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Generalized cluster structure on the Drinfeld double of GL_n

Structures d'algébres amassées généralisées sur le double de Drinfeld du group GL_n

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ABSTRACT

We construct a generalized cluster structure compatible with the Poisson bracket on the Drinfeld double of the standard Poisson–Lie group GL_n and derive from it a generalized cluster structure in GL_n compatible with the push-forward of the dual Poisson–Lie bracket.

RÉSUMÉ

On construit des structures d'algèbres amassées généralisées compatibles avec le crochet de Poisson sur le double de Drinfeld du group GL_n muni de sa structure de Poisson–Lie usuelle. On en déduit une structure d'algèbre amassée généralisée sur GL_n compatible avec l'image directe du crochet de Poisson dual.

1. Introduction

The connection between cluster algebras and Poisson structures is documented in [6]. Among the most important examples in which this connection has been utilized are coordinate rings of double Bruhat cells in semisimple Lie groups equipped with (the restriction of) the standard Poisson–Lie structure. In [6], we applied our technique of constructing a cluster structure compatible with a given Poisson structure in this situation and recovered the cluster structure built in [2]. The standard Poisson–Lie structure is a particular case of Poisson–Lie structures corresponding to quasi-triangular Lie bialgebras. Such structures are associated with solutions to the classical Yang–Baxter equation. Their complete classification was obtained by Belavin and Drinfeld in [1] in terms of certain combinatorial data defined in terms of the corresponding root system. In [7] we conjectured that any such solution gives rise to a compatible cluster structure on the Lie group and provided several examples supporting this conjecture. Recently [8,9], we constructed the cluster structure corresponding to the Cremmer–Gervais Poisson structure in GL_n for any n. As we established in [9], the construction of cluster structures

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on a simple Poisson–Lie group $\mathcal{G}$ relies on properties of the Drinfeld double $D(\mathcal{G})$. Moreover, in the Cremmer–Gervais case generalized determinantal identities on which cluster transformations are modeled can be extended to identities valid in the double. It is not too far-fetched then to suspect that there exists a cluster structure on $D(\mathcal{G})$ compatible with the Poisson–Lie bracket induced by the Poisson–Lie bracket on $\mathcal{G}$. However, an interesting phenomenon was observed even in the first nontrivial example of $D(GL_2)$: although we were able to construct a log-canonical regular coordinate chart in terms of which all standard coordinate functions are expressed as (subtraction free) Laurent polynomials, it is not possible to define cluster transformations in such a way that all cluster variables that one expects to be mutable transform into regular functions. This problem is resolved, however, if one is allowed to use *generalized cluster transformations* previously considered in [5,6] and, more recently, axiomatized in [4].

In this note, we describe such a generalized cluster structure on the Drinfeld double in the case of the standard Poisson–Lie group GL_n. Using this structure, one can recover the standard cluster structure on GL_n and introduce a generalized cluster structure on GL_n compatible with the Poisson bracket dual to the standard Poisson–Lie bracket. Note that the log-canonical basis suggested in [3] is different from the one constructed here and does not lead to a regular cluster structure.

2. Generalized cluster structures of geometric type and compatible Poisson brackets

Let $\widetilde{B} = (b_{ij})$ be an $n \times (n+m)$ integer matrix whose principal part B is skew-symmetrizable (recall that the principal part of a rectangular matrix is its maximal leading square submatrix). Let $\mathcal{F}$ be the field of rational functions in $n+m$ independent variables with rational coefficients. There are m distinguished variables; we denote them $x_{n+1}, \dots, x_{n+m}$ and call *stable*. Finally, we define $2n$ *stable τ-monomials* $v_{i;>}$ and $v_{i;<}$, $1 \le i \le n$, via $v_{i;>} = \prod\{x_j^{b_{ij}} : n+1 \le j \le n+m, b_{ij} > 0\}$, $v_{i;<} = \prod\{x_j^{-b_{ij}} : n+1 \le j \le n+m, b_{ij} < 0\}$; here, as usual, the product over the empty set is assumed to be equal to 1.

