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Partial differential equations

Global bifurcation of vortex and dipole solutions in Bose–Einstein condensates

Bifurcation globale des solutions de type « vortex » et « dipôle » dans les condensats de Bose–Einstein

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A B S T R A C T

We prove the existence of symmetric periodic solutions to

$$i\,u_t + \Delta u - (x^2 + y^2)u - |u|^2 u = 0.$$

As a corollary we obtain the existence of dipole solutions.

R É S U M É

Dans cette note, nous prouvons l'existence de solutions périodiques symétriques de l'équation

$$i\,u_t + \Delta u - (x^2 + y^2)u - |u|^2 u = 0.$$

Comme corollaire, nous obtenons des solutions de type « dipôle ».

Version française abrégée

L'équation de Gross–Pitaevskii pour un condensat de Bose–Einstein confiné dans un piège harmonique symétrique est donnée par

$$i\,u_t + \Delta u - (x^2 + y^2)u - |u|^2\,u = 0. \tag{1}$$

Dans ce travail, nous prouvons l'existence de plusieurs branches globales de solutions de (1), dont certaines correspondent à des solutions de type « vortex » et d'autres à des solutions de type « dipôle ». Nos principaux résultats sont les suivants.

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Théorème 1.1. *Soit $m_0, n_0 \in \mathbb{N}^*$. L'équation (1) possède une branche de bifurcation globale dans Fix$(\tilde{O}(2)) \times \mathbb{R}$ (voir (3) ci-dessous) de solutions périodiques de la forme*

$$e^{-i\omega t} e^{im_0\theta} u(r),$$

à partir de $\omega = 2(m_0 + 2n_0 + 1)$. Ci-dessus, $u(r)$ est une fonction réelle qui s'annule à l'origine.

Théorème 1.2. *Soit $m_0, n_0 \in \mathbb{N}$ avec $m_0 \geq 1$, et $n_0 < m_0$, alors l'équation (1) possède une branche de bifurcation globale dans Fix$(\mathbb{Z}_2 \times \tilde{D}_{2m_0}) \times \mathbb{R}$ (voir (4) ci-dessous) de solutions de la forme*

$$e^{-i\omega t} u(r, \theta),$$

à partir de $\omega = 2(m_0 + 2n_0 + 1)$. Ci-dessus, $u(r, \theta)$ est une fonction réelle qui s'annule à l'origine et a les symétries suivantes :

$$u(r, \theta) = u(r, -\theta) = -u(r, \theta + \pi/m_0).$$

Le cas $(m_0, n_0) = (1, 0)$ correspond à des solutions de type « dipôle ».

1. Introduction

The Gross–Pitaevskii equation for a Bose–Einstein condensate (BEC) with symmetric harmonic trap is given by

$$i\, u_t + \Delta u - (x^2 + y^2)u - |u|^2 u = 0. \tag{2}$$

Periodic solutions to (2) play an important role in the understanding of the long-term behavior of its solutions. In [12], symmetric and asymmetric vortex solutions are obtained and their stability is established. Solutions with two rotating vortices of opposite vorticities are constructed in [14]. In [3], the authors prove the existence of periodic and quasi-periodic trajectories of dipoles in anisotropic condensates. The literature of the study of vortex dynamics in Bose–Einstein condensates is vast, both on the mathematical and physical sides; we refer the reader to [5–7,9,11,14] and the references therein for a more detailed account.

In this note, we prove the existence of several global branches of solutions to (2), among which there are vortex solutions and dipole solutions. Our main results are:

Theorem 1.1. *Let $m_0 \geq 1$ and n_0 be fixed non-negative integers. Eq. (2) has a global bifurcation in*

$$Fix(\tilde{O}(2)) = \{u \in H^2(\mathbb{R}^2; \mathbb{C}) : u(r, \theta) = e^{im_0\theta} u(r) \text{ with } u(r) \text{ real valued } \}. \tag{3}$$

These are periodic solutions to (2) of the form

$$e^{-i\omega t} e^{im_0\theta} u(r),$$

starting from $\omega = 2(m_0 + 2n_0 + 1)$, where $u(r)$ is a real-valued function.

