C. R. Acad. Sci. Paris, Ser. I 354 (2016) 219–223

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Calculus of variations

Asymptotic formulas for perturbations in Stokes flow due to the presence of small immiscible liquid particles

Formules de représentations asymptotiques pour des perturbations dans un écoulement de Stokes causées par la présence de particules liquides de faible volume

Mohamed Abdelwahed [a], Nejmeddine Chorfi [a], Maatoug Hassine [b]

[a] *Department of Mathematics, College of Sciences, King Saud University, Saudi Arabia*
[b] *Département de Mathématiques, Faculté des sciences de Monastir, Tunisia*

ARTICLE INFO

Article history:
Received 4 March 2015
Accepted 5 November 2015
Available online 19 January 2016

Presented by Haïm Brézis

ABSTRACT

The aim of this work is the design of an efficient method to obtain information about a finite number of small and well-separated liquid particles located in a viscous fluid from boundary measurements. The viscosity and density of particles are different from those of a background fluid governed by a Stokes flow. An asymptotic formula for the velocity perturbation is derived based on the concept of Viscous Moment Tensor analysis. This formula will be the principle for efficient computational algorithms of identification.

RÉSUMÉ

On considère l'écoulement d'un fluide visqueux incompressible décrit par les équations de Stokes. On démontre dans cette note des formules de représentations asymptotiques pour les perturbations de vitesse causées par la présence d'un nombre fini de particules liquides de faible volume. On suppose que les particules (la phase dispersée) sont distinctes et que leurs caractéristiques physiques (la viscosité et la densité) sont différentes de celle du fluide de référence (la phase continue). Nous avons introduit la notion des tenseurs visqueux. Ces formules de représentations peuvent être utilisées pour construire des algorithmes numériques très efficaces d'identification (ou d'optimisation d'emplacements) des particules à partir des données au bord sur-déterminées.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ a smooth and bounded domain occupied by a viscous incompressible fluid $\mathcal{F}$ and $\Gamma = \partial\Omega$ its boundary, considered smooth and connected (Γ is taken $\mathcal{C}^\infty$ for simplicity) (see Fig. 1). We suppose that the fluid $\mathcal{F}$

E-mail addresses: mabdelwahed@ksu.edu.sa (M. Abdelwahed), nchorfi@ksu.edu.sa (N. Chorfi), maatoug.hassine@enit.rnu.tn (M. Hassine).

http://dx.doi.org/10.1016/j.crma.2015.10.023
1631-073X/

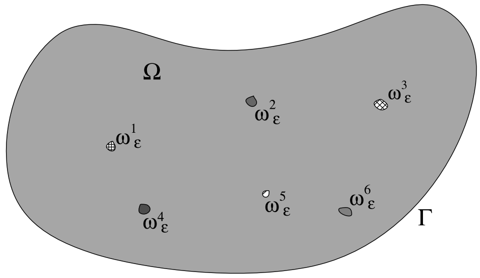

Fig. 1. Small liquid particles $\{\omega_\varepsilon^i,\ 1 \le i \le m\}$ suspended in the fluid $\mathcal{F}$ (multiphase flow).

contains a finite number of immiscible liquid particles (considered for simplicity well-separated) $\mathcal{P}^i$, $i = 1, \ldots, m$ of volume $\omega_\varepsilon^i \subset \mathbb{R}^d$, density ρ_i, kinematic viscosity ν_i, relative size $\mathcal{R}_i$ $(0 < r_1 \le \mathcal{R}_i \le r_2)$, centre of mass $z_i \in \Omega$ that satisfy

$$\left|z_i - z_j\right| \ge d_0 > 0,\ \forall j \ne i \text{ and } dist(z_i, \partial\Omega) \ge d_0 > 0\ \ i = 1, \ldots, m \tag{1}$$

and geometry form $\omega_\varepsilon^i = z_i + \varepsilon\mathcal{R}_i\omega$, where $\omega \subset \mathbb{R}^d$ is a bounded, smooth domain containing the origin and ε (the shared diameter magnitude of the particles) is chosen small enough to have disjoint particles with distance to $\mathbb{R}^d \backslash \overline{\Omega}$ greater than $d_0/2$.

