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Differential geometry

Generalized contact bundles

Sur le fibrés de contact généralisés

Luca Vitagliano [a], Aïssa Wade [b]

[a] *DipMat, Università degli Studi di Salerno & Istituto Nazionale di Fisica Nucleare, GC Salerno, via Giovanni Paolo II n° 123, 84084 Fisciano, SA, Italy*

[b] *Department of Mathematics, Penn State University, University Park, State College, PA 16802, USA*

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ABSTRACT

In this Note, we propose a line bundle approach to odd-dimensional analogues of generalized complex structures. This new approach has three main advantages: (1) it encompasses all existing ones; (2) it elucidates the geometric meaning of the integrability condition for generalized contact structures; (3) in light of new results on multiplicative forms and Spencer operators [8], it allows a simple interpretation of the defining equations of a generalized contact structure in terms of Lie algebroids and Lie groupoids.

RÉSUMÉ

Dans cette Note, nous proposons une approche des structures de contact généralisées reposant sur les fibrés vectoriels de rang 1. Cette nouvelle approche possède trois principaux avantages : (1) elle inclut toutes les autres approches connues à ce jour ; (2) elle éclaircit la signification géométrique de la condition d'intégrabilité des structures de contact généralisées ; (3) au vu de résultats récents obtenus sur les formes multiplicatives et les opérateurs de Spencer [8], elle permet une interprétation simple des équations définissant une structure généralisée de contact en termes d'algébroïdes et de groupoïdes de Lie.

1. Introduction

Generalized complex structures have been introduced by Hitchin in [11] and further investigated by Gualtieri in [10]. They can only be supported by even-dimensional manifolds and encompass symplectic structures and complex structures as extreme cases. Since both of these extreme cases have analogues in odd-dimensional geometry (namely, contact and almost contact structures, respectively), it is natural to ask if there is any natural odd-dimensional analogue of generalized complex structures.

E-mail addresses: lvitagliano@unisa.it (L. Vitagliano), wade@math.psu.edu (A. Wade).

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Several approaches to odd-dimensional analogues of generalized complex structures can be found in the literature [12, 21,17,18,1]. They are often named *generalized contact structures* and all of them include contact structures globally defined by a contact 1-form. However, none of them incorporates non-coorientable contact structures. From a conceptual point of view, contact geometry is the geometry of an hyperplane distribution and the choice of a contact form is just a technical tool making things simpler. Even more, there are interesting contact structures that do not possess any global contact form. Accordingly, it would be nice to define a generalized contact structure "independently of the choice of a contact form". This Note fills that gap. We call the proposed structure a *generalized contact bundle* to distinguish it from previously defined generalized contact structures. Generalized contact bundles are just a slight generalization of Iglesias–Wade integrable generalized almost contact structures [12] to the realm of (generically non-trivial) line bundles. Generalized contact bundles encompass (generically non-coorientable) contact structures and complex structures on the Atiyah algebroid of a line bundle as extreme cases. This new point of view on generalized contact geometry could also be useful in studying T-duality [1].

In this Note, we interpret the defining equations of a generalized contact structure in terms of Lie algebroids and Lie groupoids. As a side result we define a novel notion of multiplicative Atiyah form on a Lie groupoid and identify its infinitesimal counterpart. This could be of interest in its own right.

