C. R. Acad. Sci. Paris, Ser. I 354 (2016) 307–311

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential geometry

On the volume of the $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ shadow of a compact set ☆

Sur le volume de la $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$*-ombre d'un ensemble compact*

Amedeo Altavilla [a], Lorenzo Nicolodi [b]

[a] *Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy*
[b] *Dipartimento di Matematica e Informatica, Università degli Studi di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy*

A R T I C L E   I N F O

Article history:
Received 30 September 2015
Accepted 15 December 2015
Available online 3 February 2016

Presented by the Editorial Board

A B S T R A C T

Let $F:\mathbb{R}^{4n}\to\mathbb{R}^{4n}$ be an element of the quaternionic unitary group $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$, let K be a compact subset of $\mathbb{R}^{4n}$, and let V be a $4k$-dimensional quaternionic subspace of $\mathbb{R}^{4n}\cong\mathbb{H}^n$. The $4k$-dimensional *shadow* of the image under F of K is its orthogonal projection $P(F(K))$ onto V. We show that there exists a $4k$-dimensional quaternionic subspace W of $\mathbb{R}^{4n}$ such that the volume of the shadow $P(F(K))$ is the same as the volume of the section $K\cap W$. This is a quaternionic analogue of the symplectic linear non-squeezing result recently obtained by Abbondandolo and Matveyev.

R É S U M É

Soit $F:\mathbb{R}^{4n}\to\mathbb{R}^{4n}$ un élément du groupe unitaire quaternionnien $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$, soit K un ensemble compact dans $\mathbb{R}^{4n}$, et soit V un sous-espace vectoriel quaternionnien de dimension $4k$ dans $\mathbb{R}^{4n}\cong\mathbb{H}^n$. L'*ombre* $4k$-dimensionnelle de l'image par F de K est sa projection orthogonale $P(F(K))$ sur V. Nous montrons qu'il existe un sous-espace vectoriel quaternionnien $W\subset\mathbb{R}^{4n}$ de dimension $4k$ tel que le volume de l'ombre $P(F(K))$ est égal au volume de la section $K\cap W$. Ceci est un analogue quaternionnien du résultat de *non-squeezing* lineaire symplectique obtenu récemment par Abbondandolo et Matveyev.

1. Introduction

Consider $\mathbb{R}^{2n}$ with the canonical symplectic form Ω. Let $B_R\subset\mathbb{R}^{2n}$ be the $2n$-dimensional ball of radius R, and let ω_{2k} denote the volume of the unit $2k$-dimensional ball, $1\le k\le n$. Recently, Abbondandolo and Matveyev [1] have proved the following.

☆ Authors partially supported by FIRB 2012 "Geometria differenziale e teoria geometrica delle funzioni" (A.A.); PRIN 2010–2011 "Varietà reali e complesse: geometria, topologia e analisi armonica" (L.N.); and by the GNSAGA of INdAM.

E-mail addresses: amedeoaltavilla@gmail.com (A. Altavilla), lorenzo.nicolodi@unipr.it (L. Nicolodi).

http://dx.doi.org/10.1016/j.crma.2015.12.007
1631-073X/

Theorem 1 *(Linear non-squeezing). (See [1].) Let F be a linear symplectic automorphism of $\mathbb{R}^{2n} \cong \mathbb{C}^n$, and let $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the orthogonal projection onto a complex linear subspace $V \subset \mathbb{R}^{2n}$ of real dimension $2k$, $1 \le k \le n$. Then*

$$vol_{2k} P(F(B_R)) \ge \omega_{2k} R^{2k},$$

and equality holds if and only if the image of V under the adjoint of F is a complex linear subspace.

For $k = 1$, this result is a reformulation of Gromov's non-squeezing theorem for linear symplectic maps [8]. The proof relies essentially on Wirtinger's inequality for the kth powers $\Omega^k = \Omega \wedge \cdots \wedge \Omega$, $1 \le k \le n$, or equivalently on the fact that the $2k$-forms $\Omega^k/k!$ give rise to calibrations in the sense of Harvey–Lawson [10].

