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Complex analysis

Faber polynomial coefficients of bi-subordinate functions

Polynômes de Faber et coefficients des fonctions bi-subordonnées

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ABSTRACT

A function is said to be bi-univalent in the open unit disk $\mathbb{D}$ if both the function and its inverse map are univalent in $\mathbb{D}$. By the same token, a function is said to be bi-subordinate in $\mathbb{D}$ if both the function and its inverse map are subordinate to certain given function in $\mathbb{D}$. The behavior of the coefficients of such functions are unpredictable and unknown. In this paper, we use the Faber polynomial expansions to find upper bounds for the n-th ($n \geq 3$) coefficients of classes of bi-subordinate functions subject to a gap series condition as well as determining bounds for the first two coefficients of such functions.

RÉSUMÉ

Une fonction est dite bi-univalente dans le disque unité ouvert $\mathbb{D}$ si elle et son inverse sont univalentes dans $\mathbb{D}$. Dans le même ordre, une fonction est dite bi-subordonnée dans $\mathbb{D}$ si elle et son inverse sont subordonnées à une fonction donnée dans $\mathbb{D}$. Le comportement des coefficients de telles fonctions est imprévisible et inconnu. Dans cette Note, nous utilisons les développements en polynômes de Faber afin d'établir une borne supérieure pour le n^{e} ($n \geq 3$) coefficient d'une fonction bi-subordonnée, lorsque les $n-2$ précédents coefficients sont nuls. Nous donnons également des bornes plus précises pour les deux premiers coefficients de telles fonctions.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ and let S be the class of functions f that are analytic and univalent in $\mathbb{D}$ and are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

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For $f(z)$ and $F(z)$ analytic in $\mathbb{D}$, we say that $f(z)$ is subordinate to $F(z)$, written $f \prec F$, if there exists a Schwarz function $\varphi(z)$ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ in $\mathbb{D}$ such that $f(z) = F(\varphi(z))$. We note that $f(\mathbb{D}) \subset F(\mathbb{D})$ if f and F are in S. For real numbers A and B so that $-1 \le B < A \le 1$ we let $S[A, B]$ consist of functions $f \in \mathcal{S}$ satisfying the subordination condition

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \qquad |z| < 1.$$

For classes related to $S[A, B]$, see Janowski ([18,19]).

If f is given by the power series (1), then $g = f^{-1}$, the inverse map of the function f, has a Maclaurin series expansion in some disk about the origin (e.g. see Duren [10]). A function $f \in S$ is said to be bi-univalent if $g = f^{-1}$ also belongs to S. Certain coefficient bounds for subclasses of bi-univalent functions were obtained by several authors including Ali et al. [4], Altinkaya and Yalcin [6,5,7], Bulut [8], Deniz [9], Frasin and Aouf [12], Hamidi and Jahangiri ([13,14]), Jahangiri and Hamidi [16], Jahangiri et al. [17], Magesh and Yamini [20], Srivastava et al. ([21,22]) and Zaprawa [23]. The bi-univalency condition imposed on the function f makes the behavior of the coefficients of bi-univalent functions unpredictable. Not much is known about the higher coefficients of bi-univalent functions as Ali et al. [4] also declared the bounds for the n-th ($n \ge 4$) coefficients of bi-univalent functions an open problem. We use the Faber polynomial expansions to obtain bounds for the n-th ($n \ge 3$) coefficients of bi-subordinate functions $f \in S[A, B]$ subject to a gap series condition. We then demonstrate the unpredictability of the early coefficients a_2 and a_3 of such bi-subordinate functions. A function f in $S[A, B]$ is said to be bi-subordinate if its inverse map $g = f^{-1}$ is also in $S[A, B]$.

2. Main results

In the following theorem we use the Faber polynomials introduced by Faber [11] to obtain a bound for the general coefficients $|a_n|$ of the bi-univalent functions in $S[A, B]$ subject to a gap series condition.

