C. R. Acad. Sci. Paris, Ser. I 354 (2016) 371–375

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial differential equations/Probability theory

A Schauder estimate for stochastic PDEs

Une estimée de Schauder pour des EDPs stochastiques

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ARTICLE INFO

Article history:
Received 18 September 2015
Accepted 25 January 2016
Available online 11 February 2016

Presented by the Editorial Board

ABSTRACT

Considering stochastic partial differential equations of parabolic type with random coefficients in vector-valued Hölder spaces, we establish a sharp Schauder theory. The existence and uniqueness of solutions to the Cauchy problem is obtained.

RÉSUMÉ

Nous considérons des équations aux dérivées partielles stochastiques, du type parabolique et à coefficients aléatoires dans des espaces de Hölder à valeurs vectorielles. Nous obtenons une estimée de Schauder optimale, puis nous utilisons cette estimée pour prouver l'existence et l'unicité de la solution du problème de Cauchy.

1. Introduction

We consider the second-order stochastic partial differential equations (SPDEs) of the Itô type

$$\mathrm{d}u = (a^{ij}u_{x^ix^j} + b^i u_{x^i} + cu + f)\,\mathrm{d}t + (\sigma^{ik}u_{x^i} + \nu^k u + g^k)\,\mathrm{d}w_t^k, \tag{1.1}$$

in $\mathbf{R}^n \times (0,\infty)$, where w^k are countable independent standard Wiener processes defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, \mathbb{P})$ for $k = 1, 2, \cdots$. The matrix $a = (a^{ij})$ is symmetric, and the uniform *parabolic condition* is assumed throughout the paper, namely there is a constant $\lambda > 0$ such that

$$2a^{ij} - \sigma^{ik}\sigma^{jk} \geq \lambda\delta_{ij} \quad \text{on } \mathbf{R}^n \times (0,\infty) \times \Omega, \tag{1.2}$$

where δ_{ij} is the Kronecker delta. The random fields u, a^{ij}, b^i, f are all real-valued, while σ^i, ν and g take values in ℓ^2. One of the most important examples of (1.1) is the Zakai equation arising in the nonlinear filtering problem [15].

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[1] Research of Liu was supported by the Australian Research Council DE140101366.

http://dx.doi.org/10.1016/j.crma.2016.01.010

The regularity of solutions to (1.1) in Sobolev spaces has already been investigated by many researchers. Various aspects of L^2-theory were studied since 1970s, see [11,9,13,1] and references therein. Later on, a complete L^p-theory was established by Krylov in 1990s, see [7,8]. By using Sobolev's embedding, one then has the regularity in Hölder spaces, which is however not sharp. As an open problem mentioned in [8], one desires a sharp $C^{2+\alpha}$-theory in the sense that not only that for f, g belonging to a proper space $\mathcal{F}$, the solution belongs to some kind of stochastic $C^{2+\alpha}$-spaces, but also that every element of this stochastic space can be obtained as a solution for certain f, g belonging to the same $\mathcal{F}$.

The purpose of this paper is to establish a Schauder theory of Equation (1.1), which is sharp in the above sense. In order to state our main results, we first introduce a notion of quasi-classical solutions.

Definition 1.1. A random field u is called a *quasi-classical* solution to (1.1) if

(1) for each $t \in (0,\infty)$, $u(\cdot,t)$ is a twice strongly differentiable function from $\mathbf{R}^n$ to $L^\gamma_\omega := L^\gamma(\Omega;\mathbf{R})$ for some $\gamma \ge 2$; and
(2) for each $x \in \mathbf{R}^n$, the process $u(x,\cdot)$ satisfies (1.1) in the Itô integral form with respect to the time variable.

If furthermore, $u(\cdot,t,\omega) \in C^2(\mathbf{R}^n)$ for any $(t,\omega) \in (0,\infty)\times\Omega$, then u is a classical solution to (1.1).

