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Partial differential equations

A note on Sylvester's proof of discreteness of interior transmission eigenvalues

Une remarque sur la preuve de la distribution discrète des valeurs propres intérieures de transmission de Sylvester

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ABSTRACT

It has been shown by Sylvester (2011) [10] that the set of interior transmission eigenvalues forms a discrete set if the contrast does not change its sign in a neighborhood of the boundary. In this short note, we give a more elementary proof of this fact using the classical inf–sup conditions of Babuška–Brezzi.

RÉSUMÉ

Il a été démontré par Sylvester (2011) [10] que l'ensemble des valeurs propres intérieures de transmission constitue un ensemble discret si le contraste ne change pas de signe dans un voisinage du bord. Nous donnons une preuve plus élémentaire de ce fait en utilisant les conditions classiques « inf–sup » de Babuška–Brezzi.

1. Introduction

Transmission eigenvalue problems are non-selfadjoint problems that occur in the study of the scattering of time-harmonic waves by inhomogeneous media. The scalar case in acoustics leads to the problem to determine $k > 0$ and corresponding nontrivial pairs (u, w) such that

$$\Delta w + k^2 w = 0 \text{ in D}, \quad \Delta u + k^2(1+q)u = 0 \text{ in } D, \tag{1.1}$$

$$u = w \text{ on } \partial D, \quad \partial u/\partial \nu = \partial w/\partial \nu \text{ on } \partial D. \tag{1.2}$$

As discussed in, e.g., [7] this problem is neither self-adjoint nor elliptic. Therefore, standard results from functional analysis don't apply. The first question, answered in many papers starting with [3], concerns the discreteness of the spectrum. The

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assumption that the contrast q does not change its sign in the domain D has been weakened in [10] to the assumption that it does not change its sign on some neighborhood of the boundary ∂D. For the, in some sense simpler (because elliptic), anisotropic case, this has been assumed in, e.g., [1,6]. For an overview on transmission eigenvalue problems, we refer to [2,5,9] (see also [7]).

In this note, we want to show that Sylvester's result [10] can also be obtained by the use of the classical inf–sup conditions of Babuška–Brezzi (see [8]), which are closely related to the T-coercivity approach of, e.g., [1].

2. Discreteness of the spectrum

As it is well known the eigenvalue problem (1.1), (1.2) is degenerated in the sense that we look for $u, w \in L^2(D)$ such that $u - w \in H_0^2(D) = \{v \in H^2(D) : v = \partial v/\partial \nu = 0 \text{ on } \partial D\}$. We set $\lambda = -k^2$ and $v = (u - w)/\lambda$. Then the problem is to determine $\lambda \in \mathbb{C}$ and a nontrivial pair $(v, w) \in H_0^2(D) \times L^2(D)$ such that

$$\Delta w - \lambda w = 0 \quad \text{and} \quad \Delta v - \lambda(1+q)v = qw \quad \text{in } D \tag{2.3}$$

in the following sense:

$$\int_D \left[\Delta\overline{\psi} - \lambda\overline{\psi}\right] w \,\mathrm{d}x = 0\,, \qquad \int_D \left[\Delta v - \lambda(1+q)v - qw\right]\overline{\phi}\,\mathrm{d}x = 0$$

for all $\psi \in H_0^2(D)$ and $\phi \in L^2(D)$.

Definition 2.1. $\lambda \in \mathbb{C}$ is called interior transmission eigenvalue if there exists a non-trivial pair $(v, w) \in X = H_0^2(D) \times L^2(D)$ such that (2.3) holds in the variational sense.