The purpose of this note is to address a quaternionic analogue of this result. In the quaternionic case, the symplectic 2-form Ω is replaced by the standard quaternionic Kähler 4-form

$$\Phi = \omega_i \wedge \omega_i + \omega_j \wedge \omega_j + \omega_k \wedge \omega_k$$

on $\mathbb{R}^{4n} \cong \mathbb{H}^n$, identified by

$$(x_0^1, x_1^1, x_2^1, x_3^1, \ldots, x_0^n, x_1^n, x_2^n, x_3^n) \mapsto (x_0^1 + x_1^1 i + x_2^1 j + x_3^1 k, \ldots, x_0^n + x_1^n i + x_2^n j + x_3^n k), \tag{1}$$

where

$$\begin{cases} \omega_i = \sum_{\alpha=1}^n (\mathrm{d}x_0^\alpha \wedge \mathrm{d}x_1^\alpha - \mathrm{d}x_2^\alpha \wedge \mathrm{d}x_3^\alpha), \\ \omega_j = \sum_{\alpha=1}^n (\mathrm{d}x_0^\alpha \wedge \mathrm{d}x_2^\alpha - \mathrm{d}x_3^\alpha \wedge \mathrm{d}x_1^\alpha), \\ \omega_k = \sum_{\alpha=1}^n (\mathrm{d}x_0^\alpha \wedge \mathrm{d}x_3^\alpha - \mathrm{d}x_1^\alpha \wedge \mathrm{d}x_2^\alpha). \end{cases} \tag{2}$$

For $n \ge 2$, it is known that the subgroup of $\mathrm{GL}(4n, \mathbb{R})$ that fixes Φ is the quaternionic unitary group $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$ (cf. [4,9,12,13]). We will prove the following.

Theorem 2. *Let $F \in \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ be a linear quaternionic unitary automorphism of $\mathbb{R}^{4n} \cong \mathbb{H}^n$, and let $P : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ be the orthogonal projection onto a quaternionic linear subspace $V \subset \mathbb{R}^{4n}$ of real dimension $4k$, $k = 1, \ldots, n$. Then, for every compact set $K \subset \mathbb{R}^{4n}$, there exists a $4k$-dimensional quaternionic linear subspace $W \subset \mathbb{R}^{4n}$ such that*

$$vol_{4k} P(F(K)) = vol_{4k}(K \cap W).$$

2. Preliminaries

Let $\mathbb{H}$ be the real noncommutative algebra of quaternions, with the standard basis $\{1, i, j, k\}$. Multiplication is determined by the rules

$$i^2 = j^2 = k^2 = ijk = -1, \tag{3}$$

which imply $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. If $q \in \mathbb{H}$, we write

$$q = q_0 + q_1 i + q_2 j + q_3 k.$$

The real and imaginary parts of q are $\mathrm{Re}\, q = q_0$ and $\mathrm{Im}\, q = q_1 i + q_2 j + q_3 k$, respectively. The conjugate of q is given by $\bar{q} = \mathrm{Re}\,(q) - \mathrm{Im}\,(q)$ and the square norm of q is defined by $N(q) = q\bar{q}$. As a vector space, $\mathbb{H}$ identifies with $\mathbb{R}^4$ under the isomorphism $q_0 + q_1 i + q_2 j + q_3 k \mapsto {}^{\mathrm{t}}(q_0, q_1, q_2, q_3)$, which in turn induces an isomorphism between the subspace of imaginary quaternions $\mathrm{Im}\, \mathbb{H} = \mathrm{span}\{i, j, k\}$ and $\mathbb{R}^3$. In particular, $1, i, j, k$ identify with the elements of the canonical basis e_0, e_1, e_2, e_3 of $\mathbb{R}^4$, respectively. For $p, q \in \mathbb{H}$, we have $\overline{pq} = \bar{q}\bar{p}$ and $\bar{p}q + q\bar{p} = 2\mathrm{Re}\,(\bar{p}q) = 2\langle p, q\rangle$, where $\langle p, q\rangle$ is the standard Euclidean inner product on $\mathbb{R}^4 \cong \mathbb{H}$, under the above identification.