Analogously to classical Hölder spaces, we can define the L^γ_ω-valued Hölder spaces $C^{m+\alpha}_x(\mathcal{Q}_T; L^\gamma_\omega)$ and $C^{m+\alpha,\alpha/2}_{x,t}(\mathcal{Q}_T; L^\gamma_\omega)$, where $T>0$, $\mathcal{Q}_T = \mathbf{R}^n \times (0,T)$, and $L^\gamma_\omega := L^\gamma(\Omega;\mathbf{R})$ is a Banach space equipped with the norm $\|\xi\|_{L^\gamma_\omega} := (\mathbb{E}|\xi|^\gamma)^{1/\gamma}$. More specifically, we define $C^{m+\alpha}_x(\mathcal{Q}_T; L^\gamma_\omega)$ to be the set of all L^γ_ω-valued strongly continuous functions u such that

$$|u|_{m+\alpha;\mathcal{Q}_T} := \sup_{\substack{(x,t)\in\mathcal{Q}_T\\|\beta|\le m}} \|D^\beta u(x,t)\|_{L^\gamma_\omega} + \sup_{\substack{t,\,x\neq y\\|\beta|=m}} \frac{\|D^\beta u(x,t) - D^\beta u(y,t)\|_{L^\gamma_\omega}}{|x-y|^\alpha} < \infty. \tag{1.3}$$

Using the parabolic module $|X|_{\mathbf{p}} := |x| + \sqrt{|t|}$ for $X=(x,t)\in\mathbf{R}^n\times\mathbf{R}$, we define $C^{m+\alpha,\alpha/2}_{x,t}(\mathcal{Q}_T; L^\gamma_\omega)$ to be the set of all $u \in C^{m+\alpha}_x(\mathcal{Q}_T; L^\gamma_\omega)$ such that

$$|u|_{(m+\alpha,\alpha/2);\mathcal{Q}_T} := |u|_{m;\mathcal{Q}_T} + \sup_{|\beta|=m,\ X\neq Y} \frac{\|D^\beta u(X) - D^\beta u(Y)\|_{L^\gamma_\omega}}{|X-Y|^\alpha_{\mathbf{p}}} < \infty. \tag{1.4}$$

Similarly, we can define the norms (1.3) and (1.4) over a domain $Q = \mathcal{O}\times I$, for any domains $\mathcal{O}\subset\mathbf{R}^n$ and $I\subset\mathbf{R}$.

Our main result is the following.

Theorem 1.1. *Assume that the classical C^α_x-norms of $a^{ij}, b^i, c, \sigma^i, \sigma^i_x, \nu, \nu_x$ are all dominated by a constant K uniformly in $(t,\omega)\in(0,T)\times\Omega$, and the condition (1.2) is satisfied. If $f\in C^\alpha_x(\mathcal{Q}_T; L^\gamma_\omega)$, $g\in C^{1+\alpha}_x(\mathcal{Q}_T; L^\gamma_\omega)$ for some $\gamma\ge 2$, then Equation (1.1) with a zero initial condition admits a unique quasi-classical solution u in $C^{2+\alpha,\alpha/2}_{x,t}(\mathcal{Q}_T; L^\gamma_\omega)$.*

We remark that the problem with nonzero initial value can be easily reduced to our case by a simple transform. We also remark that by an anisotropic Kolmogorov continuity theorem (see [2]), if $\alpha\gamma > n+2$, the above obtained quasi-classical solution u has a $C^{2+\delta,\delta/2}$ modification for $0<\delta<\alpha-(n+2)/\gamma$ as a classical solution to (1.1).

In order to prove the solvability in Theorem 1.1, by means of the standard method of continuity, it suffices to establish the following a priori estimate.

