions in [13, Theorem 2.1]; see [13, Section 5.1] and [24] for more detailed discussions.

2. **Väisälä’s inequality**

The Väisälä’s inequality was first proved by Väisälä [22] and it plays an important role in the theory of quasiregular mappings, in particular, many profound value-distribution type results; see [19].

**Definition 2.1 (Väisälä’s inequality).** We say that $f$ satisfies Väisälä’s inequality with constant $K_f$ if the following condition holds: suppose $m \in \mathbb{N}$, and $\Gamma$ and $\Gamma'$ are curve families in $X$ and $Y$ respectively, such that for each $\gamma' \in \Gamma'$, there are curves $\gamma_1, \ldots, \gamma_m \in \Gamma$ such that $f(\gamma_k)$ is a subcurve of $\gamma'$ for each $k$, and for each $t \in [0,1(\gamma')]$ and each $x \in X$, we have $|k : \gamma_k(t) = x| \leq i(x, f)$. Then

$$\text{Mod}_Q(\Gamma') \leq K_f \text{Mod}_Q(\Gamma')/m.$$  

As an application of Corollary 1.2 and [23, Theorem 1.1], we obtain the following very general Väisälä’s inequality in [6].
**Theorem 2.1** (Väisälä’s inequality). Let $X$ and $Y$ be Ahlfors $Q$-regular generalized $n$-manifolds, where $X$ is LLC$^n$ and $Y$ is linearly locally connected, and suppose $f : X \to Y$ is discrete and open, with $H_f(x) < \infty$ for all $x \in X$, and $H_f(x) \leq H$ for $H^Q$-almost every $x \in X$.

Then $f$ satisfies Väisälä’s inequality for some constant $K$, depending only on $H$ and the data of $X$ and $Y$.

In the special case where $Y$ is a generalized manifold with controlled geometry and topology and $X = \mathbb{R}^n$, Theorem 2.1 was first proved by Onninen and Rajala [17, Theorem 11.1].

3. **Loewner spaces**

There is a subtlety to the observation that Corollary 1.2 generalizes the Bonk–Heinonen theorem, which gave an index-free upper bound on $\dim H^Q B_f$. In general, the linear dilatation $H_f(x)$ of a quasiregular map in $\mathbb{R}^n$ does not need to be globally bounded – it is instead finite and essentially bounded, and at any point $x \in \mathbb{R}^n$, the dilatation depends quantitatively on not merely the essential supremum of $H_f$, but also on the index $i(x, f)$. That Corollary 1.2 is an actual generalization requires the fact that $H_f(x)$ is bounded everywhere by a constant $H^*$ independent of $i(x, f)$. This latter fact was proved in the Euclidean case in [15], using the $K_0$- and Väisälä’s inequalities, as well as the Loewner property of $\mathbb{R}^n$.

In the case where $X$ and $Y$ are Loewner, however, Väisälä’s inequality allows us to generalize the corresponding result of [15], giving an index free upper bound on $H_f$.

**Theorem 3.1.** Suppose (under the standing assumptions) that $X$ and $Y$ are locally Ahlfors $Q$-regular and $Q$-Loewner, $H_f(x) < \infty$ for all $x \in X$, and $H_f(x) \leq H$ for $H^Q$-almost every $x \in X$. Then $H_f(x) \leq H^*$ for every $x \in X$, where $H^*$ depends only on $H$ and the data of $X$ and $Y$, and the sets $B_f$ and $f(B_f)$ are $\delta$-porous, for some $\delta$ depending only on $H$ and the data.

Combining Theorem 3.1 with Corollary 1.2, we obtain the following result, the first half of which is a true generalization of the Bonk–Heinonen theorem.

**Corollary 3.2.** Under the assumptions of Theorem 3.1, we have

$$\max \{ \dim H^Q(B_f), \dim H^Q(f(B_f)) \} \leq Q - \eta < Q,$$

for some constant $\eta$ depending only on $H$ and the data of $X$ and $Y$.

Corollary 3.2 answers affirmatively the open problem of Heinonen and Rickman [11, Remark 6.7 (b)] in a stronger form, namely, we obtain dimensional estimates for the class of quasiregular mappings, which is strictly larger than the class of BLD mappings.

It was asked by Heinonen and Semmes [12, Question 27] that if for a given branched covering $f : S^n \to S^n$, $n \geq 3$, there is a metric $d$ on $S^n$ so that $(S^n, d)$ is an Ahlfors $n$-regular and locally linearly contractible metric space, and $f : (S^n, d) \to S^n$ is a BLD mapping. By Corollary 3.2, the existence of such a metric $d$ necessarily implies that $B_f$ must be null with respect to the $n$-dimensional Hausdorff measure $H^n$. On the other hand, there are plenty of branched coverings $f : S^n \to S^n$ such that $H^n(B_f) > 0$ and so we have the following negative answer to this question.

**Corollary 3.3.** Not every branched covering $f : S^n \to S^n$, $n \geq 3$, can be made BLD by changing the metric in the domain but keeping the space Ahlfors $n$-regular and linearly locally contractible.

4. **Generalization of the McAuley–Robinson theorem**

One of the crucial ingredient in the proof of Theorem 1.1 is the following generalization of the McAuley–Robinson theorem [16], which is of independent interest.

**Theorem 4.1.** Let $A \subset X$, where $X$ is a $\lambda$-LLC$^n$ generalized $n$-manifold and $\dim_{\text{top}}(A) \leq n$ and let $Y$ be another generalized $n$-manifold. Let $f : X \to Y$ be a proper branched covering such that for some $x_0 \in A \setminus \partial A$, $f^{-1}(f(x_0)) = x_0$ and

$$\sup_{x \in \partial A} \frac{\diam f^{-1}(f(x))}{d(x, x_0)} \leq \frac{1}{\lambda^{2n+1}}.$$

Then $x_0 \notin B_f$.

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