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The asymptotics of the holomorphic torsion forms

L'asymptotique des formes de torsion holomorphe

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A B S T R A C T

In this note, we use the theory of Toeplitz operators to give an asymptotic formula for the holomorphic analytic torsion forms associated with a family of holomorphic vector bundles that are the direct image of $L^{\otimes p}$, where L is a line bundle. To obtain the asymptotics, we make a positivity assumption along the fibers on L. This result is the family version of the results of Bismut and Vasserot on the holomorphic torsion.

R É S U M É

Dans cette note, nous utilisons la théorie des opérateurs de Toeplitz pour donner une formule asymptotique des formes de torsion analytique holomorphe associées à une famille de fibrés vectoriels holomorphes donnés par l'image directe de $L^{\otimes p}$, où L est un fibré en droites. Pour obtenir cette asymptotique, nous faisons une hypothèse de positivité le long des fibres sur L. Ce résultat est la version en famille des résultats de Bismut et Vasserot sur la torsion holomorphe.

Version française abrégée

La torsion analytique holomorphe a été définie dans [17] par Ray et Singer comme l'analogue complexe de la torsion réelle pour les fibrés vectoriels plats. Elle apparaît notamment dans l'étude du déterminant de l'image directe d'un fibré vectoriel holomorphe par une fibration holomorphe propre menée par Bismut, Gillet et Soulé dans [4].

La torsion analytique a une extension dans le cas des familles : les formes de torsion analytique, définies d'abord par Bismut, Gillet et Soulé [3], puis par Bismut et Köhler [5] et Bismut [2], à divers degrés de généralité. La partie de degré zéro de ces formes n'est autre que la torsion de Ray–Singer le long de la fibre. Les formes de torsion analytique ont eu beaucoup d'applications, comme par exemple la contribution analytique au formalisme de l'image directe pour les fibrations Kähler en géométrie d'Arakelov.

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Dans [7], Bismut et Vasserot ont calculé le comportement asymptotique de la torsion analytique associée à des puissances tensorielles croissantes d'un fibré en droites positif et, dans [8], ils ont étendu leur résultat au cas où les puissances du fibré en droites sont remplacées par les puissances symétriques d'un fibré positif (de rang quelconque). Ces résultats ont joué un rôle important dans un résultat d'amplitude arithmétique de Gillet et Soulé [12].

Dans cette note, nous donnons une version en famille, et au niveau des formes, des résultats de Bismut et Vasserot pour les formes de torsion analytique. Soit $\pi_1 : N \to M$ et $\pi_2 : M \to B$ deux fibrations holomorphes de fibres compactes et soit ω^M une forme de type $(1,1)$ sur M induisant une métrique sur les fibres de π_2. Soit (L, h^L) un fibré en droites holomorphe hermitien sur N, positif le long des fibres de $\pi_3 := \pi_2 \circ \pi_1$. On pose $F_p = R^\bullet \pi_{1*} L^{\otimes p}$, que l'on suppose localement libre pour p grand. Nous calculons alors les deux termes dominants dans l'asymptotique des formes de torsion $\mathcal{T}(\omega^M, h^{F_p})$ quand $p \to +\infty$, où h^{F_p} est la métrique L^2 naturelle sur F_p (correctement normalisée). Ces termes sont donnés par l'intégrale le long des fibres de π_3 de termes locaux définis à partir de la courbure de Chern de (L, h^L). Pour obtenir ce résultat, nous utilisons la théorie de opérateurs de Toeplitz.

Si B est un point, on retrouve [7] dans le cas où $\pi_1 = \mathrm{Id}$ et [8] dans le cas où $N = \mathbb{P}(E^*)$ et $L = \mathcal{O}(1)$, avec E un fibré positif sur M.

Les résultats annoncés dans cette note sont démontrés dans [16].