Theorem 1.2. *Under the hypotheses of Theorem 1.1, letting $u\in C^{2,0}_{\mathrm{loc}}(\mathcal{Q}_T; L^\gamma_\omega)$ be a quasi-classical solution to (1.1) and $u(\cdot,0)=0$, there is a positive constant C depending only on $n, \lambda, \gamma, \alpha$, and K such that*

$$|u|_{(2+\alpha,\alpha/2);\mathcal{Q}_T} \le Ce^{CT}(|f|_{\alpha;\mathcal{Q}_T} + |g|_{1+\alpha;\mathcal{Q}_T}). \tag{1.5}$$

The Hölder regularity in spaces $C^{m+\alpha}_x(\mathcal{Q}_T; L^\gamma_\omega)$ for Equation (1.1) was previously investigated by Rozovsky [12], and later was improved by Mikulevicius [10]. However, both works addressed only the equations with nonrandom coefficients and with no derivatives of the unknown function in the stochastic term, namely a^{ij} is deterministic and $\sigma^{ik}\equiv 0$. Moreover, both previous works did not obtain the time-continuity of second-order derivatives of u, comparing to our estimate (1.5) and Theorem 1.1.

The Schauder estimate we obtained in Theorem 1.2 is sharp in the sense that mentioned in [8], and is for the general form (1.1) with natural assumptions, where all coefficients are random. The approach to $C^{2+\alpha}$-theory in [10] was based on several delicate estimates for the heat kernel. Our method is completely different and more straightforward by combining certain integral estimates and a perturbation argument of Wang [14]. A sketch of proof of Theorem 1.2 is given in Section 2. Full details in addition to applications and further remarks are contained in our separate paper [3].

2. Schauder estimates

In this section, we give an outline of the proof of our main estimate (1.5). For simplicity we will first deal with a simplified model equation, and then extend to the general ones.

Consider the model equation

$$\mathrm{d}u = (a^{ij}u_{ij} + f)\,\mathrm{d}t + (\sigma^{ik}u_i + g^k)\,\mathrm{d}w_t^k, \tag{2.1}$$

where a^{ij}, σ^{ik} are predictable processes, independent of x, satisfying the condition (1.2). We shall consider the model equation in the entire space $\mathbf{R}^n \times \mathbf{R}$. Suppose that $f(t,\cdot)$ and $g_x(t,\cdot)$ are Dini continuous with respect to x uniformly in t, namely

$$\int_0^1 \frac{\varpi(r)}{r}\,\mathrm{d}r < \infty,$$

where

$$\varpi(r) = \sup_{t\in\mathbf{R},\ |x-y|\le r} \left(\|f(t,x)-f(t,y)\|_{L^\gamma_\omega} + \|g_x(t,x)-g_x(t,y)\|_{L^\gamma_\omega}\right).$$

For any $r>0$, we denote

$$B_r(x) = \{y\in\mathbf{R}^n : |y-x|<r\}, \quad Q_r(x,t) = B_r(x)\times(t-r^2,t), \tag{2.2}$$

and further define $B_r = B_r(0)$ and $Q_r = Q_r(0,0)$.

Lemma 2.1. *Let $u \in C^{2,0}_{x,t}(Q_1; L^\gamma_\omega)$ be a quasi-classical solution to (2.1). Then there is a positive constant C, depending only on n, λ and γ such that for any $X, Y \in Q_{1/4}$,*

$$\|u_{xx}(X) - u_{xx}(Y)\|_{L^\gamma_\omega} \le C\left[\delta M_1 + \int_0^\delta \frac{\varpi(r)}{r}\,\mathrm{d}r + \delta\int_\delta^1 \frac{\varpi(r)}{r^2}\,\mathrm{d}r\right], \tag{2.3}$$

where $\delta = |X-Y|_{\mathbf{p}}$ and $M_1 = |u|_{0;Q_1} + |f|_{0;Q_1} + |g|_{1;Q_1}$.

