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# A remark on the convergence of the inverse $\sigma_k$-flow

## *Une remarque sur la convergence du $\sigma_k$-flot inverse*

Jian Xiao [1]

*Institute of Mathematics, Fudan University, 200433 Shanghai, China*

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ABSTRACT

We study the positivity of cohomology classes related to the convergence problem of the inverse $\sigma_k$-flow, according to a conjecture proposed by Lejmi and Székelyhidi.


RÉSUMÉ

Nous étudions la positivité des classes de cohomologie liée au problème de la convergence du $\sigma_k$-flot inverse, suivant une conjecture proposée par Lejmi et Székelyhidi.


## 1. Introduction

From the point of view that relates the existence of canonical Kähler metrics with algebro-geometric stability conditions, Lejmi and Székelyhidi [12] proposed a numerical criterion characterizing when the inverse $\sigma_k$-flow converges. We aim to study the positivity of related cohomology classes in their conjecture. We generalize their conjecture by weakening the numerical condition on $X$ a little bit.

**Conjecture 1.1.** *(See [12, Conjecture 18].) Let $X$ be a compact Kähler manifold of dimension $n$, and let $\omega, \alpha$ be two Kähler metrics over $X$ satisfying*

$$\int_X \omega^n - \frac{n!}{k!(n-k)!}\omega^{n-k} \wedge \alpha^k \geq 0. \tag{1.1}$$

*Then there exists a Kähler metric $\omega' \in \{\omega\}$ such that*

$$\omega'^{n-1} - \frac{(n-1)!}{k!(n-k-1)!}\omega'^{n-k-1} \wedge \alpha^k > 0 \tag{1.2}$$

*E-mail address:* jian.xiao@ujf-grenoble.fr.
[1] Current address: Institut Fourier, Université Joseph-Fourier, 38402 Saint-Martin-d'Hères, France.

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*as a smooth $(n-1, n-1)$-form if and only if*

$$\int_V \omega^p - \frac{p!}{k!(p-k)!}\omega^{p-k} \wedge \alpha^k > 0 \tag{1.3}$$

*for every irreducible subvariety of dimension $p$ with $k \le p \le n-1$.*

For previous works closely related to this conjecture, we refer the reader to [9,3,4,17,10]. And in this note we mainly concentrate on the case when $k = 1$ and $k = n-1$.

For $k = 1$, [5, Theorem 3] confirmed this conjecture for toric manifolds. Over a general compact Kähler manifold, it is not hard to see that the implication (1.2) $\Rightarrow$ (1.3) holds. In the reverse direction, we prove that $\{\omega - \alpha\}$ must be a Kähler class under the numerical conditions in Conjecture 1.1 for $k = 1$; indeed, this is a necessary condition of (1.2) and [12, Proposition 14] proved this over Kähler surfaces.

**Theorem 1.1.** *Let $X$ be a compact Kähler manifold of dimension $n$, and let $\omega, \alpha$ be two Kähler metrics over $X$ satisfying the numerical conditions in Conjecture 1.1 for $k = 1$. Then $\{\omega - \alpha\}$ is a Kähler class.*

For $k = n-1$, we have the following similar result.

**Theorem 1.2.** *Let $X$ be a compact Kähler manifold of dimension $n$, and let $\omega, \alpha$ be two Kähler metrics over $X$ satisfying the numerical conditions in Conjecture 1.1 for $k = n-1$. Then the class $\{\omega^{n-1} - \alpha^{n-1}\}$ lies in the closure of the Gauduchon cone, i.e. it has nonnegative intersection number with every pseudoeffective $(1,1)$-class.*

## 2. Proof of the main results

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2.

### 2.1. *Theorem 1.1*

**Proof.** The first observation is that, when $k = 1$, the inequalities in the numerical conditions are just the right-hand side in weak transcendental holomorphic Morse inequalities. Recall that Demailly's conjecture on weak transcendental holomorphic Morse inequalities (see, e.g., [1, Conjecture 10.1]) is stated as follows:

*Let $X$ be a compact complex manifold of dimension $n$, and let $\gamma, \beta$ be two nef classes over $X$. Then we have*

$$\mathrm{vol}\,(\gamma - \beta) \ge \gamma^n - n\gamma^{n-1} \cdot \beta.$$

*In particular, $\gamma^n - n\gamma^{n-1} \cdot \beta > 0$ implies the class $\gamma - \beta$ is big, that is, $\gamma - \beta$ contains a Kähler current.*

Note that the last statement has been proved for Kähler manifolds by [15] (see also [18]), that is, if $X$ is a compact Kähler manifold then $\gamma^n - n\gamma^{n-1} \cdot \beta > 0$ implies that there exists a Kähler current in the class $\gamma - \beta$.