We equip X with the norm $\|(v, w)\|_X = \|v\|_{H^2(D)} + \|w\|_{L^2(D)}$ and the corresponding inner product $\langle\cdot,\cdot\rangle_X$.
For any $\lambda \in \mathbb{C}$ we define the sesquilinear form $a_\lambda : X \times X \to \mathbb{C}$ by

$$a_\lambda(v, w; \psi, \phi) \;=\; \int_D (\Delta\overline{\psi} - \lambda\overline{\psi})\, w\,\mathrm{d}x \;+\; \int_D (\Delta v - \lambda(1+q)v)\,\overline{\phi} - q\,w\,\overline{\phi}\,\mathrm{d}x$$

for $(v, w) \in X$ and $(\psi, \phi) \in X$.
Then λ is an eigenvalue if there exists a nontrivial pair $(v, w) \in X$ with $a_\lambda(v, w; \psi, \phi) = 0$ for all $(\psi, \phi) \in X$.
We define also the following auxiliary form $\hat{a}_\lambda$ by

$$\hat{a}_\lambda(v, w; \psi, \phi) \;=\; \int_D (\Delta\overline{\psi} - \lambda\overline{\psi})\, w\,\mathrm{d}x \;+\; \int_D (\Delta v - \lambda v)\,\overline{\phi} - q\,w\,\overline{\phi}\,\mathrm{d}x$$

for $(v, w), (\psi, \phi) \in X$. The representation theorem of Riesz yields the existence of bounded operators $A_\lambda, \hat{A}_\lambda : X \to X$ such that

$$a_\lambda(v, w; \psi, \phi) \;=\; \big\langle A_\lambda(v, w); (\psi, \phi)\big\rangle_X \quad \text{for all } (v, w), (\psi, \phi) \in X\,, \tag{2.4}$$

and, analogously, the operator $\hat{A}_\lambda$ is defined. We note that λ is an eigenvalue if, and only if, A_λ fails to be injective.
We make the following assumption:

Assumption 2.2. There exists $q_0 > 0$ and some neighborhood[1] R of ∂D such that $q \ge q_0$ on R or $q \le -q_0$ on R.

We will need the following lemma from the theory of the Helmholtz equation.

Lemma 2.3. *Let $q \in L^\infty(D)$ satisfy Assumption 2.2. Then there exist $\hat{c} > 0$ and $d > 0$ such that for all $\lambda > 0$ the following estimate holds:*

$$\int_{D\setminus R} |w|^2\,\mathrm{d}x \;\le\; \hat{c}\,\mathrm{e}^{-2d\sqrt{\lambda}} \int_R |q|\,|w|^2\,\mathrm{d}x \tag{2.5}$$

for all solutions $w \in L^2(D)$ of $\Delta w - \lambda w = 0$ in D.

[1] That is, an open subdomain $R \subset D$ with $\partial D \subset \overline{R}$.

Proof. We choose a neighborhood R' of ∂D with $d = \mathrm{dist}(D \setminus R, R') > 0$ and a function $\rho \in C^\infty(D)$ with compact support in D and $\rho = 1$ in $D \setminus R'$. We apply Green's representation theorem (see, e.g., [4]) to ρw in D where w satisfies $\Delta w - \lambda w = 0$ in D which yields

$$\begin{aligned}
\rho(x)\, w(x) &= -\int\limits_D \big[\Delta(\rho w)(y) - \lambda(\rho w)(y)\big] \frac{\exp(-\sqrt{\lambda}|x-y|)}{4\pi\,|x-y|}\,\mathrm{d}y \\
&= -\int\limits_{R'} \big[2\,\nabla\rho(y)\cdot\nabla w(y) + w(y)\,\Delta\rho(y)\big] \frac{\exp(-\sqrt{\lambda}|x-y|)}{4\pi\,|x-y|}\,\mathrm{d}y \\
&= \int\limits_{R'} \left[2\,\mathrm{div}_y\left(\nabla\rho(y)\,\frac{\exp(-\sqrt{\lambda}|x-y|)}{4\pi\,|x-y|}\right) - \Delta\rho(y)\,\frac{\exp(-\sqrt{\lambda}|x-y|)}{4\pi\,|x-y|}\right] w(y)\,\mathrm{d}y\,.
\end{aligned}$$