2. The Atiyah algebroid associated with a contact distribution

For a better understanding of the concept of a generalized contact bundle, we briefly discuss a line bundle approach to contact geometry. By definition, a contact structure on an odd-dimensional manifold M is a maximally non-integrable hyperplane distribution $H \subset TM$. In a dual way, any hyperplane distribution H on M can be regarded as a nowhere vanishing 1-form $\theta : TM \to L$ (its *structure form*) with values in the line bundle $L = TM/H$, such that $H = \ker\theta$. Now, consider the so called *Atiyah algebroid* $DL \to M$ (also known as *gauge algebroid* [14,5]) of the line bundle L [22, Sections 2, 3]. Recall that sections of DL are *derivations* of L, i.e. $\mathbb{R}$-linear operators $\Delta : \Gamma(L) \to \Gamma(L)$ such that there exists a, necessarily unique, vector field $\sigma\Delta \in \mathfrak{X}(M)$, called the *symbol of* Δ, such that $\Delta(f\lambda) = (\sigma\Delta)(f)\lambda + f\Delta(\lambda)$ for all $f \in C^\infty(M)$ and $\lambda \in \Gamma(L)$. This is a transitive Lie algebroid whose Lie bracket is the commutator, and whose anchor $DL \to TM$ is the symbol σ. Additionally, L carries a *tautological representation* of DL given by the action of an operator on a section. Any k-cochain in the de Rham complex $(\Omega_L^\bullet := \Gamma(\wedge^\bullet(DL)^* \otimes L), d_{DL})$ of DL with coefficients in L will be called an L-valued *Atiyah k-form*. There is a one-to-one correspondence between contact structures H with $TM/H = L$ and non-degenerate, d_{DL}-closed, L-valued Atiyah 2-forms. Contact structure H, with structure form $\theta : TM \to L$, corresponds to the Atiyah 2-form $\omega := d_{DL}\sigma^*\theta$, where $\sigma^*\theta(\Delta) := \theta(\sigma\Delta)$.

For more details on Atiyah forms as well as their functorial properties, see [22, Section 3].

3. Generalized contact bundles and contact-Hitchin pairs

Recall that a generalized almost complex structure on a manifold M is an endomorphism $\mathcal{J} : \mathbb{T}M \to \mathbb{T}M$ of the *generalized tangent bundle* $\mathbb{T}M := TM \oplus T^*M$ such that (1) $\mathcal{J}^2 = -\mathrm{id}$, and (2) $\mathcal{J}$ is skew-symmetric with respect to the natural pairing on $\mathbb{T}M$. If, additionally, (3) the $\sqrt{-1}$-eigenbundle of $\mathcal{J}$ in the complexification $\mathbb{T}M \otimes \mathbb{C}$ in involutive relative to the Dorfman (equivalently, the Courant) bracket, then $\mathcal{J}$ is said to be *integrable*, and $(M, \mathcal{J})$ is called a *generalized complex manifold* (see [10] for more details).

Replacing the tangent algebroid with the Atiyah algebroid of a line bundle in the definition of a generalized complex manifold, we obtain the notion of *generalized contact bundle*. More precisely, let $L \to M$ be a line bundle. For a vector bundle $V \to M$, there is an obvious L-valued (duality) pairing $\langle -, - \rangle_L : V \otimes (V^* \otimes L) \to L$, and for every vector bundle morphism $F : V \to W$ (covering the identity) there is an adjoint morphism $F^\dagger : W^* \otimes L \to V^* \otimes L$ uniquely determined by $\langle F^\dagger(\phi), v\rangle_L = \langle \phi, F(v)\rangle_L$, $\phi \in W^* \otimes L$, $v \in V$. Clearly, $(DL)^* \otimes L = J^1 L$, the first jet bundle of L. The direct sum $\mathbb{D}L := DL \oplus J^1 L$ is a contact-Courant algebroid (in the sense of Grabowski [9]), and is called an *omni-Lie algebroid* in [5]. We denote by $[\![-, -]\!] : \Gamma(\mathbb{D}L) \times \Gamma(\mathbb{D}L) \to \Gamma(\mathbb{D}L)$ and $\langle\!\langle -, - \rangle\!\rangle : \mathbb{D}L \otimes \mathbb{D}L \to L$, the Dorfman–Jacobi bracket and the L-valued symmetric pairing, respectively. Namely, for all $\Delta, \nabla \in \Gamma(DL)$, $\phi, \psi \in \Gamma(J^1 L)$,

$$\langle\!\langle (\Delta, \phi), (\nabla, \psi) \rangle\!\rangle := \langle \Delta, \psi\rangle_L + \langle \nabla, \phi\rangle_L,$$

and

$$[\![(\Delta, \phi), (\nabla, \psi)]\!] := ([\Delta, \nabla], \mathcal{L}_\Delta \psi - i_\nabla d_{DL}\phi).$$

See e.g. [22] for the main properties of these structures.

Definition 3.1. A *generalized almost contact bundle* is a line bundle $L \to M$ equipped with a *generalized almost contact structure*, i.e. an endomorphism $\mathcal{I} : \mathbb{D}L \to \mathbb{D}L$ such that $\mathcal{I}^2 = -\mathrm{id}$, and $\mathcal{I}^\dagger = -\mathcal{I}$.