A *seed* (of *geometric type*) in $\mathcal{F}$ is a triple $\Sigma = (\mathbf{x}, \widetilde{B}, \mathcal{P})$, where $\mathbf{x} = (x_1, \dots, x_n)$ is a transcendence basis of $\mathcal{F}$ over the field of fractions of $\bar{\mathbb{A}} = \mathbb{Z}[x_{n+1}^{\pm 1}, \dots, x_{n+m}^{\pm 1}]$ (here we write $x^{\pm 1}$ instead of x, x^{-1}), and $\mathcal{P}$ is a set of n *strings*. The ith string is a collection of monomials $p_{ir} \in \bar{\mathbb{A}}$, $0 \le r \le d_i$, such that d_i is a factor of $\gcd\{b_{ij} : 1 \le j \le n\}$, $p_{i0} = p_{id_i} = 1$, and $\hat{p}_{ir} = \left(p_{ir} v_{i;>}^r v_{i;<}^{d_i - r}\right)^{1/d_i}$ belong to the polynomial ring $\mathbb{A} = \mathbb{Z}[x_{n+1}, \dots, x_{n+m}]$, $0 \le r \le d_i$; it is called *trivial* if $d_i = 1$, and hence both elements of the string are equal to one.

Matrices B and $\widetilde{B}$ are called the *exchange matrix* and the *extended exchange matrix*, respectively. The n-tuple $\mathbf{x}$ is called a *cluster*, and its elements $x_1, \dots, x_n$ are called *cluster variables*. The monomials p_{ir} are called *coefficients*. We say that $\widetilde{\mathbf{x}} = (x_1, \dots, x_{n+m})$ is an *extended cluster*, and $\widetilde{\Sigma} = (\widetilde{\mathbf{x}}, \widetilde{B}, \mathcal{P})$ is an *extended seed*.

In certain cases it is convenient to represent the data $(\widetilde{B}, d_1, \dots, d_n)$ by a quiver. Assume that the matrix obtained from $\widetilde{B}$ by replacing each b_{ij} by b_{ij}/d_i for $1 \le j \le n$ and retaining it for $n+1 \le j \le n+m$ has a skew-symmetric principal part. We say that the corresponding quiver Q represents $(\widetilde{B}, d_1, \dots, d_n)$ and write $\Sigma = (\mathbf{x}, Q, \mathcal{P})$. A vertex with $d_i \ne 1$ is called *special*, and d_i is said to be its *order*. A stable vertex j such that $b_{ij} = 0$, $1 \le i \le n$, is called *isolated*.

Given a seed as above, the *adjacent cluster* in direction k, $1 \le k \le n$, is defined by $\mathbf{x}' = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\}$, where the new cluster variable x'_k is given by the *generalized exchange relation*

$$x_k x'_k = \sum_{j=0}^{d_k} \hat{p}_{kj} u_{k;>}^j u_{k;<}^{d_k - j}$$

with *cluster τ-monomials* $u_{k;>}$ and $u_{k;<}$ defined by $u_{k;>} = \prod\{x_i^{b_{ki}/d_k} : 1 \le i \le n, b_{ki} > 0\}$, $u_{k;<} = \prod\{x_i^{-b_{ki}/d_k} : 1 \le i \le n, b_{ki} < 0\}$.

We say that $\widetilde{B}'$ is obtained from $\widetilde{B}$ by a *matrix mutation* in direction k if $b'_{ij} = -b_{ij}$ for $i = k$ or $j = k$ and $b'_{ij} = b_{ij} + (|b_{ik}| b_{kj} + b_{ik} |b_{kj}|)/2$ otherwise. Note that $\gcd\{b_{ij} : 1 \le j \le n\} = \gcd\{b'_{ij} : 1 \le j \le n\}$, and for its arbitrary factor d, $b_{ij} = b'_{ij} \bmod d$ for $n+1 \le j \le n+m$.

The *coefficient mutation* in direction k is given by $p'_{ir} = p_{i, d_i - r}$ for $i = k$ and $p'_{ir} = p_{ir}$ otherwise.

Given a seed $\Sigma = (\mathbf{x}, \widetilde{B}, \mathcal{P})$, we say that a seed $\Sigma' = (\mathbf{x}', \widetilde{B}', \mathcal{P}')$ is *adjacent* to Σ (in direction k) if $\mathbf{x}'$, $\widetilde{B}'$ and $\mathcal{P}'$ are as above. Two seeds are *mutation equivalent* if they can be connected by a sequence of pairwise adjacent seeds. The set of all seeds mutation equivalent to Σ is called the *generalized cluster structure* (of geometric type) in $\mathcal{F}$ associated with Σ and denoted by $\mathcal{GC}(\Sigma)$; in what follows, we usually write $\mathcal{GC}(\widetilde{B}, \mathcal{P})$, or even just $\mathcal{GC}$ instead. Clearly, by taking $d_i = 1$ for $1 \le i \le n$, and hence making all strings trivial, we get an ordinary cluster structure.