For $x \in D \setminus R$ we conclude that

$$\big|w(x)\big| \;\le\; c_1\,\mathrm{e}^{-d\sqrt{\lambda}} \int\limits_{R'} \big|w(y)\big|\,\mathrm{d}y$$

for some $c_1 > 0$ which depends only on D, R, R', and ρ, and thus

$$\big|w(x)\big|^2 \;\le\; c_1^2\,\mathrm{e}^{-2d\sqrt{\lambda}}\,|R| \int\limits_R \big|w(y)\big|^2\,\mathrm{d}y \;\le\; \frac{c_1^2\,|R|}{q_0}\,\mathrm{e}^{-2d\sqrt{\lambda}} \int\limits_R |q(y)|\,\big|w(y)\big|^2\,\mathrm{d}y\,.$$

Integration with respect to x over $D \setminus R$ yields the assertion. $\square$

We show the following inf–sup condition.

Theorem 2.4. *There exists $\lambda_0 > 0$ and $c > 0$ such that for all $\lambda \ge \lambda_0$*

$$\sup_{(\psi,\phi)\neq 0} \frac{\big|\hat{a}_\lambda(v,w;\psi,\phi)\big|}{\|(\psi,\phi)\|_X} \;\ge\; c\,\|(v,w)\|_X \quad \textit{for all } (v,w) \in X\,. \tag{2.6}$$

Proof. We fix λ_0 such that

$$\int\limits_{D\setminus R} |q|\,|w|^2\,\mathrm{d}x \;\le\; \|q\|_\infty \int\limits_{D\setminus R} |w|^2\,\mathrm{d}x \;\le\; \frac{1}{2}\int\limits_R |q|\,|w|^2\,\mathrm{d}x \tag{2.7}$$

for all solutions to $\Delta w - \lambda w = 0$ in D and all $\lambda \ge \lambda_0$. This is possible by the estimate (2.5) of Lemma 2.3. If a constant c with (2.6) does not exist, there exists a sequence $(v_j, w_j) \in X$ with $\|(v_j, w_j)\|_X = 1$ and

$$\sup_{(\psi,\phi)\neq 0} \frac{\big|\hat{a}_\lambda(v_j,w_j;\psi,\phi)\big|}{\|(\psi,\phi)\|_X} \;\longrightarrow\; 0\,, \quad j\to\infty\,. \tag{2.8}$$

There exist weakly convergent subsequences $w_j \rightharpoonup w$ in $L^2(D)$ and $v_j \rightharpoonup v$ in $H^2(D)$ for some $(v,w) \in X$. From (2.8) we observe that (v,w) satisfies $\Delta w - \lambda w = 0$ and $\Delta v - \lambda v = qw$ in D.

In the first part, we show again that v and w vanish.

From $\mathrm{Re}\,\hat{a}_\lambda(v,w;-v,w) = 0$ we conclude that $\int_D q\,|w|^2 \mathrm{d}x = 0$. The estimate (2.7) yields

$$\int\limits_R |q|\,|w|^2\mathrm{d}x = \left|\int\limits_R q\,|w|^2\mathrm{d}x\right| \;=\; \left|\int\limits_{D\setminus R} q\,|w|^2\mathrm{d}x\right| \;\le\; \int\limits_{D\setminus R} |q|\,|w|^2\mathrm{d}x \;\le\; \frac{1}{2}\int\limits_R |q|\,|w|^2\mathrm{d}x$$

and thus $w = 0$ on R. Analytic continuation yields $w = 0$ in all of D and thus also $v = 0$ by $0 = \hat{a}_\lambda(v,w;0,v) = \int_D(\Delta v - \lambda v)\overline{v}\,\mathrm{d}x = -\int_D(|\nabla v|^2 + \lambda|v|^2)\,\mathrm{d}x$.

In the second part, we prove a contradiction.