Remark 1. Similarly, as for generalized almost complex structures, it is easy to see that, in the case $L = M \times \mathbb{R}$, one recovers [12, Definition 4.1] which is equivalent to Sekiya's generalized f-almost contact structures [18]. In particular, Poon–Wade's generalized almost contact pairs are special cases of Definition 3.1.

Using direct sum decomposition $\mathbb{D}L = DL \oplus J^1L$, one sees that every generalized almost contact structure on L is of the form

$$\mathcal{I} = \begin{pmatrix} \varphi & J^\sharp \\ \omega^\flat & -\varphi^\dagger \end{pmatrix} \tag{1}$$

where

(i) $\varphi : DL \to DL$ is a vector bundle endomorphism,
(ii) $J : \wedge^2 J^1L \to L$ is a 2-form with associated morphism $J^\sharp : J^1L \to DL$, and
(iii) $\omega : \wedge^2 DL \to L$ is a 2-form with associated morphism $\omega^\flat : DL \to J^1L$,

satisfying the relations:

$$\varphi J^\sharp = J^\sharp \varphi^\dagger; \quad \varphi^2 = -\,\mathrm{id} - J^\sharp \omega^\flat; \quad \text{and} \quad \omega^\flat \varphi = \varphi^\dagger \omega^\flat. \tag{2}$$

Conversely, every triple (φ, J, ω) as above determines a generalized almost contact structure via (1). From the third equation in (2), putting $\omega_\varphi(\Delta, \nabla) := \omega(\varphi\Delta, \nabla)$, we get a well defined Atiyah 2-form ω_φ. Following [12] we introduce the:

Definition 3.2. A generalized almost contact structure $\mathcal{I}$ on L is *integrable* if its *Nijenhuis torsion* $\mathcal{N}_\mathcal{I} : \Gamma(\mathbb{D}L) \times \Gamma(\mathbb{D}L) \to \Gamma(\mathbb{D}L)$, defined by $\mathcal{N}_\mathcal{I}(\alpha, \beta) := [\![\mathcal{I}\alpha, \mathcal{I}\beta]\!] - [\![\alpha, \beta]\!] - \mathcal{I}[\![\mathcal{I}\alpha, \beta]\!] - \mathcal{I}[\![\alpha, \mathcal{I}\beta]\!]$, vanishes identically. A *generalized contact structure* is an integrable generalized almost contact structure. A *generalized contact bundle* is a line bundle equipped with a generalized contact structure.

Now, a section $J \in \Gamma(\wedge^2 (J^1L)^* \otimes L)$ defines both a skew-symmetric bracket $\{-,-\}_J$ on $\Gamma(L)$ and a skew-symmetric bracket $[-,-]_J$ on $\Gamma(J^1L)$ via

$$\{\lambda, \mu\}_J := J(j^1\lambda, j^1\mu) \quad \text{and} \quad [\phi, \psi]_J := \mathcal{L}_{J^\sharp\phi}\psi - \mathcal{L}_{J^\sharp\psi}\phi - d_{DL}J(\phi, \psi).$$

It is easy to see that $(L, \{-,-\}_J)$ is a Jacobi bundle (see, e.g. [16]) if and only if $(J^1L, [-,-]_J, \sigma J^\sharp)$ is a Lie algebroid (see, e.g. [7,15]), in this case we say that J is a *Jacobi structure on L*.

Proposition 3.3. *Let $\mathcal{I}$ be a generalized almost contact structure on L. It is integrable if and only if, for all $\sigma, \tau \in \Gamma(J^1L)$, $\Delta, \nabla, \Box \in \Gamma(DL)$,*

$$J^\sharp[\phi, \psi]_J = [J^\sharp\phi, J^\sharp\psi]; \tag{3}$$

$$\varphi^\dagger[\phi, \psi]_J = \mathcal{L}_{J^\sharp\phi}\varphi^\dagger\psi - \mathcal{L}_{J^\sharp\psi}\varphi^\dagger\phi - d_{DL}J(\varphi\phi, \psi); \tag{4}$$