Let $\mathbb{H}^n$ be the space of column n-tuples of quaternions endowed with its right $\mathbb{H}$-vector space structure, $x \cdot q = {}^{\mathrm{t}}(x_1, \ldots, x_n) \cdot q = {}^{\mathrm{t}}(x_1 q, \ldots, x_n q)$, its standard quaternionic Hermitian product

$$h(x, y) = {}^{\mathrm{t}}\bar{x} y = \sum_{i=1}^n \bar{x}_i y_i,$$

and its real scalar product $\langle x, y\rangle := \mathrm{Re}\, h(x, y)$, which equals the standard Euclidean inner product on $\mathbb{R}^{4n} \cong \mathbb{H}^n$.

The subgroup of $\mathrm{GL}(n, \mathbb{H})$ that fixes h is the *symplectic group*

$$\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n, \mathbb{H}) \mid h(Ax, Ay) = h(x, y)\}.$$

The group $\mathrm{Sp}(1) = S^3 = \{q \in \mathbb{H} \mid \bar{q}q = 1\}$ of unit quaternions acts on $\mathbb{H}^n \cong \mathbb{R}^{4n}$ by right multiplication and the corresponding $\mathbb{R}$-linear maps preserve the Euclidean inner product, i.e.,

$$\langle x, y\rangle = \langle x \cdot q, y \cdot q\rangle. \tag{4}$$

In fact, for $x, y \in \mathbb{H}^n$ and $q \in \mathrm{Sp}(1)$,

$$2\langle xq, yq\rangle = h(xq, yq) + h(yq, xq) = \bar{q}h(x, y)q + \bar{q}h(y, x)q = 2\bar{q}\langle x, y\rangle q = 2\langle x, y\rangle.$$

If $S^2 = \{u \in \mathrm{Im}\,\mathbb{H} \mid \bar{u}u = -u^2 = 1\}$ is the sphere of unit imaginary quaternions, right multiplication by $u \in S^2$, $R_u x := x \cdot u$, defines an orthogonal complex structure on $\mathbb{R}^{4n}$, i.e., $R_u^2 = -\mathrm{Id}$.

Note that $\mathrm{Sp}(1)$ as a group of right multiplications by unit quaternions is not a subgroup of $\mathrm{Sp}(n)$. However, $\mathrm{Sp}(1)$ and $\mathrm{Sp}(n)$ consist of $\mathbb{R}$-linear maps of $\mathbb{H}^n$ which preserve the Euclidean inner product of $\mathbb{H}^n \cong \mathbb{R}^{4n}$ and their intersection is $\mathbb{Z}_2 \cong \{\pm\mathrm{Id}\}$. The *quaternionic unitary group* is the enhancement

$$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) := \mathrm{Sp}(n) \times \mathrm{Sp}(1)/\mathbb{Z}_2$$

of $\mathrm{Sp}(n)$ by $\mathrm{Sp}(1)$. It consists of the $\mathbb{R}$-linear automorphisms $T_{A,q}$ of $\mathbb{H}^n$ defined by

$$T_{A,q}(x) := Ax \cdot q, \quad x \in \mathbb{H}^n,$$

where $A \in \mathrm{Sp}(n)$ and $q \in \mathrm{Sp}(1)$. The group $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ is a subgroup of $\mathrm{SO}(4n)$. In particular, for each $\lambda \in \mathbb{H}$, $T_{A,q}(x \cdot \lambda) = T_{A,q}(x) \cdot q^{-1}\lambda q$.

Remark 1. If $F \in \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, its adjoint F^{T} with respect to the Euclidean inner product is still an element of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$. In fact, for $F = T_{A,q} \in \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ and $x, y \in \mathbb{H}^n$,

$$\langle Fx, y\rangle = \langle Ax \cdot q, y\rangle = \langle x \cdot q, {}^{\mathrm{t}}\bar{A}y\rangle = \langle x, {}^{\mathrm{t}}\bar{A}y \cdot \bar{q}\rangle$$

by (4). Thus, $F^{\mathrm{T}} = T_{A^*,\bar{q}}$, where $A^* = {}^{\mathrm{t}}\bar{A} \in \mathrm{Sp}(n)$ and $\bar{q} \in \mathrm{Sp}(1)$.