We choose a neighborhood R' of ∂D with closure in $R \cup \partial D$ and a non-negative function $\rho_1 \in C^\infty(D)$ with $\rho_1 = 0$ in $D \setminus R$ and $\rho_1 = 1$ in R' and substitute $\psi = \rho_1 v_j$ and $\phi = -\rho_1 w_j$ in (2.8). Then, because $(-\rho_1 w_j, \rho_1 v_j)$ is bounded in X,

$$\int\limits_R \big[\Delta(\rho_1\overline{v_j}) - \lambda\rho_1\overline{v_j}\big]\,w_j\,\mathrm{d}x - \int\limits_R (\Delta v_j - \lambda v_j)\,\rho_1\,\overline{w_j} - q\rho_1|w_j|^2\,\mathrm{d}x$$

tends to zero, thus

$$\operatorname{Re}\int_R \big[2\,w_j\,\nabla\rho_1\cdot\nabla\overline{v_j}+\overline{v_j}\,w_j\,\Delta\rho_1+q\,\rho_1|w_j|^2\big]\,\mathrm{d}x \longrightarrow 0\,. \tag{2.9}$$

Since v_j converges weakly to zero in $H^2(D)$, it converges to zero in the norm of $H^1(D)$. Therefore, the first two terms converge to zero, thus also $\int_R q\,\rho_1\,|w_j|^2\,\mathrm{d}x \to 0$. Since q is of one sign on R and $|q|\rho_1 \geq q_0$ on R' we conclude that w_j tends to zero in $L^2(R')$.

Now we choose a third neighborhood R'' of ∂D with closure in $R'\cup\partial D$ and a non-negative function $\rho_2\in C^\infty(D)$ with $\rho_2=0$ in R'' and $\rho_2=1$ in $D\setminus R'$. We determine $z_j\in H^2(D)$ with $\Delta z_j-\lambda z_j=w_j$ in D and $z_j=0$ on ∂D. We substitute $\phi=0$ and $\psi=\rho_2 z_j$ in (2.8) which yields (note that $(\rho_2 z_j)$ is bounded in $H^2(D)$)

$$\int_{D\setminus R''}\big[\Delta(\rho_2\overline{z_j})-\lambda\,\rho_2\overline{z_j}\big]\,w_j\,\mathrm{d}x \longrightarrow 0\,,$$

that is,

$$\int_{D\setminus R''}\big[\rho_2|w_j|^2+2\,(\nabla\rho_2\cdot\nabla\overline{z_j})\,w_j+\overline{z_j}\,\Delta\rho_2\,w_j\big]\,\mathrm{d}x \longrightarrow 0\,.$$

Since $w_j\rightharpoonup 0$ in $L^2(D)$, we conclude that $z_j\rightharpoonup 0$ in $H^2(D)$ and thus $z_j\to 0$ in $H^1(D)$. Furthermore, we note that $\rho_2=1$ in $D\setminus R'$ and thus $\int_{D\setminus R'}|w_j|^2\mathrm{d}x \longrightarrow 0$.

Altogether, we have shown that $w_j\to 0$ in $L^2(D)$.

Finally, set $\psi=0$ and $\phi=(\Delta v_j-\lambda v_j)$ in (2.8) which yields

$$\frac{1}{\|\Delta v_j-\lambda v_j\|_{L^2(D)}}\int_D |\Delta v_j-\lambda v_j|^2-q\,w_j(\Delta\overline{v_j}-\lambda\overline{v_j})\,\mathrm{d}x \longrightarrow 0$$

that is,

$$\|\Delta v_j-\lambda v_j\|_{L^2(D)} - \int_D q\,w_j\,\frac{\Delta\overline{v_j}-\lambda\overline{v_j}}{\|\Delta v_j-\lambda v_j\|_{L^2(D)}}\,\mathrm{d}x \longrightarrow 0\,,$$

which implies convergence $\Delta v_j-\lambda v_j\to 0$ in $L^2(D)$. Therefore, Δv_j tends to zero in $L^2(D)$ which is equivalent to $v_j\to 0$ in $H^2(D)$.