$$\mathcal{N}_\varphi(\Delta, \nabla) = J^\sharp(i_\Delta i_\nabla d_{DL}\omega); \tag{5}$$

and

$$(d_{DL}\omega_\varphi)(\Delta, \nabla, \Box) = (d_{DL}\omega)(\varphi\Delta, \nabla, \Box) + (d_{DL}\omega)(\Delta, \varphi\nabla, \Box) + (d_{DL}\omega)(\Delta, \nabla, \varphi\Box), \tag{6}$$

where $\mathcal{N}_\varphi(\Delta, \nabla) := [\varphi\Delta, \varphi\nabla] + \varphi^2[\Delta, \nabla] - \varphi[\varphi\Delta, \nabla] - \varphi[\Delta, \varphi\nabla]$.

Equations (2) and (3)–(6) should be seen as *structure equations of a generalized contact structure*.

Example 1. Let $\mathcal{I}$ be a generalized almost contact structure on $L \to M$ given by (1). As for generalized complex structures there are two extreme cases. The first one is when $\varphi = 0$, hence $J^\sharp = (\omega^\flat)^{-1}$, and $\mathcal{I}$ is completely determined by ω which is a non-degenerate Atiyah 2-form. Now, $\mathcal{I}$ is integrable if and only if $d_{DL}\omega = 0$, hence ω corresponds to a contact structure H on M such that $TM/H = L$. The second extreme case is when $J = \omega = 0$, hence $\varphi^2 = -\,\mathrm{id}$, i.e. φ is an almost complex structure on the Atiyah algebroid DL, and $\mathcal{I}$ is integrable if and only if φ is a complex structure [2].

Equation (3) says that J is a Jacobi structure. So every generalized contact bundle has an underlying Jacobi structure. Equation (4) describes a compatibility condition between J and φ. Equation (5) measures the non-integrability of φ while (6) is a compatibility condition between φ and ω.

It is useful to characterize those generalized contact structures such that J is non-degenerate. In this case there is a unique non-degenerate Atiyah 2-form ω_J, also denoted J^{-1}, such that $J^\sharp\omega_J^\flat = \mathrm{id}$ and (3) says that ω_J is d_{DL}-closed. Hence it comes from a contact structure $H_J \subset TM$ such that $TM/H_J = L$. Following Crainic [6] we introduce the following notion:

Definition 3.4. A *contact-Hitchin pair* on a line bundle $L \to M$ is a pair (H, Φ) consisting of a contact structure $H \subset TM$ with $TM/H = L$, and an endomorphism $\Phi : DL \to DL$ such that (i) $\Omega^\flat\Phi = \Phi^\dagger\Omega^\flat$ (so that the Atiyah 2-form Ω_Φ is well-defined), and (ii) $d_{DL}\Omega_\Phi = 0$, where Ω is the Atiyah 2-form corresponding to H, i.e. $\Omega := d_{DL}\sigma^*\theta$, and $\theta : TM \to L$ is the structure form of H, i.e. $H = \ker\theta$.

Proposition 3.5. *There is a one-to-one correspondence between generalized contact structures on L given by (1), with J non-degenerate, and contact Hitchin pairs (H, Φ) on L. In this correspondence H is the contact structure corresponding to $\omega_J = J^{-1}$, and moreover:*

$$\Phi = \varphi \quad \text{and} \quad \omega = -(\omega_J + \varphi^*\omega_J), \quad \text{where} \quad (\varphi^*\omega_J)(\Delta, \nabla) := \omega_J(\varphi\Delta, \varphi\nabla).$$

The proof of Proposition 3.5 is similar to that of Proposition 2.6 in [6].

4. Multiplicative Atiyah forms on Lie groupoids and generalized contact structures

As we have seen above, every generalized contact bundle $(L, \mathcal{I})$ has an underlying Jacobi structure J. Jacobi structures are the infinitesimal counterparts of multiplicative contact structures on Lie groupoids [7] (see also [13] for the case $L = M \times \mathbb{R}$). So, it is natural to ask: *are the remaining components φ, ω of $\mathcal{I}$ also infinitesimal counterparts of suitable (multiplicative) structures on $\mathcal{G}$?* Theorem 4.6 below answers this question (see [6, Theorems 3.2, 3.3, 3.4] for the generalized complex case). In order to state it, we need to introduce a novel notion of *multiplicative Atiyah form*.