We apply this bigness criterion to the classes $\{\omega\}$ and $\{\alpha\}$, then the numerical condition (1.3) implies that $\{\omega - \alpha\}_{|V}$ is a big class on every proper irreducible subvariety $V$. More precisely, if $V$ is singular, then by some resolution of the singularities, we have a proper modification $\pi : \widehat{V} \to V$ with $\widehat{V}$ smooth, and by (1.3) we know

$$\pi^*\{\omega\}^p_{|V} - p\pi^*\{\omega\}^{p-1}_{|V} \cdot \pi^*\{\alpha\}_{|V} > 0,$$

thus the class $\pi^*\{\omega - \alpha\}_{|V}$ contains a Kähler current over $\widehat{V}$. Hence, by applying the pushforward map $\pi_*$ we obtain that the class $\{\omega - \alpha\}_{|V}$ is big over $V$.

In particular, by (1.1) and (1.3) the restriction of the class $\{\omega - (1-\epsilon)\alpha\}$ is big on every irreducible subvariety (including $X$ itself) for any sufficiently small $\epsilon > 0$.

We claim that this yields $\{\omega - (1-\epsilon)\alpha\}$ is a Kähler class over $X$ for any $\epsilon > 0$ small. Indeed, our proof implies the following fact.

- Assume $\beta$ is a big class over a compact complex manifold (or compact complex space) and its restriction to every irreducible subvariety is also big, then $\beta$ is a Kähler class over $X$.

To this end, we will argue by induction on the dimension of $X$. If $X$ is a compact complex curve, then this is obvious. For the general case, we need a result of Mihai Păun (see [14,13]):

*Let $X$ be a compact complex manifold (or compact complex space), and let $\beta = \{T\}$ be the cohomology class of a Kähler current $T$ over $X$. Then $\beta$ is a Kähler class over $X$ if and only if the restriction $\beta_{|Z}$ is a Kähler class on every irreducible component $Z$ of the Lelong sublevel set $E_c(T)$.*

As $\{\omega-(1-\epsilon)\alpha\}$ is a big class on $X$, by Demailly's regularization theorem [7], we can choose a Kähler current $T \in \{\omega-(1-\epsilon)\alpha\}$ such that $T$ has analytic singularities on $X$. Then the singularities of $T$ are just the Lelong sublevel set $E_c(T)$ for some positive constant $c$. For every irreducible component $Z$ of $E_c(T)$, by (1.3) the restriction $\{\omega-(1-\epsilon)\alpha\}_{|Z}$ is a big class. After resolution of the singularities of $Z$ if necessary, we obtain a Kähler current $T_Z \in \{\omega-(1-\epsilon)\alpha\}_{|Z}$ over $Z$ with its analytic singularities contained in a proper subvariety of $Z$, and for every irreducible subvariety $V \subseteq Z$ the restriction $\{\omega-(1-\epsilon)\alpha\}_{|V}$ is also a big class. By induction on the dimension, we get that $\{\omega-(1-\epsilon)\alpha\}_{|Z}$ is a Kähler class over $Z$. So the above result of [14,13] implies $\{\omega-(1-\epsilon)\alpha\}$ is a Kähler class over $X$, finishing the proof of our claim.

As $\epsilon > 0$ is arbitrary, we infer that $\{\omega-\alpha\}$ is a nef class on $X$. Next we prove that $\{\omega-\alpha\}$ is a big class. By [8, Theorem 2.12], we only need to show that

$$\mathrm{vol}(\{\omega-\alpha\}) = \int_X (\omega-\alpha)^n > 0.$$

Since $\{\omega-\alpha\}$ is nef, we can compute the derivative of the function $\mathrm{vol}(\omega-t\alpha)$ for any $t \in [0,1)$. Thus we have