It will be clear from the proof, and standard bifurcation theory, that for small amplitudes a, we have the local expansion

$$u(r) = a v_{m_0,n_0}(r) + \sum_{n\in\mathbb{N}} u_{m_0,n} v_{m_0,n}(r) \text{ and } u_{m_0,n}(a) = O(a^2),$$

where the $v_{m,n}$'s are the eigenfunctions introduced in (5) and the $u_{m,n}$'s are Fourier coefficients. Thus, the number m_0 is the degree of the vortex at the origin and n_0 is the number of nodes of $u(r)$ in $(0, \infty)$.

Our second theorem is concerned with the existence of multi-pole solutions.

Theorem 1.2. *Let $m_0 \geq 1$ and $n_0 < m_0$ be two fixed non-negative integers, then Eq. (2) has a global bifurcation in*

$$Fix(\mathbb{Z}_2 \times \tilde{D}_{2m_0}) = \{u \in H^2(\mathbb{R}^2; \mathbb{C}) : u(r, \theta) = \bar{u}(r, \theta) = u(r, -\theta) = -u(r, \theta + \pi/m_0)\}. \tag{4}$$

These are periodic solutions to (2) of the form

$$e^{-i\omega t} u(r, \theta),$$

starting from $\omega = 2(m_0 + 2n_0 + 1)$, where $u(r, \theta)$ is a real function vanishing at the origin, enjoying the symmetries

$$u(r, \theta) = u(r, -\theta) = -u(r, \theta + \pi/m_0).$$

The requirement $n_0 < m_0$ in Theorem 1.2 is a non-resonance condition. The solutions to the previous theorem for $(m_0, n_0) = (1, 0)$ correspond to dipole solutions. This follows locally from the estimate

$$u(r,\theta) = a(\mathrm{e}^{\mathrm{i}m_0\theta} + \mathrm{e}^{-\mathrm{i}m_0\theta}) v_{m_0,n_0}(r) + \sum_{m\in\{m_0,3m_0,5m_0,\ldots\}} \sum_{n\in\mathbb{N}} u_{m,n}(\mathrm{e}^{\mathrm{i}m\theta} + \mathrm{e}^{-\mathrm{i}m\theta}) v_{m,n}(r)$$

for small amplitude a where $u_{m,n} = O(a^2)$.

For $n_0 = 0$, since the function $v_{m_0,0}$ is positive for $r \in (0, \infty)$ and $u_{m,n} = O(a^2)$, $u(r,\theta)$ is zero only when $\theta = (k + 1/2)\pi/m_0$. Moreover, as u is real, then the lines $\theta = (k + 1/2)\pi/m_0$ correspond to zero-density regions and the phase has a discontinuous jump of π at those lines.

A difficulty when trying to obtain the dipole solution, that is for $(m_0, n_0) = (1, 0)$, is the fact that to carry out a local inversion, one has to deal with a linearized operator with a repeated eigenvalue corresponding to $(m_0, n_0) = (1, 0)$ and $(-1, 0)$. We overcome this by restricting our problem to a natural space of symmetries that we identify below. In this space our linearized operator only encounters a simple bifurcation which yields the global existence result thanks to a topological degree argument; see Theorem 2.1.

2. Reduction to a bifurcation in a subspace of symmetries

The group of symmetries of (2) is a triple torus, corresponding to rotations, phase and time invariances. The analysis of the group representations leads to two kinds of isotropy groups, one corresponding to vortex solutions and the other to dipole solutions. A fixed point argument on restricted subspaces and Leray–Schauder degree yield the global existence of these branches.

In [12], the authors study the case $(m_0, n_0) = (1, 0)$, which is bifurcation of a vortex of degree one. They also obtain second branch stemming from this one and analyze its stability. The bifurcation from the case $(m_0, n_0) = (0, 0)$ is the ground state (see [12]). The proof of existence of dipole-like solutions was left open. In the present work, we use the symmetries of the problem, classifying the spaces of irreducible representations, to obtain these as global branches provided a non-resonance condition is satisfied.