We associate with $\mathcal{GC}(\widetilde{B}, \mathcal{P})$ two algebras of rank n over the ground ring $\mathbb{A}$: the *generalized cluster algebra* $\mathcal{A} = \mathcal{A}(\mathcal{GC}) = \mathcal{A}(\widetilde{B}, \mathcal{P})$, which is the $\mathbb{A}$-subalgebra of $\mathcal{F}$ generated by all cluster variables in all seeds in $\mathcal{GC}(\widetilde{B}, \mathcal{P})$, and the *generalized upper cluster algebra* $\overline{\mathcal{A}} = \overline{\mathcal{A}}(\mathcal{GC}) = \overline{\mathcal{A}}(\widetilde{B}, \mathcal{P})$, which is the intersection of the rings of Laurent polynomials over $\mathbb{A}$ in cluster variables taken over all seeds in $\mathcal{GC}(\widetilde{B}, \mathcal{P})$. The generalized *Laurent phenomenon* [4] claims the inclusion $\mathcal{A}(\mathcal{GC}) \subseteq \overline{\mathcal{A}}(\mathcal{GC})$.

Let V be a quasi-affine variety over $\mathbb{C}$, $\mathbb{C}(V)$ be the field of rational functions on V, and $\mathcal{O}(V)$ be the ring of regular functions on V. Let $\mathcal{GC}$ be a generalized cluster structure in $\mathcal{F}$ as above. Assume that $\{f_1, \dots, f_{n+m}\}$ is a transcendence basis of $\mathbb{C}(V)$. Then the map $\theta : x_i \mapsto f_i$, $1 \le i \le n+m$, can be extended to a field isomorphism $\theta : \mathcal{F}_{\mathbb{C}} \to \mathbb{C}(V)$, where $\mathcal{F}_{\mathbb{C}} = \mathcal{F} \otimes \mathbb{C}$ is obtained from $\mathcal{F}$ by extension of scalars. The pair $(\mathcal{GC}, \theta)$ is called a generalized cluster structure *in* $\mathbb{C}(V)$,

$\{f_1, \ldots, f_{n+m}\}$ is called an extended cluster in $(\mathcal{GC}, \theta)$. Sometimes we omit direct indication of θ and say that $\mathcal{GC}$ is a generalized cluster structure *on* V. A generalized cluster structure $(\mathcal{GC}, \theta)$ is called *regular* if $\theta(x)$ is a regular function for any cluster variable x. The two algebras defined above have their counterparts in $\mathcal{F}_{\mathbb{C}}$ obtained by extension of scalars; they are denoted $\mathcal{A}_{\mathbb{C}}$ and $\overline{\mathcal{A}}_{\mathbb{C}}$. If, moreover, the field isomorphism θ can be restricted to an isomorphism of $\mathcal{A}_{\mathbb{C}}$ (or $\overline{\mathcal{A}}_{\mathbb{C}}$) and $\mathcal{O}(V)$, we say that $\mathcal{A}_{\mathbb{C}}$ (or $\overline{\mathcal{A}}_{\mathbb{C}}$) is *naturally isomorphic* to $\mathcal{O}(V)$.

Let $\{\cdot,\cdot\}$ be a Poisson bracket on the ambient field $\mathcal{F}$, and $\mathcal{GC}$ be a generalized cluster structure in $\mathcal{F}$. We say that the bracket and the generalized cluster structure are *compatible* if any extended cluster $\widetilde{\mathbf{x}} = (x_1, \ldots, x_{n+m})$ is *log-canonical* with respect to $\{\cdot,\cdot\}$, that is, $\{x_i, x_j\} = \omega_{ij} x_i x_j$, where $\omega_{ij} \in \mathbb{Z}$ are constants for all i, j, $1 \le i, j \le n+m$; it follows that all monomials p_{ir} are Casimirs of the bracket. The notion of compatibility extends to Poisson brackets on $\mathcal{F}_{\mathbb{C}}$ without any changes.