1. Introduction

The holomorphic analytic torsion was defined in [17] by Ray and Singer as the complex analogue of its real version for flat vector bundles. It appears in the study by Bismut, Gillet, and Soulé of the determinant of the direct image of a holomorphic vector bundle by a proper holomorphic fibration in [4].

Holomorphic torsion has an extension in the families setting: the holomorphic analytic torsion forms, defined in various degrees of generality by Bismut, Gillet, and Soulé [3], Bismut and Köhler [5] and Bismut [2]. The 0-degree component of these forms is the analytic torsion of Ray–Singer along the fiber. The analytic torsion forms have found many applications, as for instance the analytic counterpart of the direct image for Kähler fibrations in Arakelov geometry. Indeed, the torsion appears in the arithmetic Riemann–Roch theorem [12], and the torsion forms in the arithmetic Riemann–Roch–Grothendieck theorem in higher degrees [11].

In [7], Bismut and Vasserot computed the asymptotics of the analytic torsion associated with increasing tensor powers of a positive line bundle. They also extended their result, in [8], to the case where the powers of the line bundle are replaced by the symmetric powers of a positive bundle. These asymptotics have played an important role in a result of arithmetic ampleness by Gillet and Soulé [12] (see also [18, Chp. VIII]).

In this paper, we give the family versions at the level of the forms of the results obtained by Bismut and Vasserot for the analytic torsion forms, using the theory of Toeplitz operators (see [10], [9], and [14]).

2. Definition of the torsion forms

Let $\pi : M \to B$ be a holomorphic fibration with compact fiber X. Let (E, h^E) be a holomorphic Hermitian bundle on M. Let ω^M be a $(1,1)$-form on M such that $\omega^M(J^{T_{\mathbb{R}}X}\cdot, \cdot)|_{T_{\mathbb{R}}X \times T_{\mathbb{R}}X}$ (with $J^{T_{\mathbb{R}}X}$ the complex structure of X) defines a metric on $T_{\mathbb{R}}X$. Let h^{TX} be the induced Hermitian metric on the holomorphic tangent bundle of the fibers of π. Let $T^H_B M := TX^{\perp, \omega^M}$ be the orthogonal bundle to TX with respect to ω^M, which we will take as horizontal vector subbundle for the fibration $M \to B$.

Let B_u be the Bismut superconnection associated with ω^M and (E, h^E), defined in [2], Definitions 3.3.1, 3.6.1 and 4.2.1. When ω^M is closed, i.e., when π and $T^H_B M$ define a Kähler fibration in the sense of [3], then this superconnection has been defined in [3,5]. By [2, Thm. 3.9.3], B_u^2 is a fiberwise elliptic second-order differential operator on $\mathcal{C}^\infty\big(M, \pi^*\Lambda^\bullet(T^*_{\mathbb{R}}B) \otimes \Lambda^{0,\bullet}(T^*X) \otimes E\big)$ and its heat kernel $\exp(-B_u^2) \in \Omega^\bullet\big(B, \mathrm{End}(\mathcal{C}^\infty(X, \Lambda^{0,\bullet}(T^*X) \otimes E|_X))\big)$ is well defined.

Let τ be the involution defining the $\mathbb{Z}_2$-grading on $\mathcal{C}^\infty(X, \Lambda^{0,\bullet}(T^*X) \otimes E|_X)$. We define the supertrace Tr_s on $\mathrm{End}(\mathcal{C}^\infty(X, \Lambda^{0,\bullet}(T^*X) \otimes E|_X))$ by $\mathrm{Tr}_s[\cdot] = \mathrm{Tr}[\tau \cdot]$ and we extend it naturally to get a map $\mathrm{Tr}_s : \Omega^\bullet(B, \mathcal{C}^\infty(X, \Lambda^{0,\bullet}(T^*X) \otimes E|_X)) \to \Omega^\bullet(B)$.

