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Partial differential equations

When "blow-up" does not imply "concentration": A detour from Brézis–Merle's result

Lorsque blow-up *ne signifie pas « concentration » : un détour par rapport au résultat de Brézis–Merle*

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A R T I C L E   I N F O

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A B S T R A C T

The pioneering work by Brézis–Merle [3] applied to mean-field equations of Liouville type (1) (see below) implies that any unbounded sequence of solutions (i.e. a sequence of blow-up solutions) must exhibit only finitely many points (blow-up points) around which their "mass" concentrate. In this note, we describe some examples of blow-up solutions that violate such conclusion, in the sense that their mass may spread, as soon as we consider situations which mildly depart from Brézis–Merle's assumptions. The presence of a "residual" mass in blow-up phenomena was pointed out by Ohtsuka–Suzuki in [12], although such possibility was not substantiated by any explicit examples. We mention that for systems of Toda-type, this new phenomenon occurs rather naturally and it makes the calculation of the Leray Schauder degree much harder than the resolution of the single mean-field equation.

R É S U M É

Le travail pionnier de Brézis–Merle [3] appliqué aux équations de champ moyen de type Liouville (1) (voir ci-dessous) implique que toute suite non bornée (*blow-up suite*) montre un nombre fini de points (points de *blow-up*) autour desquels leur masse se concentre. Dans cette note, nous donnons quelques exemples de *blow-up suites* qui ne satisfont pas cette conclusion, dans le sens où leur masse s'étale au moment où on considère des situations qui s'écartent légèrement des hypothèses du travail de Brézis–Merle. La presence de masse « residuelle » dans les phénomènes de *blow-up* avait été remarquée auparavant par Ohtsuka–Suzuki [12] ; en revanche, aucun example explicite n'avait été proposé. Par rapport au system de Toda, ce nouveau phénomène apparaît plutôt naturellement et rend

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le calcul du degré de Leray–Schauder plus difficile que la résolution de la simple équation de champ moyen.

1. Mean field equation

The paper [3] by Brézis and Merle provided a pioneering study of the bubbling phenomenon for solutions to semilinear elliptic equations with exponential nonlinearity in two dimensions. As it turns out, their results apply rather nicely to mean-field equations of Liouville type over compact Riemann surfaces, as they arise naturally in several areas of mathematics and physics. More precisely, for (M, g) a given compact Riemann surface with area 1 and $\Delta_g = \Delta$, the corresponding Beltrami–Laplace operator, we consider the mean-field equation:

$$\Delta u + \rho\Big(\frac{h(x)\,\mathrm{e}^{u(x)}}{\int_M h(x)\,\mathrm{e}^{u(x)}} - 1\Big) = 4\pi \sum_{j=1}^{N} \alpha_j(\delta_{p_j} - 1) \text{ in } M, \tag{1}$$

where $h(x)$ is a positive continuous function, δ_{p_j} is the Dirac measure supported at p_j and $\alpha_j > -1$, for every $j = 1, \dots, N$.

Brézis–Merle's result applies to solutions to (1) when the Dirac measures are neglected (i.e. $\alpha_j = 0, \forall j = 1, \dots N$) and implies the following:

Theorem A *(Brézis–Merle). Suppose $\alpha_j = 0, \forall i = 1, \dots N$; and let u_k be a sequence of solutions for (1), then (along a subsequence) either*

(i) *$u_k \to u$ uniformly locally in $C^2(M\backslash\{p_1, \cdots, p_N\})$ or*
(ii) *u_k blows up in "sup-norm" and for a non-empty finite set S (blow-up set), the following holds:*

$$\rho \frac{h\,\mathrm{e}^{u_k}}{\int_M h\,\mathrm{e}^{u_k}} \to \sum_{q_j \in S} \gamma_j \cdot \delta_{q_j}, \text{ and } 4\pi \le \gamma_j.$$

