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Semi-regular varieties and variational Hodge conjecture

Variétés semi-régulières et conjecture de Hodge variationnelle

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ABSTRACT

Following [1,2], we know that semi-regular sub-varieties satisfy the variational Hodge conjecture, i.e., given a family of smooth projective varieties $\pi : \mathcal{X} \to B$, a special fiber $\mathcal{X}_o$ and a semi-regular subvariety $Z \subset \mathcal{X}_o$, the cohomology class corresponding to Z remains a Hodge class (as $\mathcal{X}_o$ deforms along B) if and only if Z remains an algebraic cycle. In this article, we investigate examples of such sub-varieties. In particular, we prove that any smooth projective variety Z of dimension n is a semi-regular sub-variety of a smooth projective hypersurface in $\mathbb{P}^{2n+1}$ of large enough degree.

RÉSUMÉ

D'après [1,2] nous savons que les sous-variétés semi-régulières satisfont la conjecture de Hodge variationnelle, c'est-à-dire qu'étant données une famille de variétés projectives lisses $\pi : \mathcal{X} \to B$, une fibre spéciale $\mathcal{X}_o$ et une sous-variété semi-régulière $Z \subset \mathcal{X}_o$, la classe de cohomologie correspondant à Z reste une classe de Hodge si et seulement si Z reste un cycle algébrique (lorsque $\mathcal{X}_o$ se déforme le long de B). Nous étudions ici des exemples de telles sous-variétés. En particulier, nous montrons que toute variété projective lisse Z de dimension n est une sous-variété semi-régulière d'une hypersurface projective lisse de $\mathbb{P}^{2n+1}$ de degré suffisamment grand.

1. Introduction

The aim of this article is to study examples of semi-regular varieties. The semi-regularity for a curve on a surface was first introduced by [5]. This was later generalized to arbitrary divisors on a complex manifold by Kodaira–Spencer in [4]. In [1],

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Bloch extended the notion to cycles corresponding to local complete intersection subschemes. This was further generalized by Buchweitz and Flenner in [2].

One of the motivations for the study of semi-regular varieties comes from the variational Hodge conjecture, namely these varieties satisfy the variational Hodge conjecture. In particular, Bloch in [1] and Buchweitz and Flenner in [2] noticed that, for a smooth projective variety X and a semi-regular local complete intersection subscheme Z in X, any infinitesimal deformation of X lifts the cohomology class of Z (which is a Hodge class) to a Hodge class if and only if Z lifts to a local complete intersection subscheme (in the deformed scheme).

In the case of a smooth hypersurface X in $\mathbb{P}^3$, an effective divisor C in X is said to be *semi-regular* if $h^1(\mathcal{O}_X(C)) = 0$. If C is smooth and $\deg(X) > \deg(C) + 4$, then Serre duality implies that $h^1(\mathcal{O}_X(C)) = h^1(\mathcal{O}_X(-C)(d-4))$, which is equal to zero because the Castelnuovo–Mumford regularity of C is at most $\deg(C)$. But the description of the semi-regularity for subschemes that are not divisors is more complicated, as we see below in Section 2. The main result of this article generalizes the above case of divisors to higher codimension subvarieties (see Section 3). In particular, we prove Theorem 1.1.

Theorem 1.1. *Let Z be a smooth subscheme in $\mathbb{P}^{2n+1}$ of codimension $n+1$. Then for $d \gg 0$, there exists a smooth degree d hypersurface in $\mathbb{P}^{2n+1}$ containing Z such that Z is semi-regular in X.*

We finally observe in Remark 3.5 that for such a choice of Z and X, the cohomology class of Z in $H^{n,n}(X, \mathbb{Z})$ satisfies the variational Hodge conjecture for a family of degree d hypersurfaces in $\mathbb{P}^{2n+1}$ with a special fiber X.

2. Bloch's semi-regularity map

2.1. In [1], Bloch generalizes the above definition of semi-regularity for divisors to any local complete intersection subscheme in a smooth projective variety over an algebraically closed field. We briefly recall the definition. Let X be a smooth projective variety of dimension n and Z be a local complete intersection subscheme in X of codimension q. Consider the composition morphism

$$\Omega_X^{n-q+1} \times \bigwedge^{q-1} \mathcal{N}_{Z|X}^\vee \xrightarrow{1\times\bigwedge^{q-1}\bar{d}} \Omega_X^{n-q+1} \times \Omega_X^{q-1} \otimes \mathcal{O}_Z \xrightarrow{\bigwedge} K_X \otimes \mathcal{O}_Z,$$

where

$$\bar{d} : \mathcal{N}_{Z|X}^\vee \cong \mathcal{I}_{Z|X}/\mathcal{I}_{Z|X}^2 \to \Omega_X^1 \otimes \mathcal{O}_Z$$