2.1. Setting of the problem

We rewrite (2) it in polar coordinates:

$$\mathrm{i}\,u_t + \Delta_{(r,\theta)} u - r^2 u - |u|^2\, u = 0,$$

where $\Delta_{(r,\theta)} = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$. Periodic solutions of the form $u(t,r,\theta) = \mathrm{e}^{-\mathrm{i}\omega t} u(r,\theta)$ are zeros of the map

$$f(u,\omega) = -\Delta_{(r,\theta)} u + (r^2 - \omega + |u|^2)u.$$

Let X be the space of functions in H^2 for which $\|u\|_X^2 = \|u\|_{H^2}^2 + \left\|r^2 u\right\|_{L^2}^2$ is finite. The eigenvalues and eigenfunctions of the linear Schrödinger operator

$$L = -\Delta_{(r,\theta)} + r^2 : X \to L^2(\mathbb{R}^2;\mathbb{C}),$$

are found in Chapter 6, complement D, pp. 727–737, on [1]. The operator L has eigenfunctions $v_{m,n}(r)\,\mathrm{e}^{\mathrm{i}m\theta}$, which form an orthonormal basis of $L^2(\mathbb{R}^2;\mathbb{C})$, and eigenvalues

$$\lambda_{m,n} = 2(|m| + 2n + 1) \text{ for } (m,n) \in \mathbb{Z}\times\mathbb{N},$$

where $v_{m,n}(r)$ is a solution to

$$\left(-(\partial_r^2 + r^{-1}\partial_r - r^{-2}m^2) + r^2\right) v_{m,n}(r) = \lambda_{m,n} v_{m,n}(r) \tag{5}$$

with $v_{m,n}(0) = 0$ for $m \neq 0$. We have that

$$u = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}} u_{m,n} v_{m,n}(r)\,\mathrm{e}^{\mathrm{i}m\theta}, \qquad Lu = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}} \lambda_{m,n} u_{m,n} v_{m,n}(r)\,\mathrm{e}^{\mathrm{i}m\theta}.$$

Moreover, we know that $v_{m,n}(r)\,\mathrm{e}^{\mathrm{i}m\theta}$ are orthogonal functions, where n is the number of nodes of $v_{m,n}(r)$ in $(0,\infty)$, see section 2.9 in [8].

Remark 1. Notice that this is a slightly different orthonormal system than the one in [12], which is more suited for anisotropic traps: $V(x,y) = \alpha x^2 + \beta y^2$ with $\alpha \neq \beta$.

We have that the norm of u in $L^2(\mathbb{R}^2;\mathbb{C})$ is $\|u\|_{L^2}^2 = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}} |u_{m,n}|^2$. Then, the inverse operator $K = L^{-1}$: $L^2(\mathbb{R}^2;\mathbb{C}) \to X$ is continuous and given by $Ku = \sum \lambda_{m,n}^{-1} u_{m,n} v_{m,n}(r)\,\mathrm{e}^{im\theta}$. Moreover, the operator $K : H^2(\mathbb{R}^2;\mathbb{C}) \to H^2(\mathbb{R}^2;\mathbb{C})$ is compact. Observe that $H^2(\mathbb{R}^2)$ is a Banach algebra, then $\left\| |u|^2 u \right\|_{H^2} \le c\,\|u\|_{H^2}^3$. We then see that $g(u) := K(|u|^2 u) = \mathcal{O}(\|u\|_{H^2}^3)$ is a *nonlinear compact map* such that $g : H^2 \to H^2$. Therefore, we obtain an equivalent formulation for the bifurcation as zeros of the map

$$Kf(u,\omega) = u - \omega K u + g(u) : H^2 \times \mathbb{R} \to H^2.$$

This formulation has the advantage that allows us to appeal to the global Rabinowitz alternative [13] (Theorem 2.1 below).