Subsequently, the result above was completed by Li–Shafrir [11] who proved that actually all $\gamma_j = 8\pi$ (mass quantisation). The "concentration" property of blow-up solutions to (1) around finitely many points, as expressed by (ii), is the most important aspect of Brézis–Merle's analysis. Subsequently, both the "concentration" phenomenon as well as the "quantization" property were confirmed by Bartolucci–Tarantello [2] also when we take into account in (1) the Dirac measures.More precisely, if the Dirac measures are supported at the pole p_i, with weight $\alpha_i > -1, i = 1, \dots, N$; and blow-up occurs exactly at one of those points, say $p_i \in S$ for some $i = 1, \dots, N$ then the corresponding "local mass" γ is no longer equal to 8π, but instead $\gamma = 8(\alpha_i + 1)\pi$, see [2] and [1,4,6] for further details. In this short note, we wish to point out that such a "concentration phenomenon" may no longer hold when blow-up is caused by the collapse of two of the poles in (1) or when we consider solutions corresponding to a smooth approximations of the Dirac measures in (1).

More precisely, let $p_1 = p_1(t)$ depend on a parameter t and suppose that

$$p_1(t) \notin \{p_2, \cdots, p_N\} \text{ and } p_1(t) \to p_2 \text{ as } t \to 0. \tag{2}$$

For simplicity, we assume $\alpha_j \in \mathbb{N}$ and $h(x) \in C^1(M)$ to prove:

Theorem 1.1. *Assume that $\alpha_j \in \mathbb{N}$, $h(x) \in C^1(M)$ and $\rho \in (8\pi, 16\pi)$. Let $t_k \to 0$, then (1) with $p_1 = p_1(t_k)$ admits a solution u_k which blows up in sup-norm. Furthermore, along a subsequence, $u_k \to w$ uniformly locally in $C^2(M\backslash p_2)$ with w satisfying:*

$$\Delta w + (\rho - 8\pi)\Big(\frac{h(x)\,\mathrm{e}^{w(x)}}{\int_M h(x)\,\mathrm{e}^{w(x)}} - 1\Big) = 4\pi \sum_{j=3}^{N} \alpha_j(\delta_{p_j} - 1) + 4\pi(\alpha_1 + \alpha_2 - 2)(\delta_{p_2} - 1).$$

Demonstration. We provide only a sketch of the proof. To this purpose, let us consider (1) with $p_1 = p_1(t)$, $t > 0$, and compare it with the equation after collapse as given by:

$$\Delta u + \rho\Big(\frac{h(x)\,\mathrm{e}^{u(x)}}{\int_M h(x)\,\mathrm{e}^{u(x)}} - 1\Big) = 4\pi(\alpha_1 + \alpha_2)(\delta_{p_2} - 1) + \sum_{j=3}^{N} \alpha_j(\delta_{p_j} - 1). \tag{3}$$

These two equations (1) and (3) have different topological degrees, see [5,7]. Thus for any fixed $\rho \in (8\pi, 16\pi)$, there exists a blow-up sequence of solution u_k of (1) with $t_k \to 0$ as $k \to +\infty$. If u_k does not blow up at p_2, then by the Brézis–Merle result or the Bartolucci–Tarantello one we see that u_k must concentrate and $\rho \in 8\pi\mathbb{N}$, a contradiction with the assumption that $\rho \in (8\pi, 16\pi)$. Hence u_k must blow up at p_2.

Let us define the "local mass" of u_k at the blowup point p_2 by

$$\sigma(p_2) = \lim_{r\to 0}\lim_{k\to+\infty} \rho \int_{B(p_2,r)} \frac{h(x)\, e^{u_k(x)}}{\int_M h(x)\, e^{u_k(x)}}.$$

Claim:

$$\sigma(p_2) \in 8\pi\mathbb{N}. \tag{4}$$

Claim (4) is highly non-trivial, as it involves a Pohozaev identity and the Riemann–Hurwitz formulae, see [10]. As a consequence of (4) and the assumption $\rho \in (8\pi, 16\pi)$, we find that: $\sigma(p_2) = 8\pi$, and p_2 must be the only blow-up point. Furthermore the mass of u_k cannot concentrate around p_2.