$$\begin{aligned}\mathrm{vol}(\{\omega\}-\{\alpha\}) - \mathrm{vol}(\{\omega\}) &= \int_0^1 \frac{d}{\mathrm{d}t}\mathrm{vol}(\{\omega\}-t\{\alpha\})\,\mathrm{d}t \\ &= -\int_0^1 n\{\omega-t\alpha\}^{n-1}\cdot\{\alpha\}\,\mathrm{d}t,\end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{vol}(\{\omega\}-\{\alpha\}) &= \mathrm{vol}(\{\omega\}) - \int_0^1 n\{\omega-t\alpha\}^{n-1}\cdot\{\alpha\}\,\mathrm{d}t \\ &\geq \int_0^1 n(\{\omega\}^{n-1}-\{\omega-t\alpha\}^{n-1})\cdot\{\alpha\}\,\mathrm{d}t.\end{aligned}$$

Here the last line follows from the equality (1.1). Since $\omega, \alpha$ are Kähler metrics, this shows $\mathrm{vol}(\{\omega-\alpha\}) > 0$. Thus $\{\omega-\alpha\}$ is a big and nef class on $X$ with its restriction to every irreducible subvariety being big and nef. By the above arguments, we know that $\{\omega-\alpha\}$ must be a Kähler class.

Finally, we give an alternative proof of the fact that the class $\{\omega-\alpha\}$ is nef using the main result of [6] instead of using [14,13]. (I would like to thank Tristan C. Collins who pointed out this to me.) Since $\{\omega\}$ is a Kähler class, the class $\{\omega-t\alpha\}$ is Kähler for $t > 0$ small. Let $s$ be the largest number such that $\{\omega-s\alpha\}$ is nef. We prove that $s \geq 1$. Otherwise, suppose that $s < 1$. Then by the numerical conditions (1.1) and (1.3), the bigness criterion given by transcendental holomorphic Morse inequalities implies that the class $\{\omega-s\alpha\}$ is big if $s < 1$, and furthermore, this holds for all irreducible subvarieties in $X$. Thus $\{\omega-s\alpha\}$ is big and nef on every irreducible subvariety $V$ in $X$. This means the null locus of the big and nef class $\{\omega-s\alpha\}$ is empty, and then the main result of [6] implies that $\{\omega-s\alpha\}$ is a Kähler class. This contradicts the definition of $s$, so we get $s \geq 1$, or equivalently, $\{\omega-\alpha\}$ must be a nef class. Then by the estimate of the volume $\mathrm{vol}(\{\omega-\alpha\})$ as above, we know that $\{\omega-\alpha\}$ is also big and nef over every irreducible subvariety of $X$. By applying [6] again, this proves that $\{\omega-\alpha\}$ must be a Kähler class. □

**Remark 2.1.** If $X$ is a smooth projective variety of dimension $n$ and $\{\omega\}$ and $\{\alpha\}$ are the first Chern classes of holomorphic line bundles, then the nefness of the class $\{\omega-\alpha\}$ just follows from Kleiman's ampleness criterion, since the numerical condition (1.3) for $p=1$ implies that the divisor class $\{\omega-\alpha\}$ has non-negative intersection against every irreducible curve.

### *2.2. Theorem 1.2*

Next we give the proof of Theorem 1.2.

**Proof.** The proof mainly depends on Boucksom's divisorial Zariski decomposition for pseudoeffective $(1,1)$-classes [2] and the bigness criterion for the difference of two movable $(n-1,n-1)$-classes [19].

Through a sufficiently small perturbation of the Kähler metric $\alpha$, e.g., by replacing $\alpha$ with

$$\alpha_\epsilon = (1-\epsilon)\alpha,\ \epsilon \in (0,1),$$

we can obtain that the inequality in (1.1) is strict for the classes $\{\omega\}$ and $\{\alpha_\epsilon\}$. We claim that in this case the $(n-1, n-1)$-class $\{\omega^{n-1}-\alpha_\epsilon^{n-1}\}$ has nonnegative intersections with all pseudoeffective $(1,1)$-classes. Then, by letting $\epsilon$ tend to zero, we conclude that the desired result for the class $\{\omega^{n-1} - \alpha^{n-1}\}$. Thus we can assume from the beginning that the inequality in (1.1) is strict for the classes $\{\omega\}$ and $\{\alpha\}$.