Theorem 2.1. *For $-1 \le B < A \le 1$ if both functions f and its inverse map $g = f^{-1}$ are in $S[A, B]$, then*

$$|a_n| \le \frac{A-B}{n-1} \quad \textit{for } a_k = 0;\ \ 2 \le k \le n-1.$$

Proof. An application of Faber polynomial expansion to the power series $f \in S[A, B]$ (e.g. see [2] or [3, equation (1.6)]) yields

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \cdots, a_n) z^{n-1} \tag{2}$$

where

$$F_{n-1}(a_2, a_3, \cdots, a_n) = \sum_{i_1+2i_2+\cdots+(n-1)i_{n-1}=n-1} A(i_1, i_2, \cdots, i_{n-1}) (a_2^{i_1} a_3^{i_2} \cdots a_n^{i_{n-1}})$$

and

$$A(i_1, i_2, \cdots, i_{n-1}) := (-1)^{(n-1)+2i_1+\cdots+ni_{n-1}} \frac{(i_1+i_2+\cdots+i_{n-1}-1)!(n-1)}{(i_1!)(i_2!)\cdots(i_{n-1}!)}.$$

The first few terms of $F_{n-1}(a_2, a_3, \cdots, a_n)$ are

$$\begin{aligned}
F_1 &= -a_2,\\
F_2 &= a_2^2 - 2a_3,\\
F_3 &= -a_2^3 + 3a_2a_3 - 3a_4,\\
F_4 &= a_2^4 - 4a_2^2a_3 + 4a_2a_4 + 2a_3^2 - 4a_5,\\
F_5 &= -a_2^5 + 5a_2^3a_3 + 5a_2^2a_4 - 5(a_3^2 - a_5)a_2 + 5a_3a_4 - 5a_6,\\
F_6 &= a_2^6 - 6a_2^4a_3 + 6a_2^3a_4 - 6(2a_3a_4 - a_6)a_2 - 2a_3^3 + 9a_2^2a_2^2 + 6a_3a_5 + 3a_4^2 - 3a_2^2a_5 - 6a_7.
\end{aligned}$$

By the same token, the coefficients of the inverse map $g = f^{-1}$ may be expressed by

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^n = w + \sum_{n=2}^{\infty} b_n w^n$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3$$
$$+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4$$
$$+ \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right]$$
$$+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6 + (-2n+5)a_3a_4] + \sum_{j\geq 7} a_2^{n-j}V_j,$$

and V_j for $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_3, a_4, \cdots, a_n$. Obviously,

$$\frac{wg'(w)}{g(w)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(b_2, b_3, \cdots, b_n)w^{n-1}. \tag{3}$$

Since, both functions f and its inverse map $g = f^{-1}$ are in $S[A, B]$, by the definition of subordination, there exist two Schwarz functions $\varphi(z) = c_1z + c_2z^2 + \cdots + c_nz^n + \ldots$, $z \in \mathbb{D}$ and $\psi(w) = d_1w + d_2w^2 + \cdots + d_nw^n + \ldots$, $w \in \mathbb{D}$, so that

$$\frac{zf'(z)}{f(z)} = \frac{1 + A\varphi(z)}{1 + B\varphi(z)} = 1 - \sum_{n=1}^{\infty}(A - B)K_n^{-1}(c_1, c_2, \cdots, c_n, B)z^n \tag{4}$$

and

$$\frac{wg'(w)}{g(w)} = \frac{1 + A\psi(w)}{1 + B\psi(w)} = 1 - \sum_{n=1}^{\infty}(A - B)K_n^{-1}(d_1, d_2, \cdots, d_n, B)w^n. \tag{5}$$

In general (e.g., see [1] and [2]), the coefficients $K_n^p(k_1, k_2, \cdots, k_n, B)$ are given by

$$K_n^p(k_1, k_2, \cdots, k_n, B) = \frac{p!}{(p-n)!n!}k_1^nB^{n-1} + \frac{p!}{(p-n+1)!(n-2)!}k_1^{n-2}k_2B^{n-2}$$
$$+ \frac{p!}{(p-n+2)!(n-3)!}k_1^{n-3}k_3B^{n-3}$$
$$+ \frac{p!}{(p-n+3)!(n-4)!}k_1^{n-4}\left[k_4B^{n-4} + \frac{p-n+3}{2}k_3^2B\right]$$
$$+ \frac{p!}{(p-n+4)!(n-5)!}k_1^{n-5}\left[k_5B^{n-5} + (p-n+4)k_3k_4B\right] + \sum_{j\geq 6}k_1^{n-j}X_j,$$

where X_j is a homogeneous polynomial of degree j in the variables $k_2, k_3, \cdots, k_n$.