The density ρ and the kinematic viscosity ν of the fluid $\mathcal{F}$ satisfy:

$$0 < c_0 \le \rho(x) \le C_0 < \infty, \quad 0 < c_1 \le \nu(x) \le C_1 < \infty \quad x \in \Omega. \tag{2}$$

In the presence of the particles, the velocity and the pressure $(u_\varepsilon, p_\varepsilon)$ are solutions to

$$\begin{aligned} -\nabla.[2\nu_\varepsilon(x)\mathcal{D}(u_\varepsilon)] + \nabla p_\varepsilon &= \rho_\varepsilon \mathcal{G} && \text{in } \Omega, \\ \nabla.u_\varepsilon &= 0 && \text{in } \Omega, \\ \sigma(u_\varepsilon, p_\varepsilon)\mathbf{n} &= g && \text{on } \Gamma, \end{aligned} \tag{3}$$

with

$$\nu_\varepsilon(x) = \begin{cases} \nu(x) & \text{if } x \in \Omega_\varepsilon = \Omega\backslash\overline{\omega_\varepsilon} \\ \nu_i & \text{if } x \in \omega_\varepsilon^i = z_i + \varepsilon\mathcal{R}_i\omega, \end{cases} \quad \rho_\varepsilon(x) = \begin{cases} \rho(x) & \text{if } x \in \Omega_\varepsilon = \Omega\backslash\overline{\omega_\varepsilon} \\ \rho_i & \text{if } x \in \omega_\varepsilon^i = z_i + \varepsilon\mathcal{R}_i\omega, \end{cases}$$

$\mathcal{D}(u_\varepsilon) = \frac{1}{2}\left(\nabla u_\varepsilon + \nabla u_\varepsilon^{\mathrm{T}}\right)$ is the deformation tensor rate, $\mathcal{G}$ is the gravitational acceleration, $\sigma(u_\varepsilon, p_\varepsilon) = 2\nu_\varepsilon\mathcal{D}(u_\varepsilon) - p_\varepsilon I$ is the stress tensor, I is the $d \times d$ unit matrix, g is a given force exerted on Γ, $\mathbf{n}$ is the unit outer normal vector.

Stress and kinematic boundary conditions are given at the surface of the particle $\partial\omega_\varepsilon^i$,

$$\left\{2\nu(x)\mathcal{D}(u_\varepsilon) - p_\varepsilon I\right\}^+ \mathbf{n} = \left\{2\nu_i\mathcal{D}(u_\varepsilon) - p_\varepsilon I\right\}^- \mathbf{n} \text{ and } u^+_{\varepsilon\ |\partial\omega_\varepsilon^i} = u^-_{\varepsilon\ |\partial\omega_\varepsilon^i},\ i = 1, \ldots, m.$$

The aim of this paper is the derivation of the asymptotic formula for the velocity perturbations caused by the presence of small and well-separated liquid particles. Our proposed method is similar to the ideas used by Cedio-Fengya, Moskow, and Vogelius in their work [4] on the identification of conductivity imperfections of small diameter by boundary measurement and later by Alves and Ammari in their asymptotic formula for the reconstruction of imperfections of small diameter in an elastic medium [1]. Asymptotic formula for perturbations due to the presence of small and well-separated inhomogeneities are derived in the case of the conduction equation [8,4], the elasticity equation [1,2], the Helmholtz equation [3], and the Maxwell equations [11]. To the best of our knowledge, the present work is the first attempt to derive an asymptotic formula in the case of the Stokes system. During our analysis we have derived an asymptotic expression for the velocity and the deformation tensor inside the particles based on the notion of viscous moment tensor. This idea is used in elasticity for the tensors of elastic moment and in electromagnetics for the tensors of polarization.