Remark 2. If $V \subset \mathbb{R}^{4n}$ is a quaternionic subspace, i.e., a right $\mathbb{H}$-linear subspace of $\mathbb{R}^{4n}$, and $F \in \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, then the image $F(V)$ is a quaternionic subspace. Now, for each $\lambda \in \mathbb{H}$, $T_{A,q}(x) \cdot \lambda = Axq\lambda = A(xq\lambda q^{-1})q = T_{A,q}(xq\lambda q^{-1})$, where $xq\lambda q^{-1} \in V$, and hence the claim. Alternatively, observe that F is a *quaternionic* $\mathbb{R}$-linear map (cf. [2], Definition 1.6), i.e., for each $u \in S^2$, $F \circ R_u = R_{u'} \circ F$, for some $u' \in S^2$. As such, it clearly takes quaternionic subspaces to quaternionic subspaces.

Let $\omega_u(x, y) := \mathrm{Re}\,h(R_u x, y) = \langle R_u x, y\rangle$ be the Kähler 2-form corresponding to the complex structure R_u, for $u \in S^2$. Then, for all $x, y \in \mathbb{H}^n$,

$$h(x, y) = \langle x, y\rangle + i\omega_i(x, y) + j\omega_j(x, y) + k\omega_k(x, y).$$

In fact, $\langle 1, h(x, y)\rangle = \mathrm{Re}\,h(x, y)$, and for $u \in S^2$,

$$2\langle u, h(x, y)\rangle = \bar{u}h(x, y) + \overline{h(x, y)}u = h(x \cdot u, y) + h(y, x \cdot u) = 2\mathrm{Re}\,h(R_u x, y).$$

According to the identification (1), the Kähler 2-forms $\omega_i, \omega_j, \omega_k$ are expressed as in (2).

3. Extremal properties of Hilbert forms and proof of Theorem 2

On $\mathbb{R}^{4n} \cong \mathbb{H}^n$, consider the quaternionic Kähler 4-form $\Phi = \omega_i \wedge \omega_i + \omega_j \wedge \omega_j + \omega_k \wedge \omega_k$.

Proposition 3. *(See Bruni [5].) The external powers of Φ and the standard powers of the polynomial $x^2 + y^2 + z^2$ have the same formal expression.*

This observation is important in view of the following classical result.

Theorem 4. *(See Hilbert [11].) For every $m \in \mathbb{N}$, the* m*th power of the sum of squares $x^2 + y^2 + z^2$ is a linear combination with coefficients in $\mathbb{Q}^+$ of* 2m*th powers of first degree polynomials with integer coefficients, i.e.,*

$$(x^2 + y^2 + z^2)^m = \sum_{t=1}^{s} p_t(a_t x + b_t y + c_t z)^{2m}, \quad p_t \in \mathbb{Q}^+,\ a_t, b_t, c_t \in \mathbb{Z}.$$

This theorem suggests the following.

Definition 1. (See Bruni [5].) Let $r_1, \dots r_s \in \mathbb{R}^+$ and $q_1, \dots, q_s \in S^2 \subset \mathrm{Im}\,\mathbb{H}$, with $q_t = q_t^1 i + q_t^2 j + q_t^3 k$ $(t = 1, \dots, s)$. For $p < n$, a *Hilbert form* is the $4p$-forms given by

$$\Psi = \sum_{t=1}^{s} r_t \left(\bigwedge^{2p} \omega_{q_t} \right) = \sum_{t=1}^{s} r_t \left(\bigwedge^{2p} \left(q_t^1 \omega_i + q_t^2 \omega_j + q_t^3 \omega_k \right) \right).$$

The *height* of Ψ is the real number $h_\Psi := \left(\sum_{t=1}^{s} r_t\right)(2p)!$.

Remark 3. Examples of Hilbert forms include the exterior powers of the quaternionic Kähler 4-form Φ. For instance,

$$\Phi^2 = \frac{9}{12}\Big[\Big(\frac{1}{\sqrt{3}}\omega_i + \frac{1}{\sqrt{3}}\omega_j + \frac{1}{\sqrt{3}}\omega_k\Big)^4 + \Big(\frac{1}{\sqrt{3}}\omega_i + \frac{1}{\sqrt{3}}\omega_j - \frac{1}{\sqrt{3}}\omega_k\Big)^4 + \Big(\frac{1}{\sqrt{3}}\omega_i - \frac{1}{\sqrt{3}}\omega_j + \frac{1}{\sqrt{3}}\omega_k\Big)^4 + \Big(-\frac{1}{\sqrt{3}}\omega_i + \frac{1}{\sqrt{3}}\omega_j + \frac{1}{\sqrt{3}}\omega_k\Big)^4\Big] + \frac{2}{3}\Big(\omega_i^4 + \omega_j^4 + \omega_k^4\Big).$$

Moreover, the heights of Φ, Φ^2, and Φ^3 are, respectively, $3 \cdot (2)!$, $5 \cdot (4)!$, and $7 \cdot (6)!$ (cf. [5]).