An important consequence of Lemma 2.1 is the fundamental Schauder estimate that the solution $u \in C^{2+\alpha,\alpha/2}_{x,t}(Q_{1/4}; L^\gamma_\omega)$ when $f\in C^\alpha_x(Q_1;L^\gamma_\omega)$ and $g\in C^{1+\alpha}_x(Q_1;L^\gamma_\omega)$ for some $\alpha\in(0,1)$.

Outline of proof. Without loss of generality, we may assume $X=0$. Let $\rho = 1/2$, and denote

$$Q^\kappa = Q_{\rho^\kappa} = Q_{\rho^\kappa}(0,0), \quad \kappa = 0,1,2,\cdots.$$

Construct a sequence of Cauchy problems

$$\begin{aligned} &\mathrm{d}u^\kappa = [a^{ij}u^\kappa_{ij} + f(0,t)]\,\mathrm{d}t + [\sigma^{ik}u^\kappa_i + g^k(0,t) + g^k_x(0,t)\cdot x]\,\mathrm{d}w^k_t \quad \text{in } Q^\kappa,\\ &u^\kappa = u \quad \text{on } \partial Q^\kappa. \end{aligned}$$

Claim 1. *For each κ, there is a unique generalised solution u^κ such that $u^\kappa(\cdot,t)\in L^\gamma(\Omega; C^m(B_\varepsilon))$ for any $m\ge 0$ and $\varepsilon\in(0,\rho^\kappa)$. Moreover, for any $r<\rho^\kappa$ there is a constant $C = C(n,\gamma)$ such that*

$$\|u\|_{L^\gamma(\Omega;L^2(Q_r))} \le C\left(r^2\|f\|_{L^\gamma(\Omega;L^2(Q_r))} + r\|g\|_{L^\gamma(\Omega;L^2(Q_r))}\right). \tag{2.4}$$

Proof. In fact, for $\gamma=2$, the unique solvability and interior smoothness of u^κ follows from [5, Theorem 2.1]. For $\gamma\ge 2$, higher order L^γ_ω-integrability (2.4) can be achieved by a truncation technique. □

Claim 2. *There is a constant $C = C(n,\lambda,\gamma)$ such that*

$$|D^m(u^\kappa - u^{\kappa+1})|_{0;Q^{\kappa+2}} \le C\rho^{(2-m)\kappa}\varpi(\rho^\kappa), \quad m = 1,2,\ldots. \tag{2.5}$$

Proof. Note that $(u^\kappa - u^{\kappa+1})$ satisfies a homogeneous equation. By a delicate computation, we have

$$|D^m(u^\kappa - u^{\kappa+1})|_{0;Q^{\kappa+2}} \le C\rho^{-m\kappa}\left\| \fint_{Q^{\kappa+1}} (u^\kappa - u^{\kappa+1})^2\,\mathrm{d}X \right\|^{1/2}_{L^{\gamma/2}_\omega} =: I_{\kappa,m}.$$

On the other hand, $(u^\kappa - u)$ satisfies a zero initial condition. By Claim 1,

$$J_\kappa := \left\| \fint_{Q^\kappa} (u^\kappa - u)^2\,\mathrm{d}X \right\|^{1/2}_{L^{\gamma/2}_\omega} \le C\rho^{2\kappa}\varpi(\rho^\kappa).$$

Thus, Claim 2 is proved, since

$$I_{\kappa,m} \le C\rho^{-m\kappa}(J_\kappa + J_{\kappa+1}) \le C\rho^{(2-m)\kappa}\varpi(\rho^\kappa).$$

It is worth remarking that instead of using the maximum principle to estimate the term $|D^m(u^\kappa - u^{\kappa+1})|_{0;Q^{\kappa+2}}$ as in [14], we obtain the inequality (2.5) by subtle integral estimates. □

Claim 3. *$\{u^\kappa_{xx}(0)\}$ converges in L^γ_ω (here $0 \in \mathbf{R}^{n+1}$), and the limit is $u_{xx}(0)$.*