The presented results will find important applications for developing highly efficient algorithms for small particles reconstructing from boundary informations. Such algorithms can be used in different applications as in [5,6,10] for molding or for the optimization of the liquid metallic mixing design and in [9,12] for colloid. It is expected also that our results will serve in the determination of rigorously very accurate effective viscosity of a suspension of general shaped particles suspended in a reference fluid.

2. Asymptotic formula

We derive in this section an asymptotic formula for $u_\varepsilon(z)$. We present first the case where Ω contains a single particle $\omega_\varepsilon = \varepsilon\omega$ centred at the origin. Then, the case of multiple particles is studied. We start with the introduction of the notion of Viscous Moment Tensor.

Definition 2.1. The Viscous Moment Tensor $\mathcal{M} \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ for the viscosity ratio $\nu(0)/\nu_1$ and domain ω is defined for all $1 \le k, l, p, q \le d$ by,

$$\mathcal{M}^{kl}_{pq} = (\frac{\nu(0)}{\nu_1} - 1)\Big\{\frac{1}{2}|\omega|(\delta_{pk}\delta_{ql} + \delta_{pl}\delta_{qk}) + (1 - \frac{\nu_1}{\nu(0)})\int_{\partial\omega} y_p\, e_q\, [\frac{\nu(0)}{\nu_1}\mathcal{D}_y(v^{k,l}) - q^{k,l}\, I]^+\, \mathbf{n}\, \mathrm{d}s(y)\Big\},$$

where y_p and $(e_q)_{q=1}^d$ denote respectively the pth component of y and the canonical basis of $\mathbb{R}^d$, $(v^{k,l}, q^{k,l})$ is solution to

$$\begin{aligned}
&-\nabla_y.[\frac{\nu(0)}{\nu_1}\mathcal{D}_y(v^{k,l})] + \nabla_y q^{k,l} = 0, \quad \nabla_y.v^{k,l} = 0, \quad \text{in } \mathbb{R}^d \backslash \overline{\omega}\\
&-\nabla_y.[\mathcal{D}_y(v^{k,l})] + \nabla_y q^{k,l} = 0, \quad \nabla_y.v^{k,l} = 0, \quad \text{in } \omega\\
&\Big(\frac{\nu(0)}{\nu_1}\mathcal{D}_y(v^{k,l}) - q^{k,l} I\Big)^+ \mathbf{n} - \Big(\mathcal{D}_y(v^{k,l}) - q^{k,l} I\Big)^- \mathbf{n} = -E^{k,l}\mathbf{n} \text{ on } \partial\omega,\\
&\lim_{|y|\to+\infty} v^{k,l}(y) = 0,
\end{aligned} \tag{4}$$

with $v^{k,l}$ is continuous across $\partial\omega$ and $\forall\, 1 \le k, l \le d$, $E^{kl} \in \mathbb{R}^d \times \mathbb{R}^d$ is a symmetric matrix given by $E^{kl}_{pq} = \frac{1}{2}(\delta_{pk}\delta_{ql} + \delta_{pl}\delta_{qk})$, $1 \le p, q \le d$, where δ_{pq} is the symbol of Kronecker.

2.1. *Single particle*

Let $(U(., z),\, P(., z)) \in (\mathbb{R}^d \times \mathbb{R}^d) \times \mathbb{R}^d$ denotes the fundamental solution to the Stokes equations corresponding to a Dirac mass at the point z and to coefficient ν. That is, for all $1 \le j \le d$, $(U^j(., z),\, P^j(., z))$ is a solution to

$$-\nabla_x.[2\nu(x)\mathcal{D}_x(U^j)(x, z)] + \nabla_x P^j(x, z) = \delta_z e_j \;\text{ in } \Omega, \;\; \nabla_x.U^j(x, z) = 0 \quad \text{in } \Omega, \tag{5}$$

where U^j denotes the jth column of U.