2.2. Equivariant bifurcation

Let us define the action of the group generated by $(\psi,\varphi) \in \mathbb{T}^2$, $\kappa \in \mathbb{Z}_2$ and $\bar\kappa \in \mathbb{Z}_2$ in L^2 as

$$\rho(\psi,\varphi)u(r,\theta) = \mathrm{e}^{\mathrm{i}\varphi} u(r,\theta+\psi); \quad \rho(\kappa)u(r,\theta) = u(r,-\theta); \quad \rho(\bar\kappa)u(r,\theta) = \bar u(r,\theta).$$

Actually, the group generated by these actions is $\Gamma = O(2) \times O(2)$, and the map Kf is Γ-equivariant.

Given a pair $(m_0,n_0) \in \mathbb{Z}\times\mathbb{N}$, the operator K has multiple eigenvalues $\lambda_{m,n}^{-1} = \lambda_{m_0,n_0}^{-1}$ for each $(m,n) \in \mathbb{Z}\times\mathbb{N}$ such that $|m| + 2n = |m_0| + 2n_0$. To reduce the multiplicity of the eigenvalue λ_{m_0,n_0}^{-1}, we assume for the moment that there is a subgroup G of Γ such that in the fixed point space,

$$Fix(G) = \{u \in H^2 : \rho(g)u = u \text{ for } g \in G\},$$

the linear map K has only one eigenvalue λ_{m_0,n_0}^{-1}. Then, we can apply the following theorem using the fact that $Kf(u,\omega)$: $Fix(G) \times \mathbb{R} \to Fix(G)$ is well defined.

Theorem 2.1. *There is a global bifurcating branch $Kf(u(\omega),\omega) = 0$, starting from $\omega = \lambda_{m_0,n_0}$ in the space $Fix(G) \times \mathbb{R}$, this branch is a continuum that is unbounded or returns to a different bifurcation point $(0,\omega_1)$.*

For a proof, see the simplified approach due to Ize in Theorem 3.4.1 of [10], or a complete exposition in [4].

We note that if $u \in H^2$ is a zero of Kf, then $u = K(\omega u - |u|^2 u) \in H^4$. Using a bootstrapping argument, we obtain that the zeros of Kf are solutions to Eq. (2) in C^∞. Next, we find the irreducible representations and the maximal isotropy groups. The fixed point spaces of the maximal isotropy groups will have the property that K has a simple eigenvalue corresponding to λ_{m_0,n_0}^{-1}. Thus, Theorems 1.1 and 1.2 will follow from Theorem 2.1 applied to $G = \tilde O(2)$ and $G = \mathbb{Z}_2 \times \tilde D_{2m_0}$, respectively.

2.3. Isotropy groups

The action of the group on the components $u_{m,n}$ is given by $\rho(\varphi,\psi)u_{m,n} = \mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{\mathrm{i}m\psi} u_{m,n}$ and $\rho(\kappa)u_{m,n} = u_{-m,n}$ and $\rho(\bar\kappa)u_{m,n} = \bar u_{-m,n}$. Then, the irreducible representations are $(z_1,z_2) = (u_{m,n}, u_{-m,n}) \in \mathbb{C}^2$, and the action of Γ in the representation (z_1,z_2) is

$$\rho(\varphi,\psi)(z_1,z_2) = \mathrm{e}^{\mathrm{i}\varphi}(\mathrm{e}^{\mathrm{i}m\psi} z_1, \mathrm{e}^{-\mathrm{i}m\psi} z_2); \quad \rho(\kappa)(z_1,z_2) = (z_2,z_1); \quad \rho(\bar\kappa)(z_1,z_2) = (\bar z_2, \bar z_1). \tag{6}$$

Actually, the irreducible representations $(u_{m_0,n}, u_{-m_0,n}) \in \mathbb{C}^2$ are similar for all $n \in \mathbb{N}$. The spaces of similar irreducible representations are of infinite dimension. We analyze only non-radial bifurcations, that is solutions bifurcating from $\omega = \lambda_{m_0,n_0}$ with $m_0 \neq 0$; the radial bifurcation with $m_0 = 0$ may be analyzed directly from the operator associated with the spectral problem (5).

Let us fix $m_0 \ge 1$ and $(z_1,z_2) = (u_{m_0,n}, u_{-m_0,n})$. Then, possibly after applying κ, we may assume $z_1 \neq 0$, unless $(z_1,z_2) = (0,0)$. Moreover, using the action of S^1, the point (z_1,z_2) is in the orbit of $(a, r\,\mathrm{e}^{\mathrm{i}\theta})$. It is known that there are only two maximal isotropy groups, one corresponding to $(a,0)$ and the other one to (a,a), see for instance [2].