In other words, $u_k \to u$ in $C^2(M\backslash p_1, \dots, p_N)$ with u satisfying:

$$\Delta u + (\rho - 8\pi)(\frac{h\, e^u}{\int h\, e^u} - 1) = 4\pi \sum_{j=3}^{N} \alpha_j(\delta_{p_j} - 1) + 4\pi(\alpha_1 + \alpha_2 - 2)(\delta_{p_2} - 1),$$

as claimed. □

Remark. The phenomenon described by Theorem 1.1 holds also for $\rho \in (8\pi k, 8\pi(k+1))$ for any $k \ge 2$. But in this case the local mass $\sigma(p_2) = 8\pi m$ with m a positive integer such that: $2m \le \alpha_1 + \alpha_2$.

A similar phenomenon occurs when we consider problem (1) over the Riemann sphere $M = (S, g_0)$, with a single point singularity located at p and multiplicity $\alpha > -1$. Actually, after rotation, we can always suppose that p coincides with the south pole, so that by using the stereographic projection from the north pole: $\pi : S^2 \to \mathbb{R}^2$, problem (1) can be equivalently formulated in terms of the function:

$$v(x) = u(y) - \ln \int_M e^u + \ln \frac{1}{(1+|x|^2)^{(\alpha+2)}}$$

with $x = \pi(y)$, $y \in S^2$, as the following problem over the plane:

$$\begin{cases} \Delta v + (1+|x|^2)^a e^v = 4\pi\alpha\delta_0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} (1+|x|^2)^a e^v = \rho, \end{cases} \tag{5}$$

where

$$a = \frac{\rho}{4\pi} - (\alpha + 2). \tag{6}$$

As shown in [14], when $-1 < \alpha \neq 0$ then a necessary condition for the solvability of (5) requires that,

$$\rho \in (0, 8\pi(1+\alpha^-)) \cup (8\pi(1+\alpha^+), \infty), \tag{7}$$

here as usual $\alpha^- = \min\{0, \alpha\}$ and $\alpha^+ = \max\{0, \alpha\}$. We consider a situation where (7) fails, more precisely we assume that

$$\alpha > 1 \quad \text{and} \quad \rho \in (8\pi, 16\pi) \cap (4\pi(1+\alpha), +\infty). \tag{8}$$

Therefore, when (8) holds then problem (5) admits no solutions. On the other hand, if we replace the Dirac measure in (5) with the standard smooth mollifier:

$$g_k(x) = \frac{\lambda_k}{\pi(1+\lambda_k|x|^2)^2}, \quad \lambda_k \to +\infty \quad \text{as } k \to +\infty, \tag{9}$$

(an approximation of δ_0 in the sense of distributions), then existence is restored in view of the available degree formula on S^2 of Chen–Lin [5,7]. So, we can claim the existence of a sequence v_k satisfying the following:

$$\begin{cases} \Delta v_k + (1+|x|^2)^a e^{v_k} = 4\alpha \frac{\lambda_k}{(1+\lambda_k|x|^2)^2} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} (1+|x|^2)^a e^{v_k} = \rho. \end{cases} \tag{10}$$

Obviously, v_k must blow-up in sup-norm and to describe its asymptotic behavior as $k \to +\infty$, we consider the new sequence:

$$\xi_k(x) = v_k(x) - \alpha \ln(\frac{1}{\lambda_k} + |x|^2)$$

satisfying:

$$\begin{cases} \Delta \xi_k = (\frac{1}{\lambda_k} + |x|^2)^\alpha (1+|x|^2)^a e^{\xi_k} \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} (\frac{1}{\lambda_k} + |x|^2)^\alpha (1+|x|^2)^a e^{\xi_k} = \rho. \end{cases} \tag{11}$$

Notice that problem (11) fails to satisfy the assumptions required by Brézis–Merle in [3], and in fact we find:

Theorem 1.2. *Assume (6) and (8) and let ξ_k satisfy (11) then (along a subsequence) the following holds:*

$$\forall \varepsilon > 0 \quad \sup_{|x|<\varepsilon} \xi_k \to +\infty \quad \textit{and} \quad \xi_k \to \xi \textit{ in } C^2_{loc}(\mathbb{R}^2 \setminus \{0\}) \tag{12a}$$

with ξ satisfying:

$$\begin{cases} -\Delta \xi = |x|^{2\alpha} (1+|x|^2)^a e^{\xi} + 8\pi \delta_0 \quad \textit{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |x|^{2\alpha} (1+|x|^2)^a e^{\xi} = \rho - 8\pi. \end{cases} \tag{12b}$$