For the coefficients of the Schwarz functions $\varphi(z)$ and $\psi(w)$ we have $|c_n| \leq 1$ and $|d_n| \leq 1$ (e.g., see Duren [10]).

Comparing the corresponding coefficients of (2) and (4) yields

$$F_{n-1}(a_2, a_3, \cdots, a_n) = (A - B)K_{n-1}^{-1}(c_1, c_2, \cdots, c_{n-1}, B) \tag{6}$$

which under the assumption $a_k = 0,\ 2 \leq k \leq n-1$ we get

$$-(n-1)a_n = -(A - B)c_{n-1}. \tag{7}$$

Similarly, comparing the corresponding coefficients of (3) and (5) gives

$$F_{n-1}(b_2, b_3, \cdots, b_n) = (A - B)K_{n-1}^{-1}(d_1, d_2, \cdots, d_{n-1}, B) \tag{8}$$

which by the hypothesis, we obtain $-(n-1)b_n = -(A - B)d_{n-1}$.

Note that, for $a_k = 0,\ 2 \leq k \leq n-1$ we have $b_n = -a_n$ and therefore

$$(n-1)a_n = -(A - B)d_{n-1}. \tag{9}$$

Taking the absolute values of either of the equations (7) or (9) and dividing by $(n-1)$ we obtain the required bound

$$|a_n| \leq \frac{A - B}{n-1}. \quad \square$$

To prove our next theorem, we shall need the following well-known lemma (e.g., see Jahangiri [15] or Duren [10]).

Lemma 2.2. *Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{A}$ be a positive real part functions so that $Re(p(z)) > 0$ for $|z| < 1$. If $\alpha \geq -1/2$ then*

$$\left|p_2 + \alpha p_1^2\right| \leq 2 + \alpha |p_1|^2. \tag{10}$$

Corollary 2.3. *Let $\varphi(z) = \sum_{n=1}^{\infty} \varphi_n z^n \in \mathcal{A}$ be a Schwarz function so that $|\varphi(z)| < 1$ for $|z| < 1$. If $\gamma \geq 0$ then*

$$\left|\varphi_2 + \gamma \varphi_1^2\right| \leq 1 + (\gamma - 1)|\varphi_1|^2. \tag{11}$$

Proof. Set $p(z) = [1 + \varphi(z)]/[1 - \varphi(z)]$ where $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ is so that $Re(p(z)) > 0$ for $|z| < 1$. Comparing the corresponding coefficients of powers of z yields $p_1 = 2\varphi_1$ and $p_2 = 2(\varphi_2 + \varphi_1^2)$. Substituting for p_1 and p_2 in (10), we obtain

$$\left|2(\varphi_2 + \varphi_1^2) + \alpha(2\varphi_1)^2\right| \leq 2 + \alpha|2\varphi_1|^2$$

or

$$\left|\varphi_2 + (1 + 2\alpha)\varphi_1^2\right| \leq 1 + 2\alpha|\varphi_1|^2.$$

Now (11) follows upon substitution of $\gamma = 1 + 2\alpha \geq 0$ in the above inequality. □

In the following theorem, we see how relaxing the restrictions imposed on Theorem 2.1 reveals the unpredictability of the coefficients of bi-subordinate functions in $S[A, B]$.