is the map induced by the differential $d : \mathcal{I}_{Z|X} \to \Omega_X^1$, with $\mathcal{I}_{Z|X}$ denoting the ideal sheaf of Z in X. By adjunction, this induces a map,

$$\Omega_X^{n-q+1} \to \bigwedge^{q-1} \mathcal{N}_{Z|X} \otimes K_X \cong \mathcal{N}_{Z|X}^\vee \otimes K_Z^0,$$

where $K_Z^0 := \bigwedge^q \mathcal{N}_{Z|X} \otimes K_X$ is the *dualizing sheaf*. Dualizing the induced map in cohomology,

$$H^{n-q-1}(X, \Omega^{n-q+1}) \to H^{n-q-1}(Z, \mathcal{N}_{Z|X}^\vee \otimes K_Z^0), \text{ gives us } \pi : H^1(\mathcal{N}_{Z|X}) \to H^{q+1}(X, \Omega_X^{q-1}).$$

Definition 2.2. The map π is called the *semi-regularity map*, and if it is injective we say that Z is *semi-regular*.

3. Proof of Theorem 1.1 and an application

3.1. Before we come to the final result of this article we recall a result by Kleiman and Altman, which tells us given a smooth subscheme in $\mathbb{P}^{2n+1}$ of codimension $n+1$, there exists a *smooth* hypersurface in $\mathbb{P}^{2n+1}$ containing it.

Notation 3.2. Let Z be a projective subscheme in $\mathbb{P}^{2n+1}$. Denote by

$$Z_e := \{z \in Z \mid \dim \Omega_{Z,z}^1 = e\}.$$

Theorem 3.3. *(See [3, Theorem 7].) If for any $e > 0$ such that $Z_e \neq \emptyset$ we have that $\dim Z_e + e$ is less than $2n+1$ then there exists a smooth hypersurface in $\mathbb{P}^{2n+1}$ containing Z. Moreover, if Z is $d-1$-regular (in the sense of Castelnuovo–Mumford), then there exists a smooth degree d such hypersurface containing Z.*

We need the following proposition.

Proposition 3.4. *Let Z be a smooth subscheme in $\mathbb{P}^{2n+1}$ of codimension $n+1$ and X be a smooth degree d hypersurface in $\mathbb{P}^{2n+1}$ containing Z for some $d \gg 0$. Then, for any integers $2 \le i < n$, $h^n\left(\bigwedge^{i-1} \mathcal{T}_Z \otimes \bigwedge^{n-i} \mathcal{N}_{Z|X}(d-4)\right) = 0$.*

Proof. Since X is a hypersurface in $\mathbb{P}^{2n+1}$, $\mathcal{N}_{X|\mathbb{P}^{2n+1}}$ is isomorphic to $\mathcal{O}_X(d)$. Under this identification, we get the following normal short exact sequence,

$$0 \to \mathcal{N}_{Z|X} \to \mathcal{N}_{Z|\mathbb{P}^{2n+1}} \to \mathcal{O}_Z(d) \to 0.$$

This gives rise to the following short exact sequence for $0 \le i \le n$:

$$0 \to \bigwedge^{n-i} N_{Z|X} \to \bigwedge^{n-i} \mathcal{N}_{Z|\mathbb{P}^{2n+1}} \to \left(\bigwedge^{n-i-1} \mathcal{N}_{Z|X}\right) \otimes \mathcal{O}_Z(d) \to 0.$$

Denote by $\mathcal{F}_{j,k} := \bigwedge^j \mathcal{T}_Z \otimes \mathcal{O}_X(k)$ for some $j, k \in \mathbb{Z}_{\ge 0}$. Since Z and X are smooth, $\mathcal{F}_{j,k}$ is $\mathcal{O}_Z$-locally free and hence $\mathcal{O}_Z$-flat. Tensoring the previous short exact sequence by $\mathcal{F}_{j,k}$ then gives us the following short exact sequence,

$$0 \to \mathcal{F}_{j,k} \otimes \bigwedge^{n-i} \mathcal{N}_{Z|X} \to \mathcal{F}_{j,k} \otimes \bigwedge^{n-i} \mathcal{N}_{Z|\mathbb{P}^{2n+1}} \to \mathcal{F}_{j,k} \otimes \bigwedge^{n-i-1} \mathcal{N}_{Z|X}(d) \to 0.$$

By Serre's vanishing theorem, for $d \gg 0$, $l > 0$ and $m \ge 1$, $H^m\left(\mathcal{F}_{j,ld-4} \otimes \bigwedge^{n-i} \mathcal{N}_{Z|\mathbb{P}^{2n+1}}\right) = 0$, hence

$$H^m\left(\mathcal{F}_{j,ld-4} \otimes \bigwedge^{n-i-1} \mathcal{N}_{Z|X}(d)\right) \cong H^{m+1}\left(\mathcal{F}_{j,ld-4} \otimes \bigwedge^{n-i} \mathcal{N}_{Z|X}\right). \tag{1}$$