Altogether we have shown $(w_j,v_j)\to 0$ in X, which is impossible since its norm is one. □

Corollary 2.5. *Let $\lambda_0>0$ such that the inf–sup condition (2.6) of Theorem 2.4 holds. Then the operator $\hat{A}_\lambda : X\to X$ is self-adjoint and an isomorphism from X onto itself.*

Proof. This follows again from a generalized Lax–Milgram theorem (see, e.g., [8]). Note that the non-degeneracy condition holds as well because $\hat{a}_\lambda$ is Hermitian. □

Theorem 2.6. *For any $\lambda,\mu\in\mathbb{R}$ the differences $A_\mu-\hat{A}_\lambda$ and $A_\mu-A_\lambda$ are compact.*

Proof. Let $(v_j,w_j)\in X$ converge to zero weakly in X and let $(\psi,\phi)\in X$ with $\|(\psi,\phi)\|_X=1$. Note that

$$(a_\mu-\hat{a}_\lambda)(v_j,w_j;\psi,\phi) \;=\; (\lambda-\mu)\int_D \overline{\psi}\,w_j\,\mathrm{d}x \;+\; \int_D\big[\lambda-\mu(1+q)\big]\,v_j\,\overline{\phi}\,\mathrm{d}x\,.$$

$v_j\rightharpoonup 0$ in $H^2(D)$ implies norm convergence $v_j\to 0$ in $L^2(D)$, and thus

$$\left|\int_D\big[\lambda-\mu(1+q)\big]\,v_j\,\overline{\phi}\,\mathrm{d}x\right| \;\leq\; \|\lambda-\mu(1+q)\|_{L^\infty(D)}\|v_j\|_{L^2(D)}\|\phi\|_{L^2(D)} \;\leq\; \|\lambda-\mu(1+q)\|_{L^\infty(D)}\|v_j\|_{L^2(D)}\,.$$

Furthermore, define $z_j\in H^1(D)$ with $\Delta z_j=w_j$ in D and $z_j=0$ on ∂D. Then $z_j\rightharpoonup 0$ in $H^1(D)$ and thus $z_j\to 0$ in $L^2(D)$.

Therefore,

$$\left|\int_D \overline{\psi}\, w_j\, dx\right| = \left|\int_D \overline{\psi}\, \Delta z_j\, dx\right| = \left|\int_D \Delta\overline{\psi}\, z_j\, dx\right| \le \|\Delta\psi\|_{L^2(D)}\|z_j\|_{L^2(D)} \le \|z_j\|_{L^2(D)}$$

and altogether

$$\sup_{\|(\psi,\phi)\|_X=1} \left|\left(a_\mu - \hat{a}_\lambda\right)(v_j, w_j; \psi, \phi)\right| \le c\left[\|z_j\|_{L^2(D)} + \|v_j\|_{L^2(D)}\right] \longrightarrow 0\,.$$

This implies compactness of $A_\mu - \hat{A}_\lambda$. The proof for $A_\mu - A_\lambda$ follows the same lines. $\square$

Theorem 2.7. *For sufficiently large $\lambda > 0$ the operator A_λ is an isomorphism from X onto itself.*

Proof. It is sufficient to prove injectivity because $\hat{A}_\lambda$ is an isomorphism and $\hat{A}_\lambda - A_\lambda$ is compact.

Assume that there exists a sequence $\lambda_j \to \infty$ and functions $(v_j, w_j) \in X$ with $\|(v_j, w_j)\|_X = 1$ and $A_{\lambda_j}(v_j, w_j) = 0$. Therefore, the functions $w_j \in L^2(D)$ and $v_j \in H_0^2(D)$ satisfy the equations

$$\Delta w_j - \lambda_j w_j = 0 \quad \text{and} \quad \Delta v_j - \lambda_j(1+q)v_j = q w_j \quad \text{in } D\,. \tag{2.10}$$