Multiplicative forms and their infinitesimal counterparts are extensively studied in [3]. Vector-bundle valued differential forms and their infinitesimal counterparts, Spencer operators, are studied in [8]. In what follows, we outline a similar theory for Atiyah forms.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, we will denote the source by s, the target by t and the unit by u. Moreover, we identify M with its image under u. Denote by $\mathcal{G}_2 = \{(g_1, g_2) \in \mathcal{G} \times \mathcal{G} : s(g_1) = t(g_2)\}$ the manifold of composable arrows and let $m : \mathcal{G}_2 \to \mathcal{G}$, $(g_1, g_2) \mapsto g_1 g_2$ be the multiplication. We denote by $\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{G}_2 \to \mathcal{G}$ the projections onto the first and second factor respectively.

Recall that the Lie algebroid A of $\mathcal{G}$ consists of tangent vectors to the source fibers at points of M. Every section α of A corresponds to a unique right invariant, s-vertical vector field α^r on $\mathcal{G}$ such that $\alpha = \alpha^r|_M$. Now let $E \to M$ be a vector bundle carrying a representation of $\mathcal{G}$. Thus, there is a flat A-connection ∇ in E. As shown in [22, Proposition 10.1], there is a canonical flat $\ker ds$-connection $\nabla^{\mathcal{G}} : \ker ds \to D(t^*E)$ in the pull-back bundle t^*E such that $\nabla_\alpha = t_*(\nabla^{\mathcal{G}}_{\alpha^r}|_M)$ for all $\alpha \in \Gamma(A)$. Additionally, there is a natural vector bundle isomorphism (covering the identity) $\mathrm{i} : (t \circ \mathrm{pr}_1)^*E \longrightarrow (t \circ \mathrm{pr}_2)^*E$ defined as follows. For $((g_1, g_2), e) \in (t \circ \mathrm{pr}_1)^*E$, $e \in E_{t(g_1)}$ put $\mathrm{i}((g_1, g_2), e) := ((g_1, g_2), g_1^{-1} \cdot e) \in (t \circ \mathrm{pr}_2)^*E$. In the following $E = L$ is a line bundle. In particular, $(t \circ \mathrm{pr}_2)^*L$-valued Atiyah forms can be pulled-back to $(t \circ \mathrm{pr}_1)^*L$-valued Atiyah forms along i.

Definition 4.1. An Atiyah form $\omega \in \Omega^\bullet_{t^*L}$ is *multiplicative* if $m^*\omega = \mathrm{pr}_1^*\,\omega + \mathrm{i}^*\,\mathrm{pr}_2^*\,\omega$.

We also need the following:

Definition 4.2. An endomorphism $\Phi : D(t^*L) \to D(t^*L)$ is *multiplicative* if, for every $\square \in D(m^*t^*L)$, there is a, necessarily unique, $\Phi D \in D(m^*t^*L)$ such that (1) $\mathrm{pr}_{1*}\,\Phi\square = \Phi\,\mathrm{pr}_{1*}\,\square$, (2) $\mathrm{pr}_{2*}\,\mathrm{i}_*\Phi\square = \Phi\,\mathrm{pr}_{2*}\,\mathrm{i}_*\square$ and (3) $m_*\Phi\square = \Phi m_*\square$.

The following definition and Theorem 4.6 provide the infinitesimal counterpart of multiplicative Atiyah forms. Let $L \to M$ be a line bundle carrying a representation of a Lie algebroid $A \to M$.