Theorem 2.4. *For $-1 \leq B < A \leq 1$ if both functions f and its inverse map $g = f^{-1}$ are in $S[A, B]$, then*

(i). $$|a_2| \leq \begin{cases} \frac{A-B}{\sqrt{1+A}}, & \text{if } B \leq 0 < A; \\ A - B, & \text{otherwise}. \end{cases}$$

(ii). $$|a_3 - a_2^2| \leq \begin{cases} \frac{A-B}{2}\left[1 - \frac{1+A}{(A-B)^2}|a_2|^2\right], & \text{if } A \leq 0; \\ \frac{A-B}{2}, & \text{if } A > 0. \end{cases}$$

Proof. For $n = 2$, (6) and (8) imply $a_2 = (A - B)c_1$ and $b_2 = (A - B)d_1 = -a_2$. Taking absolute values of both sides of either of these equations, we obtain

$$|a_2| \leq A - B.$$

For $n = 3$, the equations (6) and (8), respectively, imply

$$a_2^2 - 2a_3 = (A - B)(Bc_1^2 - c_2) \tag{12}$$

and

$$-3a_2^2 + 2a_3 = (A - B)(Bd_1^2 - d_2). \tag{13}$$

Adding (12) and (13) yields

$$-2a_2^2 = -(A - B)\left[(c_2 - Bc_1^2) + (d_2 - Bd_1^2)\right].$$

Taking absolute values of both sides of the above equation, we obtain

$$2|a_2|^2 \leq (A - B)\left[\left|c_2 + (-B)c_1^2\right| + \left|d_2 + (-B)d_1^2\right|\right].$$

If $B \leq 0$, then by (11) we have

$$2|a_2|^2 \leq (A - B)\left[1 + (-B - 1)|c_1|^2 + 1 + (-B - 1)|d_1|^2\right].$$

Upon substituting $\frac{|a_2|^2}{(A-B)^2}$ for $|c_1|^2$ and $|d_1|^2$, we obtain

$$2|a_2|^2 \leq (A - B)\left[2 - \frac{2(1 + B)}{(A - B)^2}|a_2|^2\right] = 2(A - B) - \frac{2(1 + B)}{(A - B)}|a_2|^2.$$

A simple algebraic manipulation reveals that

$$|a_2| \le \frac{A-B}{\sqrt{1+A}}.$$

Obviously, for $A > 0$ we have

$$\frac{A-B}{\sqrt{1+A}} < A-B.$$

For the second part of the Theorem 2.4, rewrite equation (13) as

$$2\left(a_3 - a_2^2\right) = (A-B)[Bd_1^2 - d_2] + a_2^2.$$

Upon substituting $(A-B)^2 d_1^2$ for a_2^2 we obtain

$$2\left(a_3 - a_2^2\right) = -(A-B)[d_2 - Ad_1^2].$$

Taking the absolute values of both sides gives

$$2\left|a_3 - a_2^2\right| \le (A-B)\left|d_2 + (-A)d_1^2\right|.$$

If $A \le 0$, then by (11) we have

$$2\left|a_3 - a_2^2\right| \le (A-B)\left(1 + (-A-1)|d_1|^2\right)$$

which upon re-substituting for $|d_1|^2 = \frac{|a_2|^2}{(A-B)^2}$ we obtain

$$\left|a_3 - a_2^2\right| \le \frac{A-B}{2}\left[1 - \frac{1+A}{(A-B)^2}|a_2|^2\right].$$

For $A > 0$, we subtract (12) from (13) to get

$$4\left(a_3 - a_2^2\right) = (A-B)\left[B(d_1^2 - c_1^2) + (c_2 - d_2)\right].$$

Using the fact that $c_1^2 = d_1^2$ and taking the absolute values of both sides of the above equation, we obtain the desired inequality

$$\left|a_3 - a_2^2\right| \le \frac{(A-B)|c_2 - d_2|}{4} \le \frac{(A-B)(|c_2| + |d_2|)}{4} \le \frac{A-B}{2}. \quad \square$$

Remark 2.5. For different values of A and B, Theorem 2.4 demonstrates the unpredictability of the coefficients of the bi-subordinate functions. Determination of extremal functions for bi-univalent functions (in general) and for bi-subordinate functions (in particular) remains a challenge.

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