From (6), we have that the isotropy group of $(a,0)$ is generated by $(\varphi, -\varphi/m_0)$ and $\kappa\bar\kappa$, that is

$$\tilde O(2) = \langle (\varphi, -\varphi/m_0), \kappa\bar\kappa \rangle.$$

While the isotropy group of (a,a) is generated by $(\pi, \pi/m_0)$, κ and $\bar\kappa$, that is

$$\mathbb{Z}_2 \times \tilde D_{2m_0} = \langle \kappa, (\pi, \pi/m_0), \bar\kappa \rangle.$$

These two groups are the only maximal isotropy groups of the representation $(z_1,z_2) \in \mathbb{C}^2$, and the fixed point spaces have real dimension one in $\mathbb{C}^2$.

3. Vortex solutions: Proof of Theorem 1.1

The functions fixed by the group $\tilde{O}(2)$ satisfy $u(r,\theta) = e^{im_0\theta}u(r)$ from the element $(\varphi, -\varphi/m_0)$, and $u(r) = \bar{u}(r)$ from the element $\kappa\bar{\kappa}$. Thus, functions in the space $Fix(\tilde{O}(2))$ are of the form

$$u(r,\theta) = \sum_{n\in\mathbb{N}} u_{m_0,n}\, e^{im_0\theta} v_{m_0,n}(r)$$

with $u_{m_0,n} \in \mathbb{R}$.

Therefore, the map $Kf(u,\omega)$ has a simple eigenvalue λ_{m_0,n_0} in the space $Fix(\tilde{O}(2)) \times \mathbb{R}$ (see (3)). Therefore, from Theorem 2.1, there is a global bifurcation in $Fix(\tilde{O}(2)) \times \mathbb{R}$ starting at $\omega = \lambda_{m_0,n_0}$.

4. Multi-pole-like solutions: Proof of Theorem 1.2

The functions fixed by $\mathbb{Z}_2 \times \tilde{D}_{2m_0}$ satisfy $u(r,\theta) = \bar{u}(r,\theta) = u(r,-\theta)$. Therefore $u_{m,n}$ is real and $u_{m,n} = u_{-m,n}$. Moreover, since $u(r,\theta) = -u(r,\theta+\pi/m_0)$, then $u_{m,n} = -e^{i\pi(m/m_0)}u_{m,n}$. This relation gives $u_{m,n} = 0$ unless $e^{i\pi(m/m_0)} = -1$ or m/m_0 is odd. Thus, functions in the space $Fix(\mathbb{Z}_2 \times \tilde{D}_{2m_0})$ are of the form

$$u(r,\theta) = \sum_{m\in\{m_0,3m_0,5m_0,\ldots\}} \sum_{n\in\mathbb{N}} u_{m,n}(e^{im\theta} + e^{-im\theta}) v_{m,n}(r),$$

where $u_{m,n}$ is real and $m_0 \geq 1$. Therefore, the map K has a simple eigenvalue $\lambda_{m_0,n_0} = 2(m_0 + 2n_0 + 1)$ in $Fix(\mathbb{Z}_2 \times \tilde{D}_{2m_0})$ if $\lambda_{lm_0,n} \neq \lambda_{m_0,n_0}$ for $n \in \mathbb{N}$ and $l = 3,5,7$. This condition is equivalent to $lm_0 + 2n \neq m_0 + 2n_0$ or $2n_0 - (l-1)m_0 \neq 2n$. Then, the eigenvalue λ_{m_0,n_0} is simple if $2n_0 < (l-1)m_0$ for $l = 3,5,\ldots$, or $n_0 < m_0$.

From Theorem 2.1, the map $Kf(u,\omega)$ has a global bifurcation in $Fix(\mathbb{Z}_2 \times \tilde{D}_{2m_0}) \times \mathbb{R}$ as ω crosses the value λ_{m_0,n_0} (see (4)).

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