3. Standard Poisson–Lie group $\mathcal{G}$ and its Drinfeld double

Let $\mathcal{G}$ be a reductive complex Lie group equipped with a Poisson bracket $\{\cdot,\cdot\}$. $\mathcal{G}$ is called a *Poisson–Lie group* if the multiplication map $\mathcal{G} \times \mathcal{G} \ni (x, y) \mapsto xy \in \mathcal{G}$ is Poisson. Denote by $\langle\ ,\ \rangle$ an invariant nondegenerate form on $\mathfrak{g}$, and by ∇^R, ∇^L the right and left gradients of functions on $\mathcal{G}$ with respect to this form. Let $\pi_{>0}, \pi_{<0}$ be projections of $\mathfrak{g}$ onto subalgebras spanned by positive and negative roots and let $R = \pi_{>0} - \pi_{<0}$. The *standard Poisson–Lie bracket* $\{\cdot,\cdot\}_r$ on $\mathcal{G}$ can be written as

$$\{f_1, f_2\}_r = \frac{1}{2}\left(\langle R(\nabla^L f_1), \nabla^L f_2\rangle - \langle R(\nabla^R f_1), \nabla^R f_2\rangle\right). \tag{1}$$

Following [10], the *Drinfeld double* of $\mathfrak{g}$ is $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}$ equipped with an invariant nondegenerate bilinear form $\langle\langle(\xi, \eta), (\xi', \eta')\rangle\rangle = \langle \xi, \xi'\rangle - \langle \eta, \eta'\rangle$. Define subalgebras $\mathfrak{d}_\pm$ of $D(\mathfrak{g})$ by $\mathfrak{d}_+ = \{(\xi, \xi) : \xi \in \mathfrak{g}\}$ and $\mathfrak{d}_- = \{(R_+(\xi), R_-(\xi)) : \xi \in \mathfrak{g}\}$, where $R_\pm \in \operatorname{End}\mathfrak{g}$ is given by $R_\pm = \frac{1}{2}(R \pm \mathrm{Id})$. The operator $R_D = \pi_{\mathfrak{d}_+} - \pi_{\mathfrak{d}_-}$ can be used to define a Poisson–Lie structure on $D(\mathcal{G}) = \mathcal{G} \times \mathcal{G}$, the double of the group $\mathcal{G}$, via

$$\{f_1, f_2\}_D = \frac{1}{2}\left(\langle\langle R_D(\nabla^L f_1), \nabla^L f_2\rangle\rangle - \langle\langle R_D(\nabla^R f_1), \nabla^R f_2\rangle\rangle\right), \tag{2}$$

where ∇^R and ∇^L are right and left gradients with respect to $\langle\langle\cdot,\cdot\rangle\rangle$. Restriction of this bracket to $\mathcal{G}$ identified with the diagonal subgroup $\{(X, X) : X \in \mathcal{G}\}$ of $D(\mathcal{G})$ (whose Lie algebra is $\mathfrak{d}_+$) coincides with the Poisson–Lie bracket (1) on $\mathcal{G}$.

A group $\mathcal{G}_r$ whose Lie algebra is $\mathfrak{d}_-$ is a Poisson–Lie subgroup of $D(\mathcal{G})$ called *the dual Poisson–Lie group of* $\mathcal{G}$. The map $\mathcal{G}_r \ni (X, Y) \mapsto U = X^{-1}Y$ induces another Poisson bracket on $\mathcal{G}$. We denote this bracket $\{\cdot,\cdot\}_*$ and refer to the Poisson manifold $(\mathcal{G}, \{\cdot,\cdot\}_*)$ as $\mathcal{G}^*$.

4. Initial seed

4.1. Log-canonical basis

In this note we only deal with the case of $\mathcal{G} = GL_n$. Let (X, Y) be a point in the double $D(GL_n)$. For $k, l \ge 1$, $k + l \le n - 1$ define a $(k+l) \times (k+l)$ matrix

$$F_{kl} = F_{kl}(X, Y) = \left[\, X^{[n-k+1,n]}\ \ Y^{[n-l+1,n]}\, \right]_{[n-k-l+1,n]}.$$

For $1 \le j \le i \le n$ define an $(n-i+1) \times (n-i+1)$ matrix $G_{ij} = G_{ij}(X) = X^{[j,j+n-i]}_{[i,n]}$. For $1 \le i \le j \le n$ define an $(n-j+1) \times (n-j+1)$ matrix $H_{ij} = H_{ij}(Y) = Y^{[j,n]}_{[i,i+n-j]}$. For $k, l \ge 1$, $k + l \le n$ define an $n \times n$ matrix

$$\Phi_{kl} = \Phi_{kl}(U) = \left[\, (U^0)^{[n-k+1,n]}\ \ U^{[n-l+1,n]}\ \ (U^2)^{[n]}\ \ldots\ (U^{n-k-l+1})^{[n]}\, \right],$$

where $U = X^{-1}Y$. Note that the definition of F_{kl} can be extended to the case $k + l = n$ yielding $F_{n-l,l} = X\Phi_{n-l,l}$.