We remark that if ν is constant, (U, P) is given by (see [7])

$$\begin{aligned}
&U(x, z) = \frac{1}{4\pi\nu}\Big(-(\log(r)I + e_r e_r^{\mathrm{T}}\Big), \quad P(x, z) = \frac{x}{2\pi r^2} \quad \text{if } d = 2,\\
&U(x, z) = \frac{1}{8\pi\nu r}\Big(I + e_r e_r^{\mathrm{T}}\Big), \quad P(x, z) = \frac{x}{4\pi r^3}, \quad \text{if } d = 3,
\end{aligned}$$

where $r = ||x - z||$, $e_r = x/r$ and e_r^{T} is the transposed vector of e_r.

In the following theorem we present the asymptotic formula describing the perturbation $u_\varepsilon(z) - u_0(z)$ caused by the presence of the particle ω_ε. The main idea of the proof is given in section 3.

Theorem 2.2. *For $z \in \overline{\Omega}\backslash\{z_1\}$, we have the following asymptotic representation formula:*

$$u^j_\varepsilon(z) - u^j_0(z) + \int_{\partial\Omega} [2\nu\mathcal{D}_x(U^j) - P^j I]\mathbf{n}(u_\varepsilon - u_0)\, \mathrm{d}s = \varepsilon^d \Big\{2\nu(0)\, \mathcal{D}_x(U^j)(0, z) : \mathcal{M}\mathcal{D}_x(u_0)(0) - (\rho(0) - \rho_1)\, |\omega|\, \mathcal{G}\, U^j(0, z)\Big\} + O(\varepsilon^{d+1/2}), \quad j = 1, .., d.$$

Note that we have used the standard notation of the matrix inner product; for matrices (X_{ij}) and (Y_{ij}), $X : Y = \sum_{i,j} X_{ij} Y_{ij}$.

2.2. *Multiple particles*

If $\omega_\varepsilon = \cup_{i=1}^m \{z_i + \varepsilon\mathcal{R}_i\omega\}$ (more than one particle), we proceed by induction using Theorem 2.2. We present, in the following theorem, the asymptotic formula of perturbations caused by the presence of a finite number of well-separated small particles.

Theorem 2.3. *Suppose that the points $z_i \in \Omega$, $1 \le i \le m$ are mutually distinct and satisfy (1). Then for any $z \in \overline{\Omega}\backslash\{z_i\}_{i=1}^m$, we have*

$$u^j_\varepsilon(z) - u^j_0(z) + \int_{\partial\Omega} [2\nu\mathcal{D}_x(U^j) - P^j I]\mathbf{n}(u_\varepsilon - u_0)\, \mathrm{d}s = \varepsilon^d \sum_{i=1}^m \Big\{2\nu(z_i)\, \mathcal{D}_x(U^j)(z_i, z) : \mathcal{M}_i\mathcal{D}_x(u_0)(z_i) - (\rho(z_i) - \rho_i)\, |\omega|\, \mathcal{G}\, U^j(z_i, z)\Big\} + O(\varepsilon^{d+1/2}), \quad j = 1, .., d,$$

where $\mathcal{M}_i$ is the viscous moment tensor for the viscosity ratio $\nu(z_i)/\nu_i$ and the domain ω_i.

3. Proof of Theorem 2.2

• *First stage*: We establish an energy estimate. Due to Green's formula and Korn's inequality we obtain the following energy estimate.

Lemma 3.1. *There exists a constant $C > 0$, independent of ε, such that*

$$\int_{\Omega} |u_\varepsilon - u_0|^2 \, \mathrm{d}x + \int_{\Omega} |\mathcal{D}(u_\varepsilon - u_0)|^2 \, \mathrm{d}x \le C\,\varepsilon^d.$$