Using Serre's vanishing theorem again for $d \gg 0$ and $i \ge 1$, $h^i\left(\bigwedge^{i-1} \mathcal{T}_Z((n-i+1)d-4)\right) = 0$. Hence, using the isomorphism (1) recursively, we get for $j = i-1$,

$$h^n\left(\bigwedge^{i-1} \mathcal{T}_Z \otimes \bigwedge^{n-i} \mathcal{N}_{Z|X}(d-4)\right) = h^{n-1}\left(\bigwedge^{i-1} \mathcal{T}_Z \otimes \bigwedge^{n-i-1} \mathcal{N}_{Z|X}(2d-4)\right) = h^i\left(\bigwedge^{i-1} \mathcal{T}_Z((n-i+1)d-4)\right) = 0.$$

This proves the proposition. □

Proof of Theorem 1.1. The existence of a smooth hypersurface in $\mathbb{P}^{2n+1}$ containing Z for $d \gg 0$ follows from Theorem 3.3. It suffices to prove that there exists a hypersurface X in $\mathbb{P}^{2n+1}$ of degree $d \gg 0$ containing Z such that the morphism from $H^{n-1}(\Omega_X^{n+1} \otimes \mathcal{O}_Z)$ to $H^{n-1}(\mathcal{N}_{Z|X}^\vee \otimes \bigwedge^n \mathcal{N}_{Z|X} \otimes K_X)$, which is the dual to the semi-regularity map π (see 2.1), is surjective.

Consider the short exact sequence,

$$0 \to \mathcal{T}_Z \to \mathcal{T}_X \otimes \mathcal{O}_Z \to \mathcal{N}_{Z|X} \to 0.$$

Consider the associated filtration,

$$0 = F^n \subset F^{n-1} \subset \ldots \subset F^0 = \bigwedge^{n-1}(\mathcal{T}_X \otimes \mathcal{O}_Z) \text{ satisfying } F^p/F^{p+1} \cong \bigwedge^p \mathcal{T}_Z \otimes \bigwedge^{n-1-p} \mathcal{N}_{Z|X}$$

for all p. Taking $p = 0$, we get the following short exact sequence

$$0 \to F^1 \to \bigwedge^{n-1}(\mathcal{T}_X \otimes \mathcal{O}_Z) \to \bigwedge^{n-1} \mathcal{N}_{Z|X} \to 0.$$

Tensoring this by K_X and looking at the associated long exact sequence, we get

$$\ldots \to H^{n-1}(\Omega_X^{n+1} \otimes \mathcal{O}_Z) \to H^{n-1}(\mathcal{N}_{Z|X}^\vee \otimes \bigwedge^n \mathcal{N}_{Z|X} \otimes K_X) \to H^n(F^1(d-4)) \to \ldots$$

It therefore suffices to prove that $h^n(F^1(d-4)) = 0$.

We claim that it is sufficient to prove $h^n(F^{n-1}(d-4)) = 0$. Indeed, suppose $h^n(F^{n-1}(d-4)) = 0$. By Proposition 3.4, for any integer $2 \le i \le n-1$, we have

$$h^n\left(\bigwedge^{i-1} \mathcal{T}_Z \otimes \bigwedge^{n-i} \mathcal{N}_{Z|X}(d-4)\right) = 0.$$

Consider the following short exact sequence, where $2 \le p \le n-1$,

$$0 \to F^p \to F^{p-1} \to \bigwedge^{p-1} \mathcal{T}_Z \otimes \bigwedge^{n-p} \mathcal{N}_{Z|X} \to 0. \tag{2}$$

Tensoring (2) by $K_X \cong \mathcal{O}_X(d-4)$ and considering the corresponding long exact sequence, we can conclude $h^n(F^{n-2}(d-4)) = 0$ (substitute $p = n-1$). Recursively substituting $p = n-2, n-3, \ldots, 2$ in (2), we observe that $h^n(F^i(d-4)) = 0$ for $i = 1, \ldots, n-2$. In particular, $h^n(F^1(d-4)) = 0$. Hence, it suffices to prove $h^n(F^{n-1}(d-4)) = 0$.

Note that, $F^{n-1} \cong \bigwedge^{n-1} \mathcal{T}_Z$ does not depend on the choice of X, hence independent of d. Therefore, by Serre's vanishing theorem, $h^n(F^{n-1}(d-4)) = 0$ for $d \gg 0$. This completes the proof of the theorem. $\square$

Remark 3.5. Notations as in Theorem 1.1. We now note that the theorem implies a very special case of the variational Hodge conjecture. Indeed, consider a family $\pi : \mathcal{X} \to S$ of smooth degree d hypersurfaces in $\mathbb{P}^{2n+1}$ with X as a special fiber. Denote by γ the cohomology class of Z in X. Then, using [1, Theorem 7.1] notice that γ remains a Hodge class if and only if Z remains an algebraic variety as X deforms along S.

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