Denote $f_{kl} = \det F_{kl}$, $g_{ij} = \det G_{ij}$, $h_{ij} = \det H_{ij}$ and $\varphi_{kl} = s_{kl}(\det X)^{n-k-l+1} \det \Phi_{kl}$, $2n^2 - n + 1$ functions in total. Here s_{kl} is a sign defined as follows: it is periodic in $k + l$ with period 4 for n odd and period 2 for n even; $s_{n-l,l} = 1$; $s_{n-l-1,l} = (-1)^l$ for n odd and $s_{n-l-1,l} = (-1)^{l+1}$ for n even; $s_{n-l-2,l} = -1$ for n odd; $s_{n-l-3,l} = (-1)^{l+1}$ for n odd. Note that the pre-factor in the definition of φ_{kl} is needed to obtain an irreducible polynomial function in matrix entries of X and Y.

Consider the polynomial $\det(X + \lambda Y) = \sum_{i=0}^n \lambda^i s_i c_i(X, Y)$, where $s_i = (-1)^i$ if n is even and $s_i = 1$ if n is odd. It is well-known that functions $c_i(X, Y)$ are Casimirs for the Poisson–Lie bracket (2) on $D(GL_n)$. Note also that $c_0(X, Y) = \det X = g_{11}$ and $c_n(X, Y) = \det Y = h_{11}$.

Theorem 4.1. *The family of functions $F_n = \{g_{ij}, h_{ij}, f_{kl}, \varphi_{kl}, c_1, \ldots, c_{n-1}\}$ forms a log-canonical coordinate system with respect to the Poisson–Lie bracket (2) on $D(GL_n)$.*

The proof exploits various invariance properties of functions in F_n.

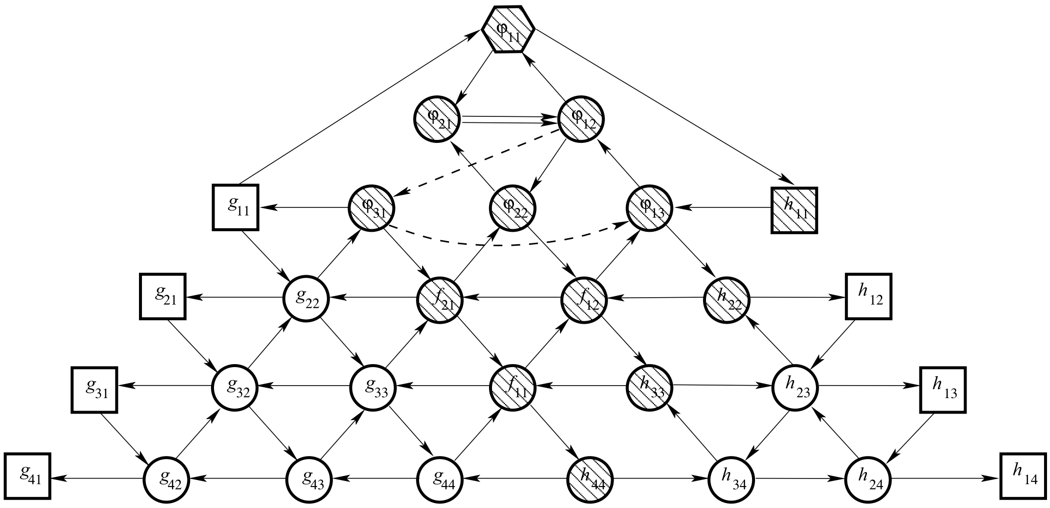

Fig. 1. Quiver Q_4.