In terms of problem (1) over the sphere, we have obtained a sequence u_k satisfying:

$$\Delta u_k + \rho(\frac{e^{u_k}}{\int_{S^2} e^{u_k}} - \frac{1}{4\pi}) = 4\pi\alpha(h_k - \frac{1}{4\pi}) \tag{13}$$

with $h_k \to \delta_p$ in the sense of distribution, such that (along a subsequence) the following holds:

Corollary 1.3.

$$\max_{S^2} u_k \to +\infty, \quad u_k \to u \textit{ in } C^2_{loc}(S^2 \setminus \{p\})$$

with u satisfying:

$$\Delta u + (\rho - 8\pi)(\frac{e^u}{\int_{S^2} e^u} - \frac{1}{4\pi}) = 4\pi(\alpha - 2)\delta_p \textit{ on } S^2.$$

Again the most delicate part in the proof of Theorem 1.2 concerns the fact that ξ_k blows-up exactly at the origin with "local mass" equal to 8π. To establish such a property, one needs to obtain uniform Harnack type estimates to be combined with a suitable Pohozaev-type inequality. Details will be provided in a forthcoming paper.

2. Toda system

The presence of a "residual mass" in blow-up phenomena (as discussed in §1), appears more significantly in the context of systems. For simplicity, we restrict to 2×2 systems. In the planar case, we mention for example the degenerate system yielding the Cosmic String Equation studied in [15] or the Toda-type systems analyzed in [13]. For those problems typically, one finds that, although both components blow-up at the same point (usually the origin), only the component with the fastest rate of blow-up concentrates, while the other component admits a residual part that passes to the limit to satisfy a "limiting" singular Liouville-type equation. It is interesting to note that the "limiting" equations obtained in this way have a geometrical meaning. Namely, their solutions provide the conformal factor for a metric in the Riemann sphere with constant Gauss curvature and conical singularities. Therefore geometrical obstructions may prevent such type of blow-up behavior. On the other hand, it has been observed in [13] for Toda-type systems that a residual mass does occur even in the blow-up behavior of radial solutions. More importantly, such a "residual mass" phenomenon represents a serious difficulty in the calculation of the Leray–Schauder degree for the conformal Toda system over a surface M. More precisely, we consider the following problem:

$$\begin{cases} \Delta u_1 + 2\rho_1(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - 1) - \rho_2(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - 1) = 4\pi \sum_{p\in S_1} \alpha_j(\delta_p - 1), \\ \Delta u_2 + 2\rho_2(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - 1) - \rho_1(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - 1) = 4\pi \sum_{p\in S_2} \beta_j(\delta_q - 1) \end{cases} \quad \text{in } M \tag{14}$$

where h_i are positive C^1 function on M. It was proved in [8] that if $\rho_i \notin 4\pi\mathbb{N}$, $i = 1, 2$, then all solutions to (14) are uniformly bounded. So that, the Leray–Schauder degree for the system (14) is well defined for such a range of parameters. On the basis of the above observations, there is no hope to reduce the calculation of such a topological degree to that of a single equation (successfully carried out in [5]), even when we fix $\rho_2 \in (4\pi k, 4\pi(k+1))$ and deform ρ_1 across the value $4\pi\ell$. Then blow-up would occur and the degree jump does correspond to the contribution to the degree by these bubbling solutions. However, while one can calculate the degree of a bubbling solution-sequence when only the first component blows up and concentrate, unfortunately such calculation becomes rather involved when (as discussed above) both components blow up, but only the first component concentrates. We discuss next an even further situation, namely we see that the concentration of mass may fail by both components. To this purpose, we let d_{jk} = the degree of (14) when $4\pi j < \rho_1 < 4\pi(j+1)$ and $4\pi k < \rho_2 < 4\pi(k+1)$. Define

$$g_j^{(2)}(x) = \sum_{k=0}^{\infty} d_{jk} x^k.$$

By a Theorem of Chen–Lin [7], we know that if $\beta_j = 1$ $\forall j$ then $g_0^{(2)}(x) = (1-x)^{\chi(M)-1}(1+x)^{|S_2|}$, where $\chi(M)$ denotes the Euler characteristic of M and $|S|$ is the cardinality of the set S. Furthermore, in [9,10] it has been proved the following theorem.