Remark 4. In [6], it is shown that every linear combination of ω_i, ω_j and ω_k is invariant under the action of $\mathrm{Sp}(n)$. This implies that all Hilbert forms are $\mathrm{Sp}(n)$-invariant. In general, they need not be invariant under $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$. However, note that the kth powers Φ^k, $1 \le k \le n$, of the quaternionic Kähler form Φ are invariant under $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$.

For the Hilbert forms, the following holds.

Theorem 5. *(See Bruni [5].) Let ξ be a simple $4p$-vector and let Ψ be a Hilbert form. Then*

$$|\Psi(\xi)| \le h_\Psi vol(\xi),$$

and equality holds if and only if the linear subspace corresponding to ξ is quaternionic.

Remark 5. From the previous theorem, it follows that any Hilbert form Ψ defines a calibration Ψ/h_Ψ. In particular, since $h_\Phi = 6$, the 4-form $\frac{1}{6}\Phi$ is a calibration (cf. [3,6,14]).

Following the idea of proof of Theorem 1 (cf. [1]), we now prove Theorem 2.

Proof of Theorem 2. For $F \in \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, let $A := P \circ F : \mathbb{R}^{4n} \to V \subset \mathbb{R}^{4n}$ and denote by $A^{\mathrm{T}} : V \to \mathbb{R}^{4n}$ the adjoint of A with respect to the Euclidean inner products. Then, $A^{\mathrm{T}}A : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ is symmetric, positive semidefinite and with kernel of codimension $4k$,

$$\mathrm{Ker}(A^{\mathrm{T}}A) = \mathrm{Ker}A = (\mathrm{Ran}A^{\mathrm{T}})^{\perp}.$$

The application $A^{\mathrm{T}}A$ restricts to an automorphism of the $4k$-dimensional space $(\mathrm{Ker}A)^\perp = \mathrm{Ran}A^{\mathrm{T}}$ and, as such, it is the composition of the two isomorphisms

$$A|_{(\mathrm{Ker}A)^\perp} : (\mathrm{Ker}A)^\perp \to V, \quad A^{\mathrm{T}} : V \to (\mathrm{Ker}A)^\perp,$$

which, being one the adjoint of each other, have the same determinant. Thus

$$\det(A^{\mathrm{T}}A|_{(\mathrm{Ker}A)^\perp}) = |\det(A|_{(\mathrm{Ker}A)^\perp})|^2.$$

Let $\xi_1, \dots, \xi_{4k}$ be a basis for $(\mathrm{Ker}A)^\perp = \mathrm{Ran}A^{\mathrm{T}} = F^{\mathrm{T}}V$, such that $|\xi_1 \wedge \dots \wedge \xi_{4k}| = 1$, where $|\xi_1 \wedge \dots \wedge \xi_{4k}|$ denotes the volume of the $4k$-parallelepiped spanned by $\xi_1, \dots, \xi_{4k}$. Since $A(K) = A(K \cap (\mathrm{Ker}A)^\perp) = A(K \cap \mathrm{Ran}A^{\mathrm{T}})$, we have (see, for instance, [1,7])

$$\frac{\mathrm{vol}_{4k}(A(K))}{\mathrm{vol}_{4k}(K \cap (\mathrm{Ker}A^\perp))} = |A\xi_1 \wedge \dots \wedge A\xi_k| = \left|\det(A|_{(\mathrm{Ker}A)^\perp})\right| = \sqrt{\det(A^{\mathrm{T}}A|_{(\mathrm{Ker}(A))^\perp})}. \tag{5}$$