Let $\omega^H = \omega^M|_{T^H_B M \times T^H_B M}$ and let N_V be the number operator defining the $\mathbb{Z}$-grading on $\Lambda^{0,\bullet}(T^*X)$. Set $N_u = N_V + i\frac{\omega^M}{u}$.

Assume that $H^j(X, E|_X) = 0$ for $j > 0$. Then one can prove (see [2] Proposition 4.6.1 and Theorem 4.10.4) that the zeta function

$$\zeta(s) = -\frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \Phi \, \mathrm{Tr}_s\Big[N_u \exp(-B_u^2)\Big] \, du \tag{2.1}$$

has a holomorphic extension on $\{|\mathrm{Re}(s)| \le 1/2\}$. Here Γ is the Euler gamma function and $\Phi : \alpha \in \Lambda^{2k}(T^*B) \mapsto (2i\pi)^{-k}\alpha$. Following Bismut [2, (4.11.3)], we define the holomorphic analytic torsion forms by

$$\mathcal{T}(\omega^M, h^E) := \zeta'(0). \tag{2.2}$$

Let $h^{H^\bullet(X,E|_X)}$ be the L^2 metric on $H^\bullet(X, E|_X)$ defined from the volume form $\mathrm{d}v_X$ associated with h^{TX} by

$$\langle s, s'\rangle = \frac{1}{(2\pi)^{\dim X}} \int_{X_b} \langle s(x), s'(x)\rangle_{h^E}(x)\, \mathrm{d}v_X(x). \tag{2.3}$$

If ω^M is closed, then we have the transgression formula proved in [5, Thm. 3.9]:

$$\frac{\bar\partial\partial}{2i\pi}\mathcal{T}(\omega^M, h^E) = \mathrm{ch}\left(H^\bullet(X, E|_X), h^{H^\bullet(X,E|_X)}\right) - \int_X \mathrm{Td}(TX, h^{TX})\mathrm{ch}(E, h^E). \tag{2.4}$$

In the general case, this equality is replaced by [2, Thm. 4.11.2]: in the right-hand side, the term $\int_X \mathrm{Td}(TX, h^{TX})\mathrm{ch}(E, h^E)$ is replaced by the integral of a non-explicit local form.

From [2, Prop. 4.6.1], we know that there are $c_j \in \Omega^\bullet(B)$ such that for $u \in]0, 1]$ and $k \in \mathbb{N}$,

$$\left| \Phi\,\mathrm{Tr}_s\left[N_u \exp(-B_u^2)\right] - \sum_{j=-m}^{k} c_j u^j \right| \leq C_k u^{k+1}. \tag{2.5}$$

Then the torsion is given by the formula

$$\begin{aligned} \mathcal{T}(\omega^M, h^E) = &-\int_0^1 \left\{ \Phi\,\mathrm{Tr}_s\left[N_u \exp(-B_u^2)\right] - \sum_{j=-m}^{0} c_j u^j \right\} \frac{\mathrm{d}u}{u} \\ &- \int_1^{+\infty} \Phi\left\{ \mathrm{Tr}_s\left[N_u \exp(-B_u^2)\right]\right\} \frac{\mathrm{d}u}{u} + \sum_{j=-m}^{-1} \frac{c_j}{j} + c_0\Gamma'(1). \end{aligned} \tag{2.6}$$

3. Statement of the result

Let $\pi_1 : N \to M$ and $\pi_2 : M \to B$ be holomorphic fibrations with compact fibers Y and X respectively. Then $\pi_3 := \pi_2 \circ \pi_1 : N \to B$ is a holomorphic fibration, whose compact fiber is denoted by Z. This is summarized in the following diagram:

$$\begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & N \\ & & \downarrow \pi_1 & & \downarrow \pi_1 \quad \searrow \pi_3 \\ & & X & \longrightarrow & M \xrightarrow{\pi_2} B \end{array}$$

Let ω^M and h^{TX} be defined as in Section 2 for the fibration π_2. We denote by $T_B^H M := TX^{\perp,\omega^M}$ the associated horizontal subbundle.