Definition 4.3. An *L-valued infinitesimal multiplicative (IM) Atiyah k-form on A* is a pair $(\boldsymbol{\mathcal{D}}, \boldsymbol{l})$, where $\boldsymbol{\mathcal{D}} : \Gamma(A) \longrightarrow \Omega^k_L$ is a first order differential operator, and $\boldsymbol{l} : A \longrightarrow \wedge^{k-1}(DL)^* \otimes L$ is a vector bundle morphism such that, for all $\alpha, \beta \in \Gamma(A)$, and $f \in C^\infty(M)$,

$$\boldsymbol{\mathcal{D}}(f\alpha) = f\boldsymbol{\mathcal{D}}(\alpha) + d_{DL} f \wedge \boldsymbol{l}(\alpha),$$

and

$$\begin{aligned} \mathcal{L}_{\nabla_\alpha}\boldsymbol{\mathcal{D}}(\beta) - \mathcal{L}_{\nabla_\beta}\boldsymbol{\mathcal{D}}(\alpha) &= \boldsymbol{\mathcal{D}}([\alpha, \beta]); \\ \mathcal{L}_{\nabla_\alpha}\boldsymbol{l}(\beta) - i_{\nabla_\beta}\boldsymbol{\mathcal{D}}(\alpha) &= \boldsymbol{l}([\alpha, \beta]); \\ i_{\nabla_\alpha}\boldsymbol{l}(\beta) + i_{\nabla_\beta}\boldsymbol{l}(\alpha) &= 0. \end{aligned}$$

Now on, $L \to M$ is a line bundle carrying a representation of a source simply connected Lie groupoid $\mathcal{G} \rightrightarrows M$, and A is the Lie algebroid of $\mathcal{G}$.

Theorem 4.4. *There is a one-to-one correspondence between t^*L-valued multiplicative Atiyah k-forms ω and L-valued IM Atiyah k-forms $(\boldsymbol{\mathcal{D}}, \boldsymbol{l})$ on A. In this correspondence*

$$\boldsymbol{\mathcal{D}}(\alpha) = u^*(\mathcal{L}_{\nabla^{\mathcal{G}}_{\alpha^r}}\omega) \quad \text{and} \quad \boldsymbol{l}(\alpha) = u^*(i_{\nabla^{\mathcal{G}}_{\alpha^r}}\omega).$$

Proof. There is a direct sum decomposition $\Omega^{\bullet}_{t^*L} = \Omega^{\bullet}(\mathcal{G}, t^*L) \oplus \Omega^{\bullet}(\mathcal{G}, t^*L)$ [1] given by $\omega \equiv (\omega_0, \omega_1)$, with $\omega = \sigma^*\omega_0 + d_{DL}\sigma^*\omega_1$, and ω is multiplicative if and only if (ω_0, ω_1) are so. Using [8, Theorem 1], we see that (ω_0, ω_1) correspond to Spencer operators on A [8, Definition 2.6]. Finally check that, similarly as for Atiyah forms, IM Atiyah forms decompose canonically into a direct sum of Spencer operators.

Remark 2. Let $H \subset T\mathcal{G}$ be a multiplicative contact structure on $\mathcal{G}$, with $T\mathcal{G}/H = t^*L$ and let Ω be the corresponding Atiyah 2-form. When specialized to Ω, Theorem 4.4 gives an isomorphism $\boldsymbol{l} : A \to J^1L$ of A with the Lie algebroid $(J^1L, [-,-]_J, \sigma J^\sharp)$ corresponding to a unique Jacobi structure J on L.

Definition 4.5. A *contact-Hitchin groupoid* is a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with

(1) a line bundle $L \to M$ carrying a representation of $\mathcal{G}$,
(2) a multiplicative contact-Hitchin pair (H, Φ) on t^*L, i.e. both H and Φ are multiplicative, and
(3) an L-valued Atiyah 2-form ω on M such that $\Omega + \Phi^*\Omega = s^*\omega - t^*\omega$,

where Ω is the Atiyah 2-form corresponding to H.

Theorem 4.6. *There is a one-to-one correspondence between contact-Hitchin groupoid structures (H, Φ, Ω) on $\mathcal{G}$ and triples (J, φ, ω) satisfying Equations (3)–(5), and the first two equations in (2). In this correspondence, J is the Jacobi structure corresponding to H (Remark 2), and $\varphi : DL \to DL$ is the (well-defined) restriction of Φ to DL.*

Theorem 4.6 can be proved using arguments similar to those in Theorems 3.3 and 3.4 in [6]. Alternatively, one could use a conceptual approach similar to that of [20], exploiting the notion of Jacobi quasi-Nijenhuis structure [19,4]. Finally, we observe that the last equation of (2) and Equation (6) have neither a global interpretation (Lie groupoid) nor an infinitesimal one (Lie algebroid).