• *Second stage*: We derive an asymptotic behaviour of u_ε and $\mathcal{D}(u_\varepsilon)$ inside the particles. Let (v, q) denotes the solution to the following auxiliary problem

$$\begin{aligned}
&-\nabla_y.[2\nu(0)\mathcal{D}_y(v)] + \nabla_y q = 0, \quad \nabla_y.v = 0, \quad \text{in } \mathbb{R}^d\backslash\overline{\omega}\\
&-\nabla_y.[2\nu_1\mathcal{D}_y(v)] + \nabla_y q = 0, \quad \nabla_y.v = 0, \quad \text{in } \omega\\
&\Big(2\nu(0)\mathcal{D}_y(v) - qI\Big)^+\mathbf{n} - \Big(2\nu_1\mathcal{D}_y(v) - qI\Big)^-\mathbf{n} = -2[\nu(0) - \nu_1]\mathcal{D}_x(u_0)(0)\mathbf{n} \text{ on } \partial\omega,\\
&v \text{ is continuous across } \partial\omega, \quad \lim_{|y|\to+\infty} v(y) = 0.
\end{aligned} \tag{6}$$

Theorem 3.2. *The velocity and the deformation tensor have the following asymptotic behaviour*

$$u_\varepsilon(\varepsilon y) = u_0(\varepsilon y) + \varepsilon v(y) + O(\varepsilon^{1/2}), \ \mathcal{D}_x(u_\varepsilon)(\varepsilon y) = \mathcal{D}_x(u_0)(\varepsilon y) + \mathcal{D}_y(v)(y) + O(\varepsilon^{1/2}), \ \forall y \in \omega.$$

The proof of the Theorem 3.2 is based essentially on the following estimate.

Lemma 3.3. *There exists a constant $C > 0$, independent of ε, such that*

$$\big\|\mathcal{D}_y[u_\varepsilon(\varepsilon y) - u_0(\varepsilon y) - \varepsilon v(y)]\big\|_{0,\widehat{\Omega}} \le C\,\varepsilon^{3/2}.$$

To prove this lemma, we derive the system satisfied by $u_\varepsilon(x) - u_0(x) - \varepsilon v(x/\varepsilon)$ and we use Green's formula, change of variable $x = \varepsilon y$, Korn's inequality, Lemma 3.1 and the behaviour of v.

• *Third stage:* We derive some preliminary results and we deduce the desired asymptotic formula. The first result is described by Lemma 3.4. It is obtained with the help of Green's formula and equations (3) and (5).

Lemma 3.4. *For $z \in \overline{\Omega}\backslash\{z_1\}$ the velocity field admits the following expansion $\forall\, j = 1, \dots, d$*

$$u_\varepsilon^j(z) - u_0^j(z) + \int_{\Gamma} [2\nu\mathcal{D}(U^j) - P^j I]\mathbf{n}\,(u_\varepsilon - u_0)\,\mathrm{d}s = \int_{\omega_\varepsilon} 2[\nu - \nu_1]\mathcal{D}(U^j) : \mathcal{D}(u_\varepsilon)\,\mathrm{d}x - \int_{\omega_\varepsilon} (\rho - \rho_1)\,\mathcal{G}\,U^j\,\mathrm{d}x.$$

Expanding $\nu(x) = \nu(0) + x.\nabla_x\nu(\eta_x)$, $\eta_x \in \omega_\varepsilon$ and $\rho(x) = \rho(0) + x.\nabla_x\rho(\xi_x)$, $\xi_x \in \omega_\varepsilon$. Using Lemma 3.1 and the fact that z is bounded away from ω_ε, in the sens that $dist(z, \omega_\varepsilon) \ge d_0 > 0$, we derive the following lemma.

Lemma 3.5. *There exists a positive constant C, independent of ε, such that for all $j = 1, .., d$*

$$\int_{\omega_\varepsilon} [\nu - \nu(0)]\mathcal{D}_x(U^j) : \mathcal{D}_x(u_\varepsilon)\,\mathrm{d}x = O(\varepsilon^{d+1}), \quad \int_{\omega_\varepsilon} [\rho - \rho(0)]\,\mathcal{G}\,U^j\,\mathrm{d}x = O(\varepsilon^{d+1}).$$

Using Green's formula, change of variable $x = \varepsilon y$, Taylor's theorem, Theorem 3.2 and system (6) we obtain the following asymptotic expansion.