Let (L, h^L) be a holomorphic Hermitian line bundle on N, with Chern curvature R^L.

Assumption 3.1. *The* $(1, 1)$*-form* $\sqrt{-1}R^L$ *is positive along the fibers of* π_3*, that is for any* $0 \neq U \in TZ$*, we have*

$$R^L(U, \overline{U}) > 0. \tag{3.1}$$

In particular, $(\pi_1, -\sqrt{-1}R^L)$ and $(\pi_3, -\sqrt{-1}R^L)$ are Kähler fibrations and we denote by $g^{T_{\mathbb{R}}Z}$ and $g^{T_{\mathbb{R}}Y}$ the corresponding metrics on $T_{\mathbb{R}}Z$, and $T_{\mathbb{R}}Y$.

We denote by L^p the pth tensor power of L. Assume temporarily that B is compact, then Assumption 3.1 and Kodaira vanishing theorem imply the existence of p_0 such that for $p \geq p_0$, the direct images $R^\bullet\pi_{1*}(L^p)$, $R^\bullet\pi_{2*}(R^\bullet\pi_{1*}(L^p))$ and $R^\bullet\pi_{3*}(L^p)$ are locally free and they vanish in positive degree. Thus for $p \geq p_0$,

$$F_p := H^0\big(Y, L^p|_Y\big) \tag{3.2}$$

is a holomorphic vector bundle on M. Let $\mathrm{d}v_Y$ be the volume form on Y associated with $g^{T_{\mathbb{R}}Y}$, we endowed F_p with the L^2 metric h^{F_p} induced by h^L and $\mathrm{d}v_Y$ as in (2.3). Finally, an easy spectral sequence argument shows that $R^i\pi_{2*}(F_p) \simeq R^i\pi_{3*}(L^p)$, so that

$$H^\bullet\big(X, F_p|_X\big) = H^0\big(X, F_p|_X\big) \simeq H^0\big(Z, L^p|_Z\big). \tag{3.3}$$

We now turn back to the case of an arbitrary B. Then we assume that a number p_0 as above can be chosen uniformly on the compact subsets of B. In the sequel, we take $p \geq p_0$.

From these data, we can define, as above, the holomorphic analytic torsion form $\mathcal{T}(\omega^M, h^{F_p})$ associated with ω^M and (F_p, h^{F_p}).

Remark 3.2. In our situation, one can always choose a closed ω^M. Indeed, let $\lambda_p = \det F_p$ be equipped with the Quillen metric associated with a fibrewise Kähler metric h^{TY} along the fibers Y and with the metric h^L on L. By the curvature theorem for Quillen metrics [4], we get $c_1(\lambda_p, \|\cdot\|_p) = \left[\int_Y \mathrm{Td}(TY, g^{TY})\, \mathrm{e}^{pc_1(L,h^L)}\right]^{(2)}$, where $\alpha^{(k)}$ denotes the component of degree k of α. Thus, as $p \to +\infty$, we have

$$c_1(\lambda_p, \|\cdot\|_p) = p^{\dim Y+1} \int_Y \frac{c_1(L,h^L)^{\dim Y+1}}{(\dim Y+1)!} + o(p^{\dim Y+1}), \tag{3.4}$$

the 2-form of the right-hand side being positive along the fibers X for p large enough because $c_1(L, h^L)$ is positive along the fibers Z.

In the sequel, we choose a non-necessarily closed form ω^M.

Let $T^H_X Z$ be the orthogonal complement with respect to g^{TZ} of TY in TZ. Then $T^H_X Z \simeq \pi_1^* TX$. Let $\dot{R}^{X,L} \in \pi_1^*\mathrm{End}(TX)$ be the Hermitian matrix such that for any $U, V \in TX$, then

$$R^L(U^H, \overline{V}^H) = \langle \dot{R}^{X,L} U, V\rangle_{h^{TX}}, \tag{3.5}$$

where $U^H, V^H \in T^H_X Z$ denotes the horizontal lifts of U, V. By Assumption 3.1, $\dot{R}^{X,L}$ is positive definite.