Theorem C. *Suppose $\alpha_i = \beta_j = 1$. Then*

$$g_1^{(2)}(x) = (1-x)^{\chi(M)-1}\left\{(1+x)^{|S_2|} - [\chi(M) - |S_1 \cup S_2|](1+x)^{|S_2|+1} - |S_2 \setminus S_1|(1+x)^{|S_2|-1}(1+x+x^2)\right\}.$$

Now, let $q \in S_1 \setminus S_2$ and $q(t) \in M$ be such that $q(t) \notin S_1 \cup S_2$ and $q(t) \to q$ as $t \to 0$. Set $S_1(t) = S_1$ and $S_2(t) = S_2 \cup \{q(t)\}$ and consider:

$$\begin{cases} \Delta u_1 + 2\rho_1\left(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - 1\right) - \rho_2\left(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - 1\right) = 4\pi \sum_{p \in S_1} (\delta_p - 1), \\ \Delta u_2 + 2\rho_2\left(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - 1\right) - \rho_1\left(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - 1\right) = 4\pi \sum_{p \in S_2(t)} (\delta_q - 1). \end{cases} \quad (15)$$

Since $S_2(t) = S_2 \cup \{q(t)\}$, then the system (15) admits a degree jump as $t \to 0$. Therefore, for any $\rho_1 \in (4\pi, 8\pi)$ and $\rho_2 \in (4\pi k, 4\pi(k+1))$, there is a blowup sequence of solutions $(u_{k,1}, u_{k,2})$ of (15) with $t = t_k \to 0$. Note that there is no collapsing of singularities like in the case of equation (1), instead one vortex $q(t_k) \in S_2(t_k) \setminus S_1$ from the second equation converges to a vortex $q \in S_1$ of the first equation. We have:

Theorem 2.1. *There exists $t_k \to 0$ and a sequence of solutions $u_k = (u_{1,k}, u_{2,k})$ of (15) such that the following holds:*

(i) *$u_{1,k}$ and $u_{2,k}$ both blow up only at q, with "local mass":*

$$\sigma_1(q) = \lim_{r \to 0} \lim_{k \to +\infty} \rho_1 \int_{B(q,r)} \frac{h_1 e^{u_{k,1}}}{\int_M h_1 e^{u_{k,1}}} = 4\pi, \quad \textit{and}$$

$$\sigma_2(q) = \lim_{r \to 0} \lim_{k \to +\infty} \rho_2 \int_{B(q,r)} \frac{h_2 e^{u_{k,2}}}{\int_M h_2 e^{u_{k,2}}} = 4\pi.$$

(ii) *$(u_{1,k}, u_{2,k})$ converges to $(\hat{u}_1, \hat{u}_2)$ in $C^2(M \setminus S_1 \cup S_2)$ and $(\hat{u}_1, \hat{u}_2)$ satisfies*

$$\begin{cases} \Delta \hat{u}_1 + 2(\rho_1 - 4\pi)\left(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - 1\right) - (\rho_2 - 4\pi)\left(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - 1\right) = 4\pi \sum_{p \in S_1 \setminus \{q\}} (\delta_p - 1), \\ \Delta \hat{u}_2 + 2(\rho_2 - 4\pi)\left(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - 1\right) - (\rho_1 - 4\pi)\left(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - 1\right) = 4\pi \sum_{p \in S_2} (\delta_p - 1). \end{cases}$$

Again, the difficult part is to establish that $\sigma_i(q) \in 4\pi\mathbb{N}$ and to see how such quantities relate to each other. The proof of this fact even in more general situations will appear later.

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