4.2. Initial quiver

The quiver Q_n contains $2n^2 - n + 1$ vertices labeled by the functions $g_{ij}, h_{ij}, f_{kl}, \varphi_{kl}$ in the log-canonical basis F_n. The Casimir functions $c_1, \dots, c_{n-1}$ correspond to isolated vertices. The vertex φ_{11} is special, and its order equals n. The vertices g_{i1}, $1 \le i \le n$, and h_{1j}, $1 \le j \le n$, are stable.

The edges of Q_n are comprised of $(n-1)(n-2)/2$ edge-disjoint triangles $h_{ij} \to h_{i+1,j+1} \to h_{i+1,j} \to h_{ij}$, $1 \le i < j \le n-1$; $n(n-1)/2$ disjoint triangles $g_{ij} \to g_{i+1,j+1} \to g_{i,j+1} \to g_{ij}$, $1 \le j \le i \le n-1$; $(n-2)(n-3)/2$ edge-disjoint triangles $f_{kl} \to f_{k-1,l} \to f_{k-1,l+1} \to f_{kl}$, $k+l \le n-1$, $k \ge 2$, $l \ge 1$; $(n-2)(n-3)/2$ edge-disjoint triangles $\varphi_{kl} \to \varphi_{k-1,l+1} \to \varphi_{k,l+1} \to \varphi_{kl}$, $k+l \le n-1$, $k \ge 2$, $l \ge 1$; the path $g_{11} \to \varphi_{11} \to \varphi_{21} \to \varphi_{12} \to \varphi_{31} \to \dots \to \varphi_{1,l-1} \to \varphi_{l1} \to \varphi_{1l} \to \dots \to \varphi_{1,n-1}$ of length $2n-3$ for $n > 2$; the path $\varphi_{1,n-1} \to \varphi_{1,n-2} \to \dots \to \varphi_{11} \to h_{11}$ of length $n-1$ for $n > 2$; the path $\varphi_{n-1,1} \to f_{n-2,1} \to \varphi_{n-2,2} \to \dots \to \varphi_{kl} \to f_{k-1,l} \to \varphi_{k-1,l+1} \to \dots \to \varphi_{1,n-1}$ of length $2(n-2)$; the path $h_{11} \to \varphi_{1,n-1} \to h_{22} \to f_{1,n-2} \to \dots \to f_{1l} \to h_{n-l+1,n-l+1} \to f_{1,l-1} \to \dots \to h_{nn}$ of length $2(n-1)$; the path $h_{nn} \to h_{n-1,n} \to \dots \to h_{1n}$ of length $n-1$; the path $h_{nn} \to g_{nn} \to g_{n,n-1} \to g_{n,n-2} \to \dots \to g_{n1}$ of length n. Above we identify $g_{i,i+1}$ with $f_{n-i,1}$ for $1 \le i \le n-1$ and $f_{k,n-k}$ with $\varphi_{k,n-k}$ for $1 \le k \le n-1$. Note that the triangle $\varphi_{21} \to \varphi_{12} \to \varphi_{22} \to \varphi_{21}$ and the first path above have a common edge; this means that for any $n > 3$ there are two edges pointing from φ_{21} to φ_{12}.

The quiver Q_4 is shown in Fig. 1. The stable vertices are shown as squares, the special vertex is shown as a hexagon, isolated vertices are not shown. It is easy to see that Q_4, as well as Q_n for any n, can be embedded into a torus. This makes possible to distinguish "special" mutations that preserve the embedding. Such mutations play an essential role in the proofs of our main results.

Remark 4.1. On the diagonal subgroup $\{(X, X) : X \in GL_n\}$ of $D(GL_n)$, $g_{ii} = h_{ii}$, $1 \le i \le n$, and functions f_{kl} and φ_{kl} vanish identically. Accordingly, vertices in Q_n that correspond to f_{kl} and φ_{kl} are erased and vertices corresponding to g_{ii} and h_{ii} are identified. As a result, one recovers a seed of the cluster structure compatible with the standard Poisson–Lie structure on GL_n, see [6, Chap. 4.3].