Theorem 3.3. *Let $k \in \{0, \dots, \dim B\}$. Then the component of degree $2k$ of the torsion form $\mathcal{T}(\omega^M, h^{F_p})$ have the following asymptotics as $p \to +\infty$:*

$$\mathcal{T}(\omega^M, h^{F_p})^{(2k)} = \frac{1}{2}\left(\int_Z \log\left[\det\left(\frac{p\dot{R}^{X,L}}{2\pi}\right)\right] \mathrm{e}^{\frac{\sqrt{-1}}{2\pi} pR^L}\right)^{(2k)} + o(p^{\dim Z+k}), \tag{3.6}$$

in the topology of $\mathcal{C}^\infty$ convergence on compact subsets of B.

Remark 3.4.

1. We can also obtain the asymptotics of the torsion if we replace the bundle $F_p = H^0(Y, L^p)$ by a twisted bundle $E \otimes H^0(Y, E' \otimes L^p)$, where $E \to M$ and $E' \to N$ are holomorphic bundles endowed with Hermitian metrics: the factor $\frac{1}{2}$ in (3.6) only has to be replaced by $\frac{\mathrm{rk}(E)\,\mathrm{rk}(E')}{2}$.
2. If B is a point, we reobtain the main result of [7] in the case where $\pi_1 = \mathrm{Id}$ and [8] in the case where $N = \mathbb{P}(E^*)$ and $L = \mathcal{O}(1)$, with E a positive bundle on M.

4. Heat kernel and Toeplitz operators

Let us point out that, when compared with the results of Bismut and Vasserot, and besides the difficulties caused by the family situation in itself, a major difference is that the dimension of F_p grows to infinity. Thus the method they used to prove the convergence of the heat kernel is not directly applicable here because, even locally, we have a family of operators acting on different spaces. To overcome this issue, we draw our inspiration from [6] and use the theory of Toeplitz operators.

Note that when (π_2, ω^M) is a Kähler fibration, one can use [5] and [13] to reduce the problem to the case of powers of a line bundle (i.e., $Y = \{\mathrm{pt}\}$) and thereby handle the direct image without using Toeplitz operators, but the convergence is then modulo ∂ and $\bar{\partial}$ exact forms on B and not at the level of forms. Thus, unless B is compact and Kähler, we are considering a convergence result modulo a non-closed space, which is irrelevant. However, for the degree zero component of the torsion forms, i.e., for the torsion, as there is no quotient to take, the result of [1] (which is [13] in degree zero) indeed enables to avoid Toeplitz operators.

We now give here the main steps of the proof of Theorem 3.3. The idea is to compute the asymptotics of the different terms appearing in the formula (2.6) applied to F_p, and then to get dominations in order to apply the dominated convergence theorem.

We now recall the definition given in [14, Def. 7.2.1] of a Toeplitz operator. Let $b \in B$. For $x \in X_b := \pi_2^{-1}(b)$, let $Y_x := \pi_1^{-1}(x)$ and, for any holomorphic Hermitian bundle E on Y_x, let $P_{p,x}$ be the orthogonal projection

$$P_{p,x} \colon L^2(Y_x, E \otimes L^p) \to H^0(Y_x, E \otimes L^p). \tag{4.1}$$

Definition 4.1. A *Toeplitz operator* on Y_x is a family of operators $T_p \in \mathrm{End}(L^2(Y_x, E \otimes L^p))$ satisfying the following two properties:

(i) for any $p \in \mathbb{N}$, we have

$$T_p = P_{p,x} T_p P_{p,x}\,; \tag{4.2}$$

(ii) there exist sections $f_r \in \mathcal{C}^\infty\big(Y, \mathrm{End}(E)\big)$ such that for $k \in \mathbb{N}$ there is $C_k > 0$ such that for the operator norm,

$$\left\| T_p - \sum_{r=0}^{k} p^{-r} P_{p,x} f_r P_{p,x} \right\| \le C_k p^{-k-1}. \tag{4.3}$$

From [14, (4.1.84), Lem. 7.2.4], we can prove the important property of Toeplitz operators:

$$\mathrm{Tr}\,|_{L^2(Y_x, E\otimes L^p)}[P_{p,x} f P_{p,x}] = p^{\dim Y} \int_{Y_x} \mathrm{Tr}\,|_E[f] \frac{c_1(L,h^L)^{\dim Y}}{\dim Y!} + O(p^{\dim Y - 1}). \tag{4.4}$$

Let $B_{p,u}$ be the Bismut superconnection associated with (ω^M, h^{F_p}) as defined in Section 2 and the references therein. Let $\exp(-B^2_{p,u/p})(x,x')$ be the Schwartz kernel of $\exp(-B^2_{p,u/p})$ with respect to $dv_{X_b}(x')$. Then for $b \in B$,

$$\exp(-B^2_{p,u/p})(x,x) \in \mathrm{End}\left(\Lambda^\bullet(T^*_{\mathbb{R},b}B) \otimes \Lambda^{0,\bullet}(T^*X_b) \otimes F_p\right). \tag{4.5}$$

In the sequel, we endow $\mathrm{End}\left(\Lambda^\bullet(T^*_{\mathbb{R},b}B) \otimes \Lambda^{0,\bullet}(T^*X_b) \otimes F_p\right)$ with the operator norm. As in Definition 4.1, we will denote this norm by $\|\cdot\|$ even if it is defined on spaces depending on p. Similarly, for $\ell \in \mathbb{N}$, the $\mathcal{C}^\ell$-norm for the parameters $(b,x) \in M$ induced by this norm on each bundle $\mathrm{End}\left(\Lambda^\bullet(T^*_{\mathbb{R},b}B) \otimes \Lambda^{0,\bullet}(T^*X_b) \otimes F_p\right)$ will be simply denoted by $\|\cdot\|_{\mathcal{C}^\ell}$.

For $a > 0$, set $\psi_a \colon \alpha \in \Lambda^q(T^*B) \mapsto a^q \alpha$. The asymptotics of this heat kernel is given by a Toeplitz operator as follows:

Theorem 4.2. *Let $\ell \in \mathbb{N}$ and $b \in B$. As $p \to +\infty$, uniformly as u varies in a compact subset of $\mathbb{R}^*_+$ and as (x,b) varies in a compact subset of M, we have the following estimate:*

$$\left\| \psi_{1/\sqrt{p}} \exp(-B^2_{p,u/p})(x,x) - \frac{p^{\dim X}}{(2\pi)^{\dim X}} P_{p,x}\, e^{-\Omega_{u,(x,\cdot)}} \frac{\det(\dot{R}^{X,L}_{(x,\cdot)})}{\det\big(1-\exp(-u\dot{R}^{X,L}_{(x,\cdot)})\big)} P_{p,x} \right\|_{\mathcal{C}^\ell} = o(p^{\dim X}). \tag{4.6}$$

Here the dot symbolizes the variable in Y_x and $\Omega_u \in \mathcal{C}^\infty\big(N, \pi_3^*\Lambda^\bullet(T^*_{\mathbb{R}}B) \otimes \pi_1^*\mathrm{End}(\Lambda^{0,\bullet}(T^*X_b))\big)$ is explicit.

The fact that Toeplitz operators arise in the asymptotics of the heat kernel ultimately comes from a result of Ma and Zhang [15, Thm 2.1], asserting that the curvature of the Chern connection of F_p is itself a Toeplitz operator. This result is the cornerstone of our approach.