Lemma 3.6. *For all $z \in \overline{\Omega} \backslash \{z_1\}$ we have*

$$\int_{\omega_\varepsilon} 2\nu_1 \mathcal{D}_x(U^j)(x,z) : \mathcal{D}_x(u_\varepsilon)(x)\,\mathrm{d}x = \varepsilon^d \Big\{ 2\nu(0)|\omega| \mathcal{D}_x(u_0)(0) : \mathcal{D}_x(U^j)(0,z)$$
$$+ \int_{\partial\omega} [2\nu(0)\mathcal{D}_y(v) - q\,I]^+(y)\mathbf{n}\mathcal{D}_x(U^j)(0,z) y\,\mathrm{d}s(y) \Big\}$$
$$+ O\left(\varepsilon^{d+1/2}\right), \quad j = 1, .., d.$$

Finally, combining Lemmas 3.4, 3.5 and 3.6 and using the fact that

$$\mathcal{D}_x(u_0)(0) = \sum_{1\le k,l \le d} [\mathcal{D}_x(u_0)(0)]_{kl}\, E^{k,l}, \text{ with } [\mathcal{D}_x(u_0)(0)]_{kl} = \frac{1}{2}\left(\frac{u_0^k}{\partial x_l}(0) + \frac{u_0^l}{\partial x_k}(0) \right)$$
$$v(y) = \frac{\nu(0) - \nu_1}{\nu_1} \sum_{1\le k,l\le d} [\mathcal{D}_x(u_0)(0)]_{kl}\, v^{k,l}, \quad q(y) = 2(\nu(0) - \nu_1) \sum_{1\le k,l\le d} [\mathcal{D}_x(u_0)(0)]_{kl}\, q^{k,l},$$

we deduce the desired asymptotic formula.

Acknowledgements

The authors would like to extend their sincere appreciation to the deanship of Scientific Research at King Saud University for funding this research group No (RG – 1435-026).

References

[1] C. Alves, H. Ammari, Boundary integral formulae for the reconstruction of imperfections of small diameter in an elastic medium, SIAM J. Appl. Math. 62 (2001) 94–106.
[2] H. Ammari, H. Kang, G. Nakamura, K. Tanuma, Complete asymptotic expansions of solutions of the system of elastostatics in the presence of an inclusion of small diameter and detection of an inclusion, J. Elast. 67 (2002) 97–129.
[3] H. Ammari, H. Kang, Boundary layer techniques for solving the Helmhholtz equation in the presence of small inhomogeneities, J. Math. Anal. Appl. 296 (2004) 190–208.
[4] D.J. Cedio-Fengya, S. Moskow, M. Vogelius, Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction, Inverse Probl. 14 (1998) 553–595.
[5] S.T. Chung, T.H. Kwon, Numerical simulation of fiber orientation in injection moulding of short-fiber-reinforced thermoplastics, Polym. Eng. Sci. 7 (35) (1995) 604–618.
[6] R. Codina, U. Schaffer, E. Oñate, Mould filling simulation using finite elements, Int. J. Numer. Methods Fluid Flow 4 (1994) 291–310.
[7] R. Dautray, J.-L. Lions, Analyse mathémathique et calcul numérique pour les sciences et les techniques, collection CEA, Masson, Paris, 1987.
[8] A. Friedman, M. Vogelius, Identification of small inhomogeneties of extreme conductivity by boundary measurements: a theorem on continuous dependence, Arch. Ration. Mech. Anal. 105 (1989) 299–326.
[9] T. Gotz, Simulating particles in Stokes flow, J. Comput. Appl. Math. 175 (2005) 415–427.
[10] K.M. Shyue, A fluid-mixture type algorithm for compressible multicomponent flow with van der Waals equation of state, J. Comput. Phys. 156 (1999) 43–88.
[11] M. Vogelius, D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities, Math. Model. Numer. Anal. 34 (2000) 723–748.
[12] H. Zhou, C. Pozrikidis, Adaptive singularity method for Stokes flow past particles, J. Comput. Phys. 117 (1995) 79–89.