Proof. By Claim 2 and the assumption of Dini continuity,

$$\sum_{\kappa\ge 1} |(u^\kappa - u^{\kappa+1})_{xx}|_{0;Q^{\kappa+2}} \le C\sum_{\kappa\ge 1}\varpi(\rho^\kappa) \le C\int_0^1 \frac{\varpi(r)}{r}\,\mathrm{d}r < \infty,$$

which implies that $u^\kappa_{xx}(0)$ converges in L^γ_ω. Since $\gamma \ge 2$, it suffices to show that

$$\lim_{\kappa\to\infty} \|u^\kappa_{xx}(0) - u_{xx}(0)\|_{L^2_\omega} = 0, \tag{2.6}$$

which can also be achieved straightforward by our integral estimates. □

Now for any $Y = (y, s) \in Q_{1/4}$ we can select an κ such that $|Y|_{\mathbf{p}} \in [\rho^{\kappa+2}, \rho^{\kappa+1})$. By decomposition, one has

$$\begin{aligned} &\|u_{xx}(Y) - u_{xx}(0)\|_{L^\gamma_\omega} \\ &\le \|u^\kappa_{xx}(Y) - u^\kappa_{xx}(0)\|_{L^\gamma_\omega} + \|u^\kappa_{xx}(0) - u_{xx}(0)\|_{L^\gamma_\omega} + \|u^\kappa_{xx}(Y) - u_{xx}(Y)\|_{L^\gamma_\omega} \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{2.7}$$

Claim 4. *$I_1 \le C\delta M_1 + C\delta\int_\delta^1 \frac{\varpi(r)}{r^2}\,\mathrm{d}r$, where $\delta := |Y|_{\mathbf{p}}$ and M_1 was given in (2.3).*

Proof. The proof is by induction. When $\kappa = 0$, note that u^0_{xx} satisfies the following homogeneous equation:

$$\mathrm{d}u^0_{xx} = a^{ij}D_{ij}u^0_{xx}\,\mathrm{d}t + \sigma^{ik}D_i u^0_{xx}\,\mathrm{d}w^k_t \quad \text{in } Q_{3/4}.$$

From interior estimates, we have

$$\|u^0_{xx}(X) - u^0_{xx}(Y)\|_{L^\gamma_\omega} \le CM_1|X - Y|_{\mathbf{p}}, \quad \forall X, Y \in Q_{1/4}. \tag{2.8}$$

When $\kappa \ge 1$, denote $h^\iota = u^\iota - u^{\iota-1}$, for $\iota = 1, 2, \ldots, \kappa$, then h^ι satisfies

$$\mathrm{d}h^\iota = a^{ij}h^\iota_{ij}\,\mathrm{d}t + \sigma^{ik}h^\iota_i\,\mathrm{d}w^k_t \quad \text{in } Q^\iota.$$

From Claim 2, we have for $-\rho^{2(\kappa+1)} \le t \le 0$ and $|x| \le \rho^{\kappa+1}$,

$$\|h^\iota_{xx}(x,t) - h^\iota_{xx}(0,0)\|_{L^\gamma_\omega} \le C\rho^{\kappa-\iota}\varpi(\rho^{\iota-1}). \tag{2.9}$$

Using (2.8) and (2.9), we can obtain the estimate

$$\begin{aligned} I_1 &\le \|u^{\kappa-1}_{xx}(Y) - u^{\kappa-1}_{xx}(0)\|_{L^\gamma_\omega} + \|h^\kappa_{xx}(Y) - h^\kappa_{xx}(0)\|_{L^\gamma_\omega} \\ &\le \|u^0_{xx}(Y) - u^0_{xx}(0)\|_{L^\gamma_\omega} + \sum_{\iota=1}^{\kappa}\|h^\iota_{xx}(Y) - h^\iota_{xx}(0)\|_{L^\gamma_\omega} \\ &\le C\delta M_1 + C\delta\int_\delta^1 \frac{\varpi(r)}{r^2}\,\mathrm{d}r. \end{aligned}$$