Defining $\rho_j = \|q\|_\infty \hat{c}\exp(-2d\sqrt{\lambda_j})$ and splitting the region of integration into R and $D \setminus R$ yields by Lemma 2.3 that

$$(1-\rho_j)\int_R |q||w_j|^2 dx \le \int_D q\,|w_j|^2 dx \le (1+\rho_j)\int_R |q||w_j|^2 dx\,. \tag{2.11}$$

Multiplication of the second equation of (2.10) by $\overline{w_j}$, integrating and using Green's second theorem yields

$$\int_D q\,\overline{w_j}\left[\lambda_j v_j + w_j\right] dx = 0\,. \tag{2.12}$$

Multiplication of the second equation of (2.10) by $\overline{v_j}$, integrating and using Green's first theorem yields

$$\int_D \left[|\nabla v_j|^2 + \lambda_j(1+q)|v_j|^2\right] dx = -\int_D q\, w_j \overline{v_j}\, dx = \frac{1}{\lambda_j}\int_D q\,|w_j|^2 dx\,. \tag{2.13}$$

Now we distinguish between two cases.

Case 1: q is negative on R. Then the right integral in (2.13) is negative as it follows from Lemma 2.3 because

$$-\int_D q\,|w_j|^2 dx \ge -\int_R q\,|w_j|^2 dx - \int_{D\setminus R} |q|\,|w_j|^2 dx \ge (1-\rho_j)\int_R |q|\,|w_j|^2 dx > 0\,.$$

This contradicts (2.13).

Case 2: q is positive on R. From (2.12), we conclude

$$\begin{aligned}
(1-\rho_j)\int_R q|w_j|^2 dx &\le \int_D q\,|w_j|^2 dx = -\lambda_j \int_D q\, w_j v_j\, dx \\
&\le \lambda_j \int_{D\setminus R} |q|\,|w_j|\,|v_j|\, dx + \lambda_j \int_R q\,|w_j|\,|v_j|\, dx \\
&\le \lambda_j \left[\int_{D\setminus R} |q|\,|w_j|^2\, dx \int_{D\setminus R} |q|\,|v_j|^2\, dx\right]^{1/2} \\
&\quad + \lambda_j \left[\int_R q\,|w_j|^2\, dx \int_R q\,|v_j|^2\, dx\right]^{1/2} \\
&\le \lambda_j \sqrt{\int_R q\,|w_j|^2\, dx}\left[\rho_j\,\|q\|_\infty^{1/2} + \sqrt{\int_R q\,|v_j|^2\, dx}\right]
\end{aligned}$$

where we used that $\int_{D\setminus R}|q|\,|v_j|^2\,dx \le \|q\|_\infty$. Therefore, we conclude that

$$\sqrt{\int\limits_R q\,|w_j|^2\,dx} \;\le\; \frac{\lambda_j}{1-\rho_j}\left[\rho_j\sqrt{\|q\|_\infty} \;+\; \sqrt{\int\limits_R q\,|v_j|^2\,dx}\,\right].$$

Now we square and use the estimate $(a+b)^2 \le \big(1+1/\rho_j)a^2+(1+\rho_j)b^2 = (1+\rho_j)\big[a^2/\rho_j+b^2\big]$ for obvious meaning of a and b. We arrive at

$$\int\limits_R q\,|w_j|^2\,dx \;\le\; \frac{(1+\rho_j)\,\lambda_j^2}{(1-\rho_j)^2}\left[\rho_j\,\|q\|_\infty \;+\; \int\limits_R q\,|v_j|^2\,dx\right]. \tag{2.14}$$