4.3. Generalized exchange relation

Proposition 4.1. *Let A be a complex $n \times n$ matrix. For $u, v \in \mathbb{C}^n$, define matrices*

$$\Gamma(u) = \left[u \; Au \; A^2u \dots A^{n-1}u\right], \quad \Gamma_1(u, v) = \left[v \; u \; Au \dots A^{n-2}u\right], \quad \Gamma_2(u, v) = \left[Av \; u \; Au \dots A^{n-2}u\right].$$

In addition, let w be the last row of the classical adjoint of $\Gamma_1(u, v)$, i.e. $w\Gamma_1(u, v) = (\det \Gamma_1(u, v))\, e_n^T$. Define $\Gamma^*(u, v)$ to be the matrix with rows $w, wA, \dots, wA^{n-1}$. Then

$$\det\Big(\det \Gamma_1(u, v) A - \det \Gamma_2(u, v)\mathbf{1}\Big) = (-1)^{\frac{n(n-1)}{2}} \det \Gamma(u) \det \Gamma^*(u, v).$$

Specializing Proposition 4.1 to the case $A = X^{-1}Y$, $u = e_n$, $v = e_{n-1}$, one obtains

Corollary 4.2. *For any $n > 2$,*

$$\det(s_{12}\varphi_{12}X + s_{21}\varphi_{21}Y) = \varphi_{11}P_n^*, \tag{3}$$

where P_n^* is a polynomial in the entries of X and Y.

Relation (3) will serve as a generalized exchange relation in our definition of a generalized cluster structure on $D(GL_n)$. More exactly, the set $\mathcal{P}_n$ contains only one nontrivial string $\{p_{ir}\}$, $1 \le r \le n-1$. It corresponds to the vertex φ_{11}, and $p_{ir} = c_r^n g_{11}^{r-n} h_{11}^{-r}$, $1 \le r \le n-1$. The strings corresponding to all other vertices are trivial.

5. Main results

Theorem 5.1. (i) *The extended seed $\widetilde{\Sigma}_n = (F_n, Q_n, \mathcal{P}_n)$ defines a generalized cluster structure in the ring of regular functions on $D(GL_n)$ compatible with the standard Poisson–Lie structure on $D(GL_n)$.*

(ii) *The corresponding generalized upper cluster algebra is naturally isomorphic to the ring of regular functions on* $\mathrm{Mat}_n \times \mathrm{Mat}_n$.

The strategy of the proof is similar to that employed, e.g., in [6, Ch. 4] or [9], although adjustments allowing for the presence of a generalized cluster transformation have to be made. The difficult part is to show that all generators of the ring of regular functions are contained in $\overline{\mathcal{A}}$.

Using Theorem 5.1, we can construct a generalized cluster structure on GL_n^*. For $U \in GL_n^*$, denote $\psi_{kl}(U) = s_{kl}\det\Phi_{kl}$, where s_{kl} are the signs defined in Section 4. The initial extended cluster F_n^* for GL_n^* consists of functions $\psi_{kl}(U)$, $k, l \ge 1$, $k+l \le n$, $h_{ij}(U)$, $2 \le i \le j \le n$, and $c_i(\mathbf{1}, U)$, $1 \le i \le n-1$. To obtain the initial seed for GL_n^*, we apply a certain sequence $\mathcal{T}$ of cluster transformations to the initial seed for $D(GL_n)$. This sequence does not involve vertices associated with functions φ_{kl}. The resulting cluster $\mathcal{T}(F_n)$ contains a subset $\{(\det X)^{\nu(f)} f : f \in F_n^*\}$ with $\nu(f) \in \mathbb{Z}_+$ (in particular, $\nu(\psi_{kl}) = n-k-l+1$). These functions are attached to a subquiver Q_n^* in the resulting quiver $\mathcal{T}(Q_n)$; it is isomorphic to the subquiver of Q_n formed by vertices associated with functions $\varphi_{kl}, f_{ij}, h_{ii}$, see Fig. 1, where the vertices of the corresponding subquiver are shaded. Functions $h_{ii}(U)$ are declared stable variables, $c_i(1, U)$ remain isolated. All exchange relations defined by mutable vertices of Q_n^* are homogeneous in $\det X$. This allows us to use $(F_n^*, Q_n^*, \mathcal{P}_n)$ as an initial seed for GL_n^*. The generalized exchanged relation associated with the cluster variable ψ_{11} now takes form $\det(s_{12}\psi_{12}\mathbf{1} + s_{21}\psi_{21}U) = \psi_{11}\Pi_n^*$, where Π_n^* is a polynomial in the entries of U.

Theorem 5.2. (i) *The generalized cluster structure on GL_n^* with the initial seed described above is compatible with $\{\cdot,\cdot\}_*$ and regular.*

(ii) *The corresponding generalized upper cluster algebra is naturally isomorphic to the ring of regular functions on* Mat_n.

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