Claim 4 is proved. □

Claim 5. *$I_i \le C\int_0^\delta \frac{\varpi(r)}{r}\,\mathrm{d}r$, for $i = 2, 3$.*

Proof. The estimate of I_2 is a refinement of convergence in Claim 3. In fact, by Claim 2 we have the precise estimate

$$I_2 = \|u_{xx}^\kappa(0) - u_{xx}(0)\|_{L^\gamma_\omega} \le \sum_{j\ge\kappa} |(u^j - u^{j+1})_{xx}|_{0;Q^{j+2}} \le C\int_0^{\rho^\kappa} \frac{\varpi(r)}{r}\,\mathrm{d}r, \tag{2.10}$$

where $C = C(n,\lambda,\gamma)$. We can obtain a similar estimate for I_3 by shifting the centre of domains. □

To sum up, Lemma 2.1 is proved. □

Having proved Lemma 2.1 we are in a position to derive the global estimate of solutions to (1.1) and complete the proof of Theorem 1.2.

Outline of proof of Theorem 1.2. The proof is by an argument of frozen coefficients. Denote $\mathcal{Q}_{r,\tau} = B_r \times (0,\tau)$, and let

$$M^\tau_{x,r}(u) = \sup_{0\le t\le\tau}\Bigg(\fint_{B_r(x)} \mathbb{E}|u(t,y)|^\gamma\,\mathrm{d}y\Bigg)^{1/\gamma} \qquad M^\tau_r(u) = \sup_{x\in\mathbf{R}^n} M^\tau_{x,r}(u).$$

By multiplying cut-off functions and applying Lemma 2.1 we can get

$$|u_{xx}|_{(\alpha,\alpha/2);\mathcal{Q}_{\rho/2,\tau}} \le C\Big(M^\tau_{0,\rho}(u) + |f|_{\alpha;\mathcal{Q}_{\rho,\tau}} + |g|_{1+\alpha;\mathcal{Q}_{\rho,\tau}}\Big), \tag{2.11}$$

for some sufficiently small $\rho > 0$. The derivation of (2.11) involves a rather delicate computation, which makes use of interpolation inequalities in Hölder spaces (see [4, Lemma 6.35] or [6, Theorem 3.2.1]). Since the centre of domains can shift to any point $x \in \mathbf{R}^n$, we obtain

$$|u|_{(2+\alpha,\alpha/2);\mathcal{Q}_\tau} \le C\Big(M^\tau_\rho(u) + |f|_{\alpha;\mathcal{Q}_\tau} + |g|_{1+\alpha;\mathcal{Q}_\tau}\Big), \tag{2.12}$$

where $C = C(n,\lambda,\gamma,\alpha)$.

To estimate $M^\tau_\rho(u)$, applying Itô's formula, and using Hölder and Sobolev–Gagliargo–Nirenberg inequalities, we can get

$$M^\tau_\rho(u) \le C_1\tau(M^\tau_\rho(u) + |u_{xx}|_{0;\mathcal{Q}_{\rho,\tau}} + |f|_{0;\mathcal{Q}_\tau} + |g|_{0;\mathcal{Q}_\tau}),$$

where $C_1 = C_1(n,\lambda,\gamma)$. Letting $\tau = (2CC_1 + C_1)^{-1}$, by virtue of (2.12) we obtain

$$|u|_{(2+\alpha,\alpha/2);\mathcal{Q}_\tau} \le C_0\big(|f|_{\alpha;\mathcal{Q}_\tau} + |g|_{1+\alpha;\mathcal{Q}_\tau}\big), \tag{2.13}$$

where $C_0 = C_0(n,\lambda,\gamma,\alpha)$.

Finally, the proof of (1.5) and Theorem 1.2 is completed by induction. □

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