We substitute this for the right hand side of (2.13):

$$\lambda_j\int\limits_R (1+q)\,|v_j|^2\,dx \;\le\; \frac{1+\rho_j}{\lambda_j}\int\limits_R q\,|w_j|^2 dx \;\le\; \frac{(1+\rho_j)^2\,\lambda_j}{(1-\rho_j)^2}\left[\rho_j\,\|q\|_\infty \;+\; \int\limits_R q\,|v_j|^2\,dx\right]$$

and thus

$$\begin{aligned}\int\limits_R |v_j|^2\,dx &\le \frac{(1+\rho_j)^2}{(1-\rho_j)^2}\,\rho_j\,\|q\|_\infty \;+\; \left(\frac{(1+\rho_j)^2}{(1-\rho_j)^2}-1\right)\underbrace{\int\limits_R q\,|v_j|^2\,dx}_{\le\,\|q\|_\infty} \\ &\le \frac{(1+\rho_j)^2}{(1-\rho_j)^2}\,\rho_j\,\|q\|_\infty \;+\; \frac{4\rho_j\,\|q\|_\infty}{(1-\rho_j)^2} \;\le\; c_1\,\rho_j\end{aligned}$$

for some $c_1>0$. From (2.14) and the observation that $\lambda_j^2\rho_j \to 0$, we note that $\int_R q\,|w_j|^2\,dx$ tends to zero and thus also $w_j \to 0$ in $L^2(D)$ by Lemma 2.3. Finally, from the (2.13) and the assumption $1+q \ge q_1 > 0$, we conclude that $q_1\lambda_j^2\|v_j\|^2_{L^2(D)} \le \int_D |q|\,|w_j|^2 dx \to 0$; that is, $\lambda_j v_j$ tends to zero in $L^2(D)$. Now we use the continuous dependence of the solution to $\Delta v_j = \lambda_j v_j + q w_j$ which yields that v_j tends to zero in $H^2(D)$, a contradiction to $\|(v_j,w_j)\|_X = 1$. $\square$

Therefore, as in the previous section we fix $\lambda_0 > 0$ such that A_{λ_0} is an isomorphism and rewrite the equation $A_\lambda(v,w) = 0$ in the form

$$(v,w) \;+\; A_{\lambda_0}^{-1}(A_\lambda - A_{\lambda_0})(v,w) \;=\; 0\,.$$

The observation that $A_\lambda - A_{\lambda_0} = (\lambda-\lambda_0)K$ for some compact operator K yields discreteness of the spectrum. We formulate the result as a theorem.

Theorem 2.8. *Let there exist $q_0 > 0$ and some neighborhood R of ∂D such that $q \ge q_0$ on R or $q \le -q_0$ on R. Then the set of transmission eigenvalues is discrete. In $\mathbb{C}$ there is no (finite) accumulation point.*

References

[1] A.-S. Bonnet-Ben Dhia, L. Chesnel, H. Haddar, On the use of T-coercivity to study the interior transmission eigenvalue problem, C. R. Acad. Sci. Paris, Ser. I 340 (2011) 647–651.
[2] F. Cakoni, H. Haddar, Transmission Eigenvalues in Inverse Scattering Theory: Inside Out II, MSRI Publications, vol. 60, 2012, pp. 527–578.
[3] D. Colton, A. Kirsch, L. Päivärinta, Far field patterns for acoustic waves in an inhomogeneous medium, SIAM J. Math. Anal. 20 (1989) 1472–1483.
[4] D.L. Colton, R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 3rd edition, Springer, 2013.
[5] D. Colton, L. Päivärinta, J. Sylvester, The interior transmission problem, Inverse Probl. Imaging 1 (2007) 13–28.
[6] E. Lakshtanov, B. Vainberg, Elliptic in the interior transmission problem in anisotropic media, SIAM J. Math. Anal. 44 (2012) 1165–1174.
[7] E. Lakshtanov, B. Vainberg, Applications of elliptic operator theory to the isotropic interior transmission eigenvalue problem, Inverse Probl. 29 (2013) 104003.
[8] P. Monk, Finite Element Methods for Maxwell's Equations, Oxford University Press, 2003.
[9] L. Päivärinta, J. Sylvester, Transmission eigenvalues, SIAM J. Math. Anal. 40 (2008) 738–753.
[10] J. Sylvester, Discreteness of transmission eigenvalues via upper triangular compact operators, SIAM J. Math. Anal. 44 (1) (2011) 341–354.