Let $\beta$ be a pseudoeffective $(1,1)$-class over $X$. By [2, Section 3], $\beta$ admits a divisorial Zariski decomposition

$$\beta = Z(\beta) + N(\beta).$$

Note that $N(\beta)$ is the class of some effective divisor (may be zero) and $Z(\beta)$ is a modified nef class. In particular, we have

$$\{\omega^{n-1} - \alpha^{n-1}\} \cdot N(\beta) \geq 0. \tag{2.1}$$

For any $\delta > 0$, we have

$$Z(\beta) + \delta\{\omega\} = \pi_*\{\widehat{\omega}\}$$

for some modification $\pi : \widehat{X} \to X$ and some Kähler metric $\widehat{\omega}$ on $\widehat{X}$ (see [2, Proposition 2.3]).

By our assumption on (1.1), we have

$$\int_{\widehat{X}} \pi^*\omega^n - n\pi^*\omega \wedge \pi^*\alpha^{n-1} > 0. \tag{2.2}$$

By [19, Theorem 3.3] (or [18, Remark 3.1]), the inequality (2.2) implies that the class $\{\pi^*\omega^{n-1} - \pi^*\alpha^{n-1}\}$ contains a strictly positive $(n-1, n-1)$-current. This implies that

$$\begin{aligned}
&\{\omega^{n-1} - \alpha^{n-1}\} \cdot (Z(\beta) + \delta\{\omega\}) \\
&= \{\omega^{n-1} - \alpha^{n-1}\} \cdot \pi_*\{\widehat{\omega}\} \\
&= \pi^*\{\omega^{n-1} - \alpha^{n-1}\} \cdot \{\widehat{\omega}\} \\
&> 0.
\end{aligned}$$

As $\delta$ is arbitrary, we get that $\{\omega^{n-1} - \alpha^{n-1}\} \cdot Z(\beta) \geq 0$. With (2.1), we show that

$$\{\omega^{n-1} - \alpha^{n-1}\} \cdot \beta \geq 0.$$

Since $\beta$ can be any pseudoeffective $(1,1)$-class, this implies $\{\omega^{n-1} - \alpha^{n-1}\}$ lies in the closure of the Gauduchon cone by [20, Proposition 2.1] (which is [11, Lemma 3.3]). □

**Remark 2.2.** We expect that $\{\omega^{n-1} - \alpha^{n-1}\}$ should have strictly positive intersection numbers with nonzero pseudoeffective $(1,1)$-classes. To show this, one only needs to verify this for modified nef classes.

**Remark 2.3.** Let $X$ be a smooth projective variety, and assume that $\{\omega^{n-1} - \alpha^{n-1}\}$ is a curve class. Then the numerical condition (1.3) in Theorem 1.2 implies that $\{\omega^{n-1} - \alpha^{n-1}\}$ is a movable class by [1, Theorem 2.2].

## 3. Further discussions

In analogy with Theorem 1.1 and Theorem 1.2, one would like to prove a similar positivity of the class $\{\omega^k - \alpha^k\}$. To generalize our results in this direction, one can apply [18, Remark 3.1] (see also [16] for further results). By [18, Remark 3.1], we know that the condition

$$\int_V \omega^p - \frac{p!}{k!(p-k)!}\omega^{p-k} \wedge \alpha^k > 0$$

implies that the class $\{\omega^k - \alpha^k\}_{|V}$ contains a strictly positive $(k,k)$-current over every irreducible subvariety $V$ of dimension $p$ with $k < p \leq n-1$. However, the difficulties appear as we know little about the singularities of positive $(k,k)$-currents for $k > 1$. We have no analogues of Demailly's regularization theorem for such currents.

Inspired by the prediction of Conjecture 1.1, we propose the following question on the positivity of positive $(k,k)$-currents.

**Question 3.1.** Let $X$ be a compact Kähler manifold (or general compact complex manifold) of dimension $n$. Let $\Omega \in H^{k,k}(X, \mathbb{R})$ be a big $(k,k)$-class, i.e. a class that can be represented by a strictly positive $(k,k)$-current over $X$. Assume that the restriction class $\Omega_{|V}$ is also big over every irreducible subvariety $V$ with $k \leq \dim V \leq n-1$, then does $\Omega$ contain a smooth strictly positive $(k,k)$-form in its Bott–Chern class? Or does $\Omega$ contain a strictly positive $(k,k)$-current with analytic singularities of codimension at least $n-k+1$ in its Bott–Chern class?

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