Remark 4.3. In the proof of Theorem 4.2, we do not need the assumption that L is positive along the fiber Z, but only along the fiber Y.

We next get the uniform short-time expansion of the heat kernel, and the fact that the coefficients are asymptotic to Toeplitz operators.

Theorem 4.4. *There exist $\{A_{p,j}\} \in \mathcal{C}^\infty\big(M, \pi_2^*\Lambda^\bullet(T^*_{\mathbb{R}}B) \otimes \mathrm{End}(\Lambda^{0,\bullet}(T^*X) \otimes F_p)\big)$ such that for any $k, \ell \in \mathbb{N}$ and b in a compact subset of B, there exists $C > 0$ such that for any $u \in\,]0,1]$ and $p \ge 1$,*

$$p^{-\dim X} \left\| \psi_{1/\sqrt{p}} \exp\left(-B^2_{p,u/p}\right)(x,x) - \sum_{j=-\dim M}^{k} A_{p,j}(x) u^j \right\|_{\mathcal{C}^\ell} \le C u^{k+1}. \tag{4.7}$$

Moreover, as $p \to +\infty$, uniformly over compact subsets of B, for any $j \ge -\dim M$

$$\left\| A_{p,j}(x) - p^{\dim X} P_{p,x} A_j(x,\cdot) P_{p,x} \right\|_{\mathcal{C}^\ell} = o(p^{\dim X}). \tag{4.8}$$

Here $A_j \in \mathcal{C}^\infty\big(N, \pi_3^*\Lambda^\bullet(T^*_{\mathbb{R}}B) \otimes \pi_1^*\mathrm{End}(\Lambda^{0,\bullet}(T^*X_b))\big)$ is explicit.

Remark 4.5. The operator convergences in Theorems 4.2 and 4.4 imply the convergence of the corresponding supertraces divided by $p^{\dim Z}$. Indeed, it is classical that $\dim F_p \le C p^{\dim Y}$ for some constant C, and thus for $D \in \mathrm{End}(\Lambda^\bullet(T^*_{\mathbb{R},b}B) \otimes \Lambda^{0,\bullet}(T^*X_b) \otimes F_p)$ and $f \in \mathrm{End}(\Lambda^\bullet(T^*_{\mathbb{R},b}B) \otimes \Lambda^{0,\bullet}(T^*X_b))$, we know that

$$\|D - p^{\dim X} P_p f P_p\| = o(p^{\dim X}) \implies |p^{-\dim Z}\,\mathrm{Tr}_s(D) - p^{-\dim Y}\,\mathrm{Tr}_s(P_p f P_p)| = o(1). \tag{4.9}$$

Thus, we can conclude using (4.4).

Finally, we get the following large time domination, for which a key ingredient is to obtain a spectral gap theorem for the Dirac operator of F_p via the Lichnerowicz formula and the expression of the Chern curvature of F_p as a Toeplitz operator [15, Thm. 2.1].

Theorem 4.6. *For $\ell \in \mathbb{N}$, there exists $C > 0$, uniform over compact subsets of B, such that for $u \geq 1$ and $p \geq 1$,*

$$p^{-\dim Z} \left| \psi_{1/\sqrt{p}} \operatorname{Tr}_s \left[N_{u/p} \exp\left(-B^2_{p,u/p} \right) \right] \right|_{\mathcal{C}^\ell} \leq \frac{C}{\sqrt{u}}. \tag{4.10}$$

Using the formula (2.6) for F_p and applying the dominated convergence theorem using Theorems 4.2, 4.4 and 4.6, we compute explicitly the derivative at zero of the limiting zeta function. This appears to be more difficult than in the result of Bismut and Vasserot. We then get the asymptotics of the torsion, i.e., Theorem 3.3.

For more details and the proofs of the results announced here, we refer the reader to [16].

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