References

[1] M. Aldi, D. Grandini, Generalized contact geometry and T-duality, J. Geom. Phys. 92 (2015) 78–93, arXiv:1312.7471.
[2] U. Bruzzo, V.N. Rubtsov, Cohomology of skew-holomorphic Lie algebroids, Theor. Math. Phys. 165 (2010) 1598–1609, arXiv:1003.1823.
[3] H. Bursztyn, A. Cabrera, Multiplicative forms at the infinitesimal level, Math. Ann. 353 (2012) 663–705, arXiv:1001.0534.
[4] R. Caseiro, A. De Nicola, J.M. Nunes da Costa, Jacobi quasi-Nijenhuis algebroids and Courant–Jacobi algebroid morphisms, J. Geom. Phys. 60 (2010) 951–961, arXiv:1003.0373.
[5] Z. Chen, Z.-J. Liu, Omni-Lie algebroids, J. Geom. Phys. 60 (2010) 799–808, arXiv:0710.1923.
[6] M. Crainic, Generalized complex structures and Lie brackets, Bull. Braz. Math. Soc. 42 (2011) 559–578, arXiv:math/0412097.
[7] M. Crainic, M.A. Salazar, Jacobi structures and Spencer operators, J. Math. Pures Appl. 103 (2015) 504–521, arXiv:1309.6156.
[8] M. Crainic, M.A. Salazar, I. Struchiner, Multiplicative forms and Spencer operators, Math. Z. 279 (2015) 939–979, arXiv:1309.6156.
[9] J. Grabowski, Graded contact manifolds and contact Courant algebroids, J. Geom. Phys. 68 (2013) 27–58, arXiv:1112.0759.
[10] M. Gualtieri, Generalized complex geometry, Ann. Math. 174 (2011) 75–123, arXiv:math/0703298.
[11] N. Hitchin, Generalized Calabi–Yau manifolds, Quart. J. Math. 54 (2003) 281–308, arXiv:math/0209099.
[12] D. Iglesias-Ponte, A. Wade, Contact manifolds and generalized complex structures, J. Geom. Phys. 53 (2005) 249–258, arXiv:math/0404519.
[13] Y. Kerbrat, Z. Souici-Benhammadi, Variétés de Jacobi et groupoïdes de contact, C. R. Acad. Sci. Paris, Ser. I 317 (1993) 81–86.
[14] Y. Kosmann-Schwarzbach, K.C.H. Mackenzie, Differential operators and actions of Lie algebroids, in: Quantization, Poisson Brackets and Beyond, Manchester, 2001, in: Contemp. Math., vol. 315, Amer. Math. Soc., Providence, RI, USA, 2002, pp. 213–233, arXiv:math.DG/0209337.
[15] H.V. Lê, Y.-G. Oh, A.G. Tortorella, L. Vitagliano, Deformations of coisotropic submanifolds in abstract Jacobi manifolds, arXiv:1410.8446.
[16] C.M. Marle, On Jacobi manifolds and Jacobi bundles, in: Symplectic Geometry, Groupoids, and Integrable Systems, Berkeley, CA, 1989, in: Math. Sci. Res. Inst. Publ., vol. 20, Springer, New York, 1991, pp. 227–246.
[17] Y.S. Poon, A. Wade, Generalized contact structures, J. Lond. Math. Soc. 83 (2) (2011) 333–352, arXiv:0912.5314.
[18] K. Sekiya, Generalized almost contact structures and generalized Sasakian structures, Osaka J. Math. 52 (2015) 43–59, arXiv:1212.6064.
[19] Y. Sheng, Jacobi quasi-Nijenhuis algebroids, Rep. Math. Phys. 65 (2010) 271–287, arXiv:0903.4332.
[20] M. Stiénon, P. Xu, Poisson quasi-Nijenhuis manifolds, Commun. Contemp. Math. 270 (2007) 709–725, arXiv:math/0602288.
[21] I. Vaisman, Generalized *CRF*-structures, Geom. Dedic. 133 (2008) 129–154, arXiv:0705.3934v2.
[22] L. Vitagliano, Dirac–Jacobi bundles, arXiv:1502.05420.