From this identity and Theorem 5, it follows that

$$\left(\frac{\mathrm{vol}_{4k}(A(K))}{\mathrm{vol}_{4k}(K \cap (\mathrm{Ker}A^\perp))}\right)^2 = \det(A^{\mathrm{T}}A|_{(\mathrm{Ker}(A))^\perp}) = \left|A^{\mathrm{T}}A\xi_1 \wedge \dots \wedge A^{\mathrm{T}}A\xi_{4k}\right|$$
$$\ge \frac{1}{h_{\Phi^k}}\left|\Phi^k(A^{\mathrm{T}}A\xi_1 \wedge \dots \wedge A^{\mathrm{T}}A\xi_{4k})\right| = \frac{1}{h_{\Phi^k}}\left|\Phi^k(F^{\mathrm{T}}A\xi_1 \wedge \dots \wedge F^{\mathrm{T}}A\xi_{4k})\right|, \tag{6}$$

and equality holds if and only if the subspace generated by $A^{\mathrm{T}}A\xi_1, \ldots, A^{\mathrm{T}}A\xi_{4k}$, i.e. the subspace $F^{\mathrm{T}}V$, is quaternionic.

According to Remark 1, if $F \in \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, also $F^{\mathrm{T}} \in \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$. Hence, since Φ^k is invariant under $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$,

$$\Phi^k(F^{\mathrm{T}}A\xi_1 \wedge \cdots \wedge F^{\mathrm{T}}A\xi_{4k}) = \Phi^k(A\xi_1 \wedge \cdots \wedge A\xi_{4k}). \tag{7}$$

Now, since the restriction of Φ^k to the quaternionic subspace V is h_{Φ^k}-times the standard volume form, by (5), we have

$$\frac{1}{h_{\Phi^k}}\left|\Phi^k(A\xi_1 \wedge \cdots \wedge A\xi_{4k})\right| = |A\xi_1 \wedge \cdots \wedge A\xi_{4k}| = \frac{\mathrm{vol}_{4k}(A(K))}{\mathrm{vol}_{4k}(K \cap (\mathrm{Ker}A^{\perp}))}. \tag{8}$$

Taking into account that, by Remark 2, the subspace $F^{\mathrm{T}}V =: W$ is always a quaternionic subspace, it follows from (6), (7), and (8) that

$$\mathrm{vol}_{4k}(A(K)) = \mathrm{vol}_{4k}(K \cap (\mathrm{Ker}A^{\perp})) = \mathrm{vol}_{4k}(K \cap W). \qquad \square$$

References

[1] A. Abbondandolo, R. Matveyev, How large is the shadow of a symplectic ball? J. Topol. Anal. 5 (2013) 87–119.

[2] D.V. Alekseevsky, S. Marchiafava, Quaternionic structures on a manifold and subordinated structures, Ann. Mat. Pura Appl. (4) 171 (1996) 205–273.

[3] M. Berger, Du côté de chez Pu, Ann. Sci. Éc. Norm. Supér. (4) 5 (1972) 1–44.

[4] A.L. Besse, Einstein Manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 10, Springer-Verlag, Berlin, 1987.

[5] M. Bruni, Forme di Hilbert e proprietà estremali in una varietà a struttura quaternionale generalizzata, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (8) 65 (1978) 63–68.

[6] R. Bryant, R. Harvey, Submanifolds in hyper-Kähler geometry, J. Amer. Math. Soc. 2 (1989) 1–31.

[7] H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften, vol. 153, Springer-Verlag, New York, 1969.

[8] M. Gromov, Pseudoholomorphic curves in symplectic manifolds, Invent. Math. 82 (1985) 307–347.

[9] R.F. Harvey, Spinors and Calibrations, Perspectives in Mathematics, vol. 9, Academic Press, Inc., Boston, MA, 1990.

[10] R.F. Harvey, H.B. Lawson, Calibrated geometries, Acta Math. 148 (1982) 47–157.

[11] D. Hilbert, Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen (Waringsches Problem), Math. Ann. 67 (1909) 281–300.

[12] V.Y. Kraines, Topology of quaternionic manifolds, Trans. Amer. Math. Soc. 122 (1966) 357–367.

[13] S. Salamon, Riemannian Geometry and Holonomy Groups, Pitman Research Notes in Mathematics Series, vol. 201, Longman Scientific & Technical, Harlow, 1989.

[14] H. Tasaki, Certain minimal or homologically volume minimizing submanifolds in compact symmetric spaces, Tsukuba J. Math. 